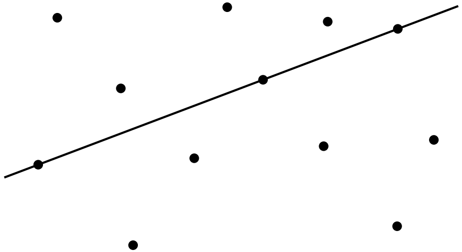
Arrangements and Duality

Computational Geometry

Lecture 11: Arrangements and Duality

Three Points on a Line

Question: In a set of n points, are there 3 points on a line?



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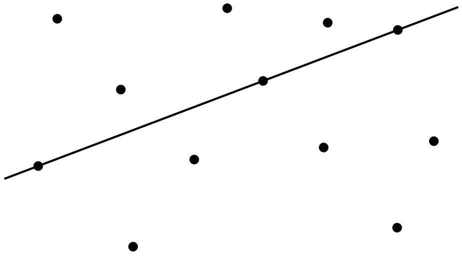

Naive algorithm: tests all triples in $O(n^3)$ time

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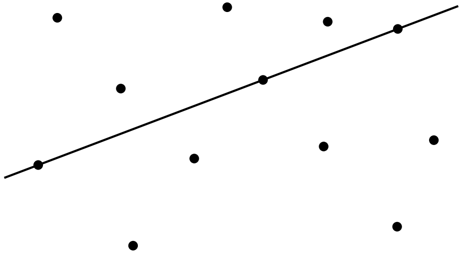

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Faster algorithm: uses duality and arrangements

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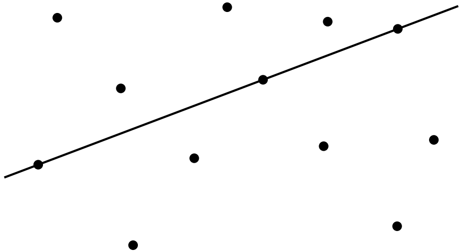
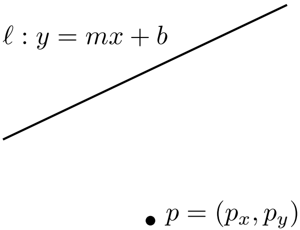


Naive algorithm: tests all triples in $O(n^3)$ time
Faster algorithm: uses duality and arrangements

Note: other motivation in chapter 8 of the book

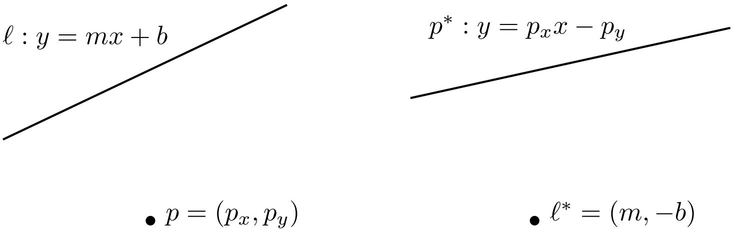
Duality



Note:

Duality

primal plane | dual plane

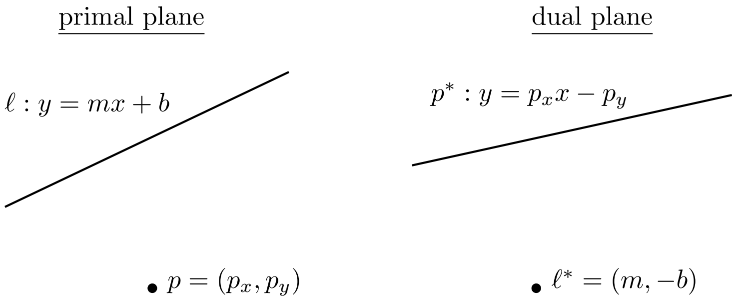


point $p = (p_x, p_y) \mapsto$ line $p^* : y = p_x x - p_y$

line $\ell : y = mx + b \mapsto$ point $\ell^* = (m, -b)$

Note:

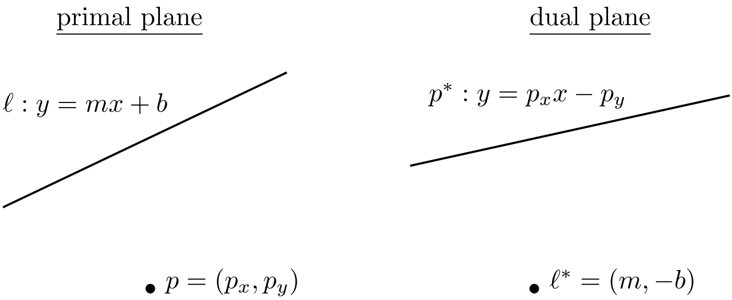
Duality



$$\text{point } p = (p_x, p_y) \mapsto \text{line } p^* : y = p_x x - p_y$$
$$\text{line } \ell : y = mx + b \mapsto \text{point } \ell^* = (m, -b)$$

Note: self inverse $(p^*)^* = p, \quad (\ell^*)^* = \ell$

Duality



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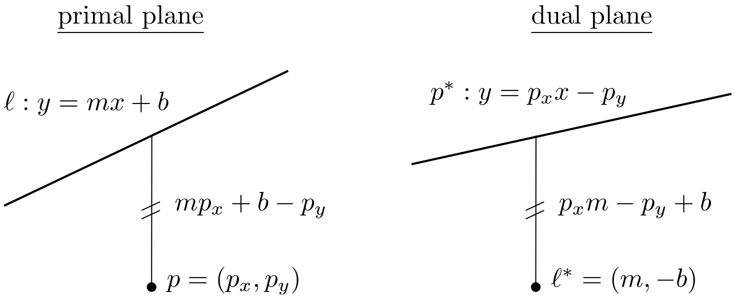
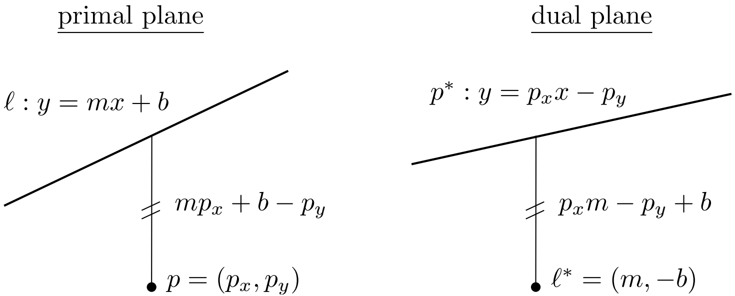
Note: does not handle vertical lines

Duality



Duality preserves vertical distances

Duality



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$\Rightarrow$ incidence preserving: $p \in \ell$ if and only if $\ell^* \in p^*$

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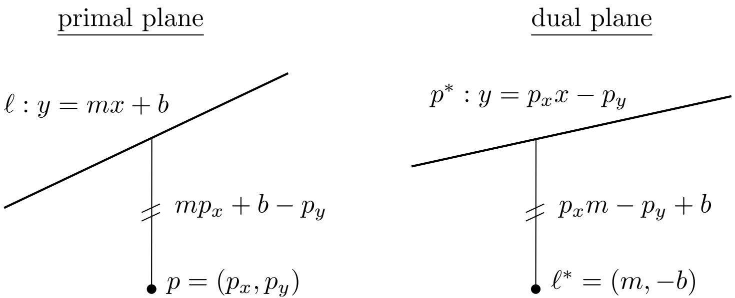


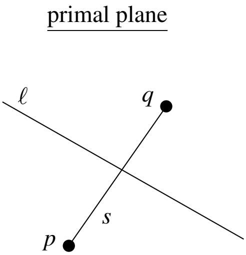
Duality preserves vertical distances

$\Rightarrow$ incidence preserving: $p \in \ell$ if and only if $\ell^* \in p^*$

$\Rightarrow$ order preserving: p lies below ℓ if and only if ℓ^* lies below p^*

Duality

It can be applied to other objects, like segments



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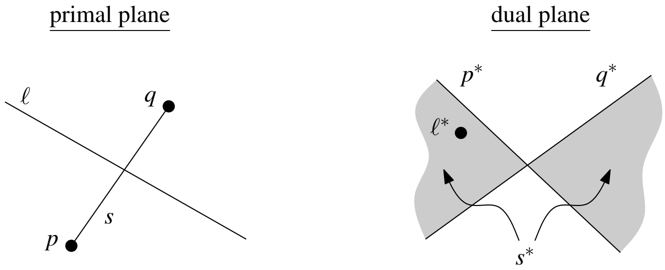


The dual of a segment is a double wedge

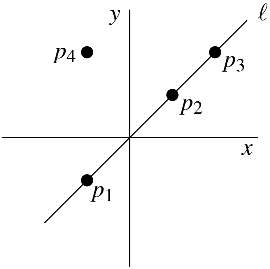
Question: What line would dualize to a point in the right part of the double wedge?

Usefulness of Duality

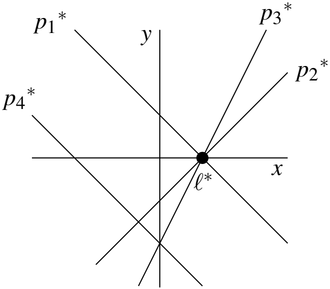
Why use duality? It gives a new perspective!

Detecting three points on a line dualizes to detecting three lines intersecting in a point

primal plane



dual plane



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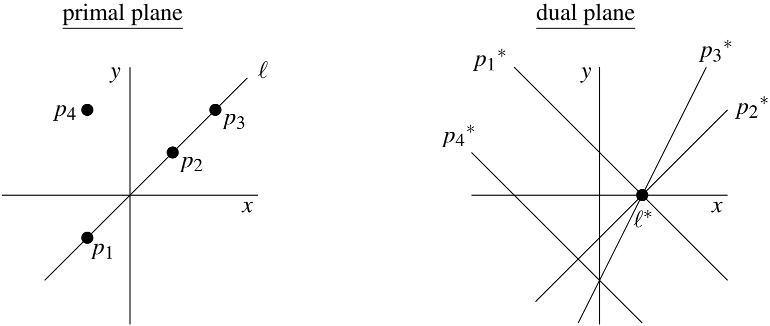
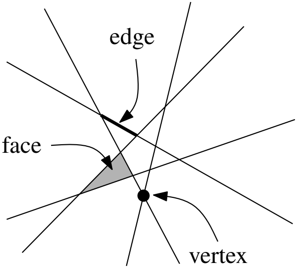


Next we use arrangements

Arrangements of Lines

Arrangement $\mathcal{A}(L)$: subdivision induced by a set of lines L

- consists of *faces*, *edges* and *vertices* (some unbounded)
- arrangements consist of other geometric objects too, like line segments, circles, higher-dimensional objects



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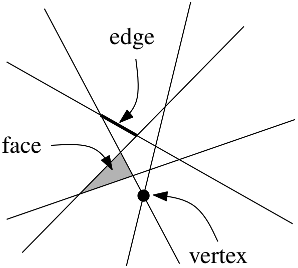


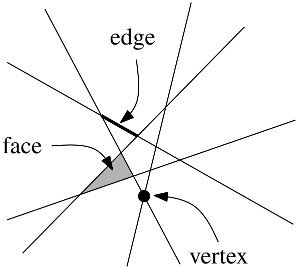
Arrangements of Lines

Combinatorial Complexity:

- $\leq n(n-1)/2$ vertices
- $\leq n^2$ edges
- $\leq n^2/2+n/2+1$ faces:
  add lines incrementally

$$1+\sum_{i=1}^{n} i = n(n+1)/2+1$$

- equality holds in *simple* arrangements



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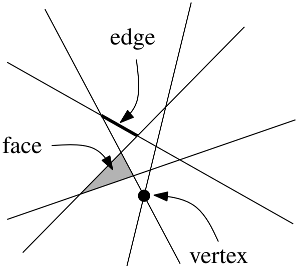

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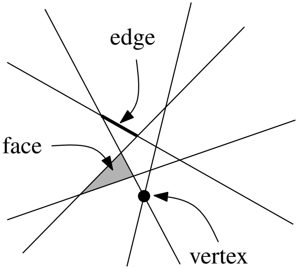

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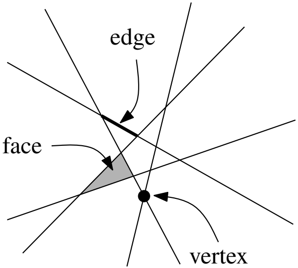

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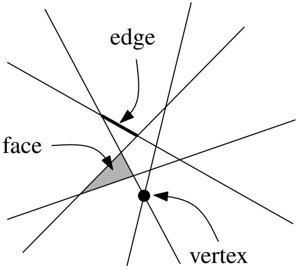
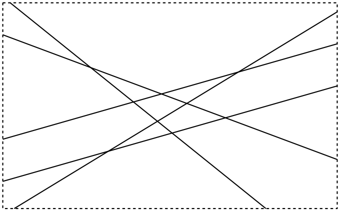
Overall $O(n^2)$ complexity



Constructing Arrangements

Goal: Compute $\mathcal{A}(L)$ in bounding box in DCEL representation



- plane sweep for line segment intersection: $O((n+k)\log n) = O(n^2 \log n)$
- faster: incremental construction

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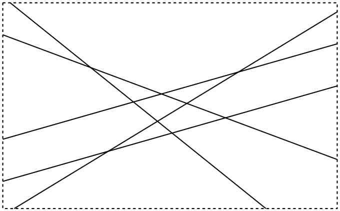

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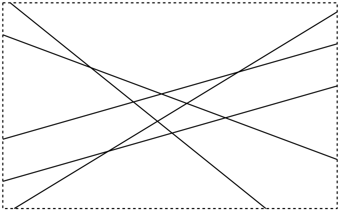

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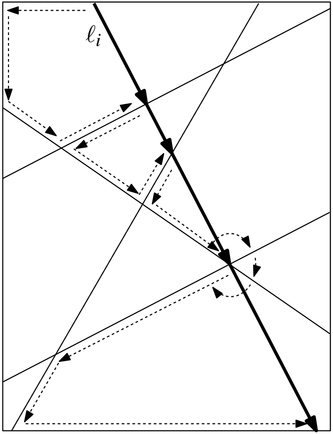
Incremental Construction



Algorithm ConstructArrangement(L)
Input. Set L of n lines
Output. DCEL for $\mathcal{A}(L)$ in $\mathcal{B}(L)$
1. Compute bounding box $\mathcal{B}(L)$
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3. **for** $i \leftarrow 1$ **to** n
4. **do** insert ℓ_i

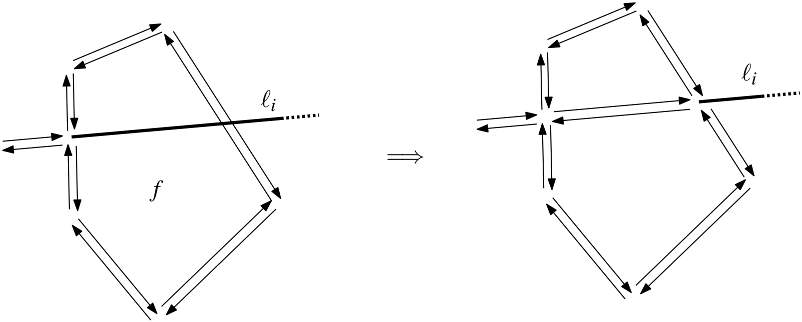
Incremental Construction

Algorithm CONSTRUCTARRANGEMENT(L)
Input. A set L of n lines in the plane
Output. DCEL for subdivision induced by $\mathcal{B}(L)$ and the part of $\mathcal{A}(L)$ inside $\mathcal{B}(L)$, where $\mathcal{B}(L)$ is a suitable bounding box

1. Compute a bounding box $\mathcal{B}(L)$ that contains all vertices of $\mathcal{A}(L)$ in its interior
2. Construct DCEL for the subdivision induced by $\mathcal{B}(L)$
3. **for** $i \leftarrow 1$ **to** n
4. **do** Find the edge e on $\mathcal{B}(L)$ that contains the leftmost intersection point of ℓ_i and $\mathcal{A}_i$
5. $f \leftarrow$ the bounded face incident to e
6. **while** f is not the unbounded face, that is, the face outside $\mathcal{B}(L)$
7. **do** Split f, and set f to be the next face intersected by ℓ_i

Incremental Construction

Face split:



Incremental Construction

Runtime analysis:

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Runtime analysis:

1. $O(n^2)$

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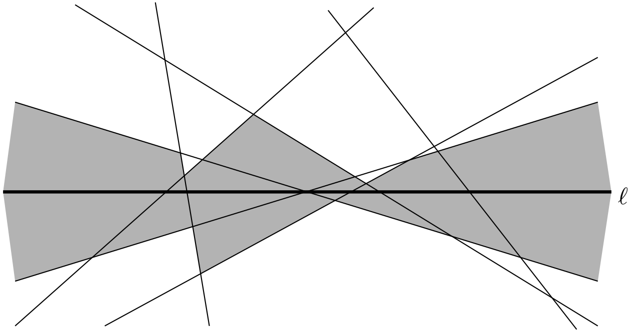
The zone of a line ℓ in an arrangement $\mathcal{A}(L)$ is the set of faces of $\mathcal{A}(L)$ whose closure intersects ℓ



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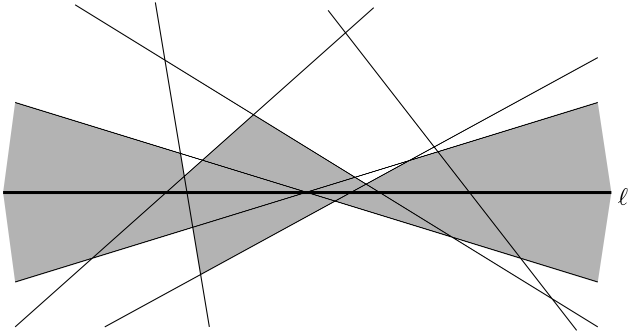

Theorem: The complexity of the zone of a line in an arrangement of m lines is $O(m)$

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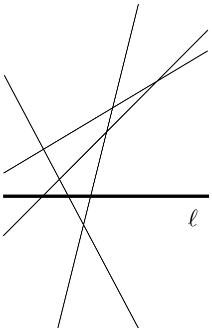
Proof:

- We can assume ℓ horizontal and no other line is horizontal
- We count number of *left-bounding* edges
- We show by induction on m that this at most $5m$:
  - $m = 1$: trivially true
    $m > 1$: only at most 3 new edges if ℓ_1 is unique.



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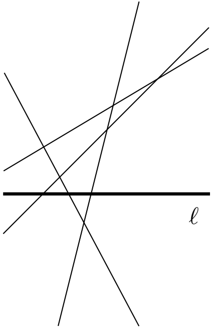

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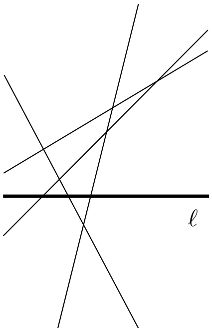

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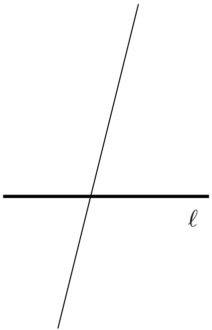

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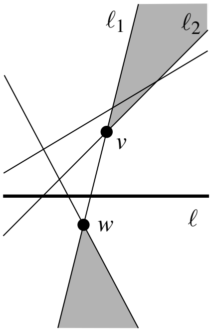

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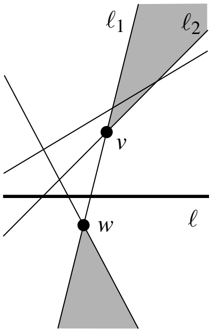

Incremental Construction

Run time analysis:

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$\sum_{i=1}^{n} O(i) = O(n^2)$

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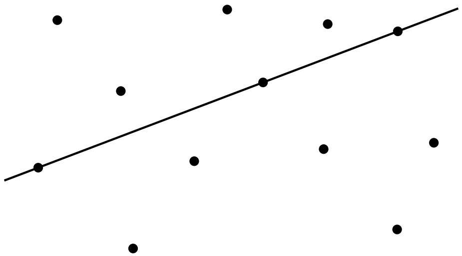

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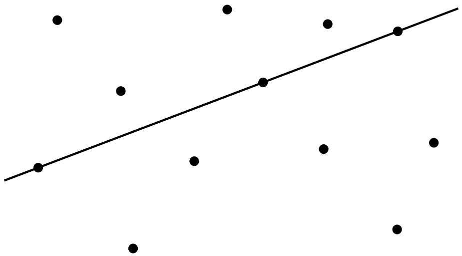


Algorithm:

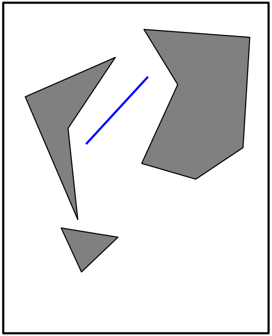
- run incremental construction algorithm for dual problem
- stop when 3 lines pass through a point

Run time: $O(n^2)$

Example: Motion Planning

Where can the rod move by translation (no rotations) while avoiding obstacles?

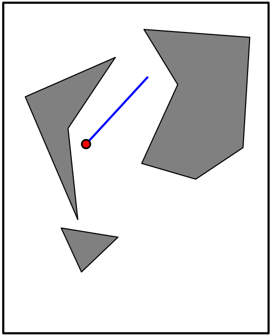
- pick a reference point:
  lower end-point of rod
- shrink rod to a point,
  expand obstacles accordingly:
  locus of semi-free placements
- reachable configurations:
  cell of initial configuration in
  arrangement of line segments



Example: Motion Planning

Where can the rod move by translation (no rotations) while avoiding obstacles?

- pick a reference point: lower end-point of rod
- shrink rod to a point, expand obstacles accordingly: locus of semi-free placements
- reachable configurations: cell of initial configuration in arrangement of line segments



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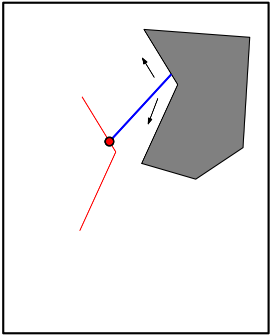

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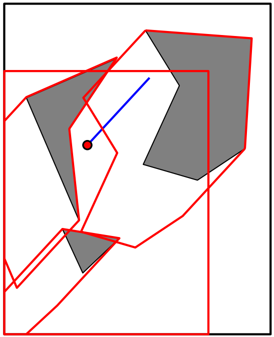

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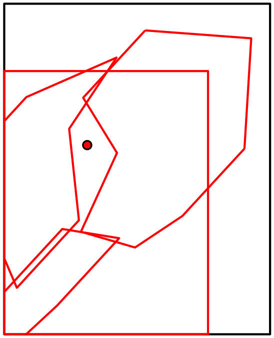
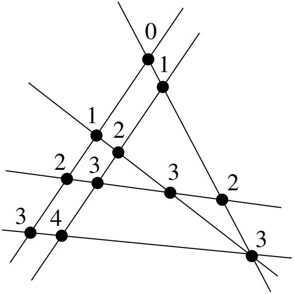


k-levels in Arrangements

The level of a point in an arrangement of lines is the number of lines strictly above it



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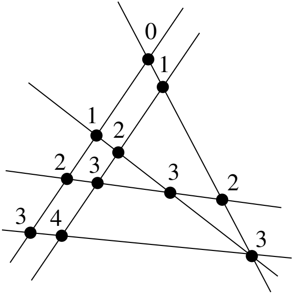

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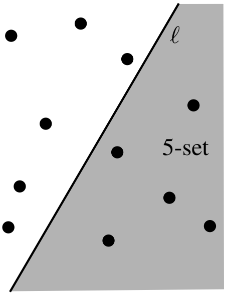

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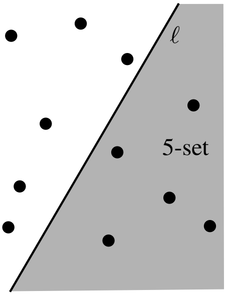

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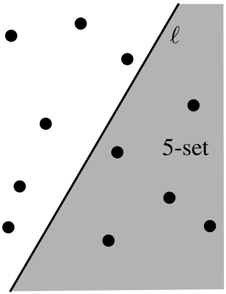


Three dimensions

In 3D, we have point-plane duality; lines dualize to other lines

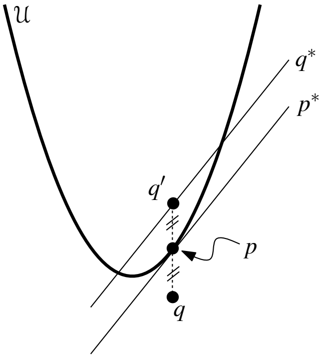
An arrangement induced by n planes in 3D has complexity $O(n^3)$

Deciding whether a set of points in 3D has four or more co-planar points can be done in $O(n^3)$ time (dualize and construct the arrangement)

More Duality

A geometric interpretation:

- parabola $\mathcal{U} : y = x^2/2$
- point $p = (p_x, p_y)$ on $\mathcal{U}$
- derivative of $\mathcal{U}$ at p is p_x, i.e., p^* has same slope as the tangent line
- the tangent line intersects y-axis at $(0, -p_x^2/2)$
- $\Rightarrow p^*$ is the tangent line at p



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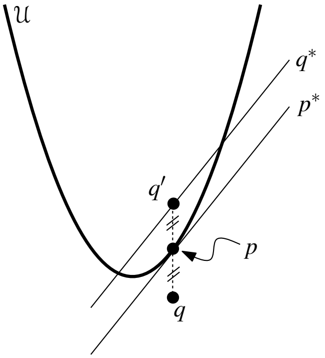

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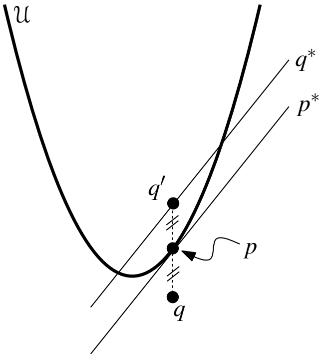

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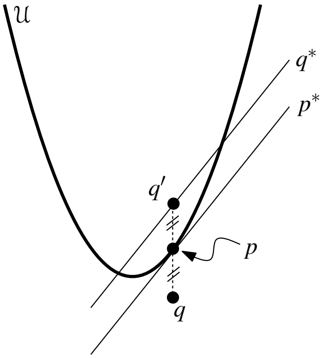

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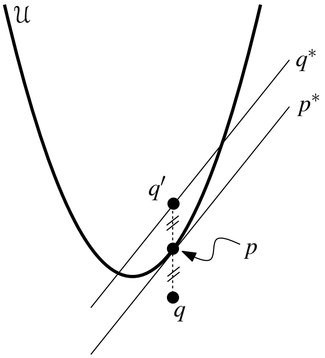

Summary

Duality is a useful tool to reformulate certain problems on points in the plane to lines in the plane, and vice versa

Dualization of line segments is especially useful

Arrangements, zones of lines in arrangements, and levels in arrangements are useful concepts in computational geometry

All of this exists in three and higher dimensional spaces too