PLANAR GRAPHS WITH BOUNDED TREEWIDTH

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- For every edge $e = (v, w) \in E$, there is a subset $X_i, i \in I$ with $v \in X_i$ and $w \in X_i$.
- For all $i, j, k \in I$, if j lies on the path from i to k, then $X_i \cap X_k \subseteq X_j$.

The treewidth of a tree-decomposition $(\{X_i | i \in\}, T)$ is $\max_{i \in I} |X_i| - 1$. The treewidth of G, denoted by treewidth(G), is the minimum treewidth of a tree-decomposition of G, taken over all possible tree-decompositions of G.

There are several alternative ways to characterize the class of graphs with treewdith $\leq k$. See e.g. [1].

Next we introduce the (new) notions of the vertex and edge remember number of maximal spanning forests of a graph.

Consider a maximal spanning forest $T = (V, F)$ of a graph $G = (V, E)$. (I.e., T contains a spanning tree of every connected component of G.) To every edge $e = (v, w) \in E - F$, we can associate its fundamental cycle, i.e. the unique cycle which consists of e and the path from v to w in T. We define the vertex remember number of G, relative to T (denoted by $vr(G, T)$), to the maximum over all $v \in V$ of the number of such fundamental cycles that use v. Similarly, the edge remember number of G, relative to T is denoted by $er(G, T)$, and defined as the maximum over all $e \in E$ of the number of such fundamental cycles that use e. We let degree(G) denote the maximum vertex degree of G.

We also use the notion of "minor". A graph H is a minor of a graph G by a number of edge-contractions, i.e., by a number of times replacing two adjacent vertices v, w by a new vertex that is adjacent to all vertices that were adjacent to v or w.

3 Preliminary results.

In this section we give some preliminary results, dealing with the notions treewidth, vertex and edge remember number and minor.

Theorem 3.1
For all graphs $G = (V, E)$ and maximal spanning forests $T = (V, F)$ of G : $er(G, T) \leq vr(G, T) \leq \frac{\text{degree}(G)}{2}.er(G, T)$.

Proof.
Suppose edge $(v, w) \in E$ is used by $er(G, T)$ fundamental cycles. Then also $er(G, T)$ fundamental cycles use vertex v, hence $vr(G, T) \geq er(G, T)$.

Next suppose vertex $v \in V$ is used by $vr(G, T)$ fundamental cycles. Suppose v is adjacent to edges $e_1, \ldots, e_k \in E$, and suppose edge e_i is used by α_i fundamental cycles ($1 \leq i \leq k$). Each fundamental cycle that uses v must use exactly 2 edges, adjacent to v, hence $\sum_{i=1}^{k} x_i = vr(G, T)$, so $er(G, T) \geq \max_{1 \leq i \leq k} \alpha_i \geq \frac{2}{k} vr(G, T) \geq \frac{2}{degree(G)} vr(G, T)$. □

The following relation will prove helpful to give bounds on the treewidth of classes of graphs.

Theorem 3.2
Let $T = (V, F)$ be a maximal spanning forest of graph $G = (V, E)$. Then treewidth $(G) \leq \max(vr(G, T), er(G, T) + 1)$.

Proof.
Let T' be the tree $(V \cup F, F')$, with $F' = \{(v,e)|v \in V, e \in F, \exists w \in V : e = (v,w)\}$, *i.e.* T' is obtained by adding an extra vertex in the middle of each edge in T. We show how to construct sets X_i, $i \in V \cup F$, such that $(\{X_i | i \in V \cup F), T' = (V \cup F, F'))$ is a tree-decomposition of G.

First, for all $v \in V$, take $v \in X_v$ and for all $(v,w) \in F$, take $v, w \in X_{(v,w)}$.

Secondly, for each edge $(v,w) \in E - F$, choose arbitrarily one of v or w, say v. Now add v to each X_x, for all vertices $x \in V, x \neq w$ that are on the fundamental cycle of (v,w). Only do not add v to X_w. Also, add v to X_e, for all edges $e \in F$ on the fundamental cycle of (v,w).

We now claim that we have a correct tree-decomposition of V. Clearly $\bigcup X_i = V$. For $(v,w) \in F, (v,w) \in X_{(v,w)}$. For $(v,w) \in E - F$, suppose v is chosen to be added to X_x, X_e for x, e on the fundamental cycle. There are two edges in this fundamental cycle adjacent to w; one of them is (v,w), the other (w,x) for some $x \in V$. Now $v \in X_{(w,x)}$ and $w \in X_{(w,x)}$.

Finally, note that for all $v \in V$, if $v \in X_v$, and $v \in X_i$, then $v \in X_j$ for all j on the path from i to v in T', by construction. It follows that for all $i,j,k \in I$, if j is on the path in T' from i to k, then $X_i \cap X_k \subseteq X_j$. So $(\{X_i | i \in V \cup F\}, T')$ is a correct tree-decomposition. For $v \in V : |X_v| \leq 1 + vr(G,T)$; and for $e \in F : |X_e| \leq 2 + er(G,T)$. So the treewidth of this tree-decomposition is at most $\max(vr(G,T), er(G,T) + 1)$. □

We also use the following well-known result.

Lemma 3.3
Let H be a minor of G. Then treewidth $(H) \leq$ treewidth (G).

Proof.
If G' is a subgraph of G, then clearly treewidth $(G') \leq$ treewidth (G). If a graph G' is obtained from G by contracting the edge (v,w) to a vertex v', then one shows that treewidth $(G') \leq$ treewidth (G) as follows. Let $(\{X_i | i \in I\}, T = (I,F))$ be a tree-decomposition of G with treewidth k. Let $X'_i = X_i$, if $v, w \notin X_i$, and $X'_i = (X_i - \{v,w\}) \cup \{v'\}$, if $v \in X_i$ or $w \in X_i$. Clearly $(\{X'_i | i \in I\}, T = (I,F))$ is a tree-decomposition of G' with treewidth $\leq k$. Now the lemma follows. □

In [12,13], Robertson and Seymour show that for every planar graph H, there is a constant w_H, such that every graph G, that does not contain H as a minor, has treewidth $\leq w_H$. However, the constants w_H are very large. For a discussion of these results, see [10].

4 Main results.

In this section we give upperbounds for the treewidth for some classes of planar graphs. First we consider the class of outerplanar graphs. A graph is outerplanar, if it is planar and it can be drawn in the plane, such that all vertices lie on the exterior face. It can be shown that every outerplanar graph is a series-parallel graph and hence has treewidth ≤ 2. The latter can also be shown in the following way.

For every outerplanar graph $G = (V,E)$, there must be a vertex v with degree$(v) = 1 \vee$ degree$(v) = 2$. Suppose degree$(v) = 2$. Let $(v,w) \in E, (v,x) \in E, w \neq x$. Now

$G' = (V - \{v\}, (E - \{(v,w),(v,x)\}) \cup \{(w,x)\})$ is an outerplanar graph, and we may assume, with induction, that we have a tree-decomposition $(\{X_i | i \in I\}, T = (I, F))$ of G with treewidth ≤ 2. There must be an $i \in I$, with $w \in X_i \wedge x \in X_i$. Now let $i^* \notin I, I^* = I \cup \{i^*\}, X_{i^*} = \{v, w, x\}$ and $T^* = (I^*, F \cup \{(i, i^*)\})$. One easily verifies that $(\{X_i | i \in I^*\}, T^*)$ is a tree-decomposition of G with treewidth at most 2.

A generalization of the outerplanar graphs are the k-outerplanar graphs. The notion of k-outerplanar graphs was introduced by Baker [4].

Definition.

- A graph $G = (V, E)$ is 1-outerplanar if and only if it is outerplanar.
- For $k \geq 2$, a graph $G = (V, E)$ is k-outerplanar if and only if it is planar and it has a planar embedding such that if all vertices on the exterior face (and all adjacent edges) are deleted, then the connected components of the remaining graph are all $(k-1)$-outerplanar.

We will show a bound of $3k - 1$ on the treewidth of k-outerplanar graph. We need a series of lemma's.

Lemma 4.1
Let $G = (V, E)$ be a planar graph with some given planar embedding. Let $H = (V, E')$ be the graph, that is obtained from G by removing all edges on the exterior face. Let $T' = (V, F')$ be a maximal spanning forest of H. Then there exists a maximal spanning forest $T = (V, F)$ of G, such that $er(G,T) \leq er(H,T') + 2$, and $vr(G,T) \leq vr(H,T') + \text{degree}(G)$.

Proof.
Consider the graph $K = (V, (E - F) \cup F')$, i.e. the graph with edges in T', or in G but not in H. Let $T = (V, F)$ be a maximal spanning forest of K, such that $T' \subseteq T$, i.e. T is obtained by adding edges from $E - F$ to T'. Note that every fundamental cycle in K, relative to T, must form the boundary of an interior face in K. As each edge is adjacent to at most 2 interior faces, and each vertex is adjacent to at most degree(G) interior faces, it follows that $er(K,T) \leq 2$, and $vr(K,T) \leq \text{degree}(G)$.

As T is also a maximal spanning forest of G, and each fundamental cycle in G either is a fundamental cycle in H, or a fundamental cycle in K, $er(G,T) \leq er(K,T) + er(H,T') \leq er(H,T') + 2$, and $vr(G,T) \leq vr(K,T) + vr(H,T') \leq vr(H,T') + \text{degree}(G)$. □

Lemma 4.2
Let $G = (V, E)$ be an outerplanar graph with degree(G) ≤ 3. Then there exists a maximal spanning forest $T = (V, F)$, with $er(G,T) \leq 2$ and $vr(G,T) \leq 2$.

Proof.
If one removes all edges on the exterior face of G, a tree or forest $T' = (V, F')$ results. Clearly $er(T',T') = vr(T',T') = 0$. The result follows now as in lemma 4.1 by observing that also each vertex is adjacent to at most 2 interior faces. □

Lemma 4.3
Let $G = (V, E)$ be a k-outerplanar graph with degree(G) ≤ 3. Then there exists a maximal spanning forest $T = (V, F)$ with $er(G,T) \leq 2k$, and $vr(G,T) \leq 3k - 1$.

Proof.
Use induction to k. The case $k = 1$ was shown in lemma 4.2. Let $k \geq 2$. If we remove all edges on the exterior face of G, then each vertex on the exterior face has degree 0 or 1, so a $(k-1)$-outerplanar graph is obtained. The result now follows with induction and lemma 4.1. □

Lemma 4.4
For every k-outerplanar graph $G = (V, E)$, there exists a k-outerplanar graph $H = (V', E')$, such that G is a minor of H, and degree$(H) \leq 3$.

Proof.
We can replace every vertex with degree $d \geq 4$ by a path of $d - 2$ vertices of degree 3, in such a way that the graph stays k-outerplanar. □

Now we are ready to prove the main results.

Theorem 4.5
For every k-outerplanar graph $G = (V, E)$, treewidth $(G) \leq 3k - 1$.

Proof.
For $k = 1$, the result is well-known. Suppose $k \geq 2$. By lemma 4.4, there exists a k-outerplanar graph H, such that G is a minor of H, and degree$(H) \leq 3$. By lemma 4.3, there exists a maximal spanning forest T of H, such that $er(H,T) \leq 2k$ and $vr(H,T) \leq 3k-1$. By lemma 3.2, treewidth $(H) \leq \max\{3k-1, 2k+1\} = 3k-1$. By lemma 3.3 treewidth $(G) \leq 3k-1$. □

Robertson and Seymour [12] define the radius of a planar graph as follows.

For every face R in a planar embedding M of G, define $d(R)$ to be the minimum value of k, such that there is a sequence of faces $R_o, \ldots, R_k$, with R_o the exterior face, and $R_k = R$, and for $1 \leq j \leq k$, there is a vertex v that is both on face R_{j-1} and R_j. The radius $\rho(M)$ of M is the minimum value of d such that $d(R) \leq d$ for all regions R of M. The radius of a planar graph is the minimum of the radii of its drawings.

In [12], it is proved that a planar graph with radius $\leq d$ has treewidth $\leq 3d+1$. As a k-outerplanar graph has radius k or $k+1$ [5], from this result it follows that a k-outerplanar graph has treewidth $\leq 3k+4$. So we improved on this bound with a constant 5. The proof on treewidth of k-outerplanar graphs can easily be modified to give the following result.

Theorem 4.6
Let $G = (V, E)$ be a planar graph with radius $\leq d, d \geq 1$. Then treewidth $(G) \leq 3d$.

Proof.
First observe that one can prove an analogue of lemma 4.4 for graphs with radius $\leq d$. Hence, it is sufficient to prove the results for graphs with degree ≤ 3. With induction to d one can show: radius $(G) \leq d$ and degree$(G) \leq 3 \Rightarrow$ there exists a maximal spanning forest $T = (V, F)$ of G with $er(G,T) \leq 2k$ and $vr(G,T) \leq 3k$. If $d = 0$, then G is a tree, and the result follows directly. For $d > 1$, note that if every edge on the exterior face is removed, then a graph with radius $\leq d - 1$ results. The proof continues as in the case of

k-outerplanar graphs. □

The last class of graphs that is considered is the class of the Halin graphs.

Definition.
A graph $G = (V, E)$ is a Halin graph, if it can be obtained by embedding a tree without vertices with degree 2 in the plane, and connecting its leaves by a cycle that crosses none of its edges, and $|V| \geq 4$.

Theorem 4.7
Let $G = (V, E)$ be a Halin graph.
The treewidth $(G) = 3$.

Proof.
By noting that radius $(G) = 1$, it follows directly from theorem 4.6 that treewidth $(G) \leq 3$. (An alternative proof can be given by observing that G is a minor of a Halin graph H, with degree $(H) = 3$, and then applying lemma 4.1.)

As G contains K_4, a clique with 4 vertices, as a minor (contract all interior vertices to one vertex, a "wheel" results, and then contract further to K_4), treewidth $(G) \geq$ treewidth $(K_4) = 3$. □

The restriction $|V| \geq 4$ is usually not given with the definition of Halin graphs. Note that we only exclude the degenerate case of a graph with 2 vertices and 3 parallel edges.

5 Final remarks.

It should be noted that all proofs are constructive, and can be transformed to polynomial algorithms of small degree, not depending on the radius of the graphs involved or the outerplanarity-number k, that actually find the tree-decompositions with the indicated treewidth. In [5] it is shown that the radius and the outerplanairity-number k can be determined in polynomial time for arbitrary planar graphs.

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