Planar point location

Computational Geometry

Lecture 9: Planar point location

Point location

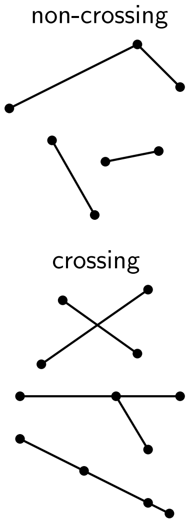
Point location problem: Preprocess a planar subdivision such that for any query point q, the face of the subdivision containing q can be given quickly (name of the face)

- From GPS coordinates, find the region on a map where you are located
- Subroutine for many other geometric problems (Chapter 13: motion planning, or shortest path computation)

Point location

Planar subdivision: Partition of the plane by a set of non-crossing line segments into vertices, edges, and faces

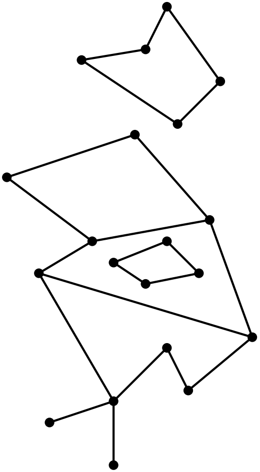
non-crossing: disjoint, or at most a shared endpoint



Point location

Data structuring question, so interest in query time, storage requirements, and preprocessing time

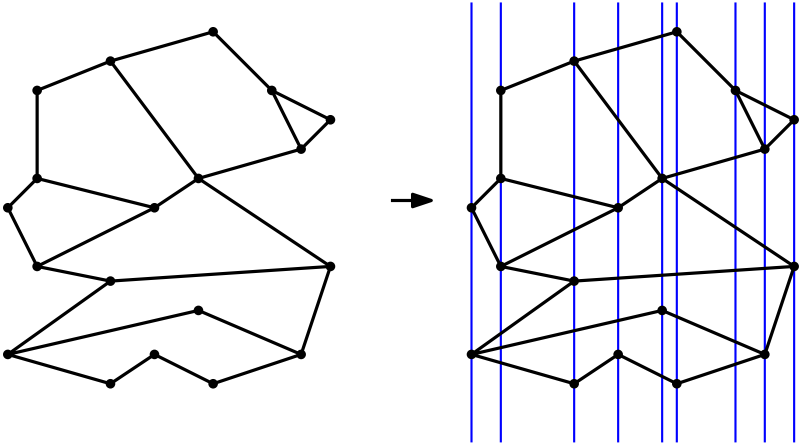
To store: set of n non-crossing line segments and the subdivision they induce



First solution

Idea: Draw vertical lines through all vertices, then do something for every vertical strip that appears

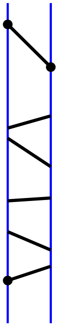
First solution



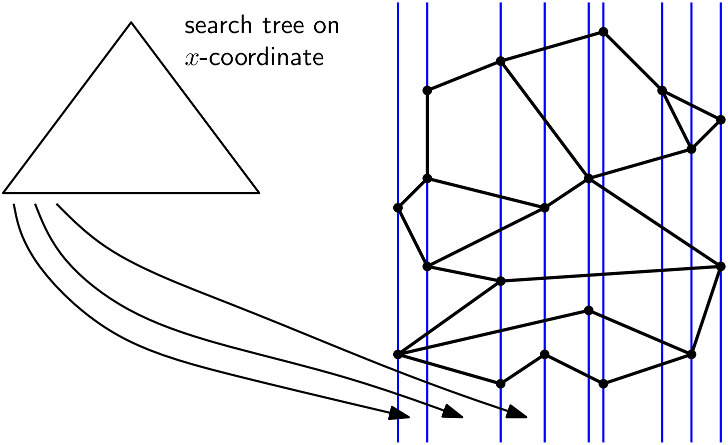
In one strip

Inside a single strip, there is a well-defined bottom-to-top order on the line segments

Use this for a balanced binary search tree that is valid if the query point is in this strip (knowing between which edges we are is knowing in which face we are)



Solution with strips

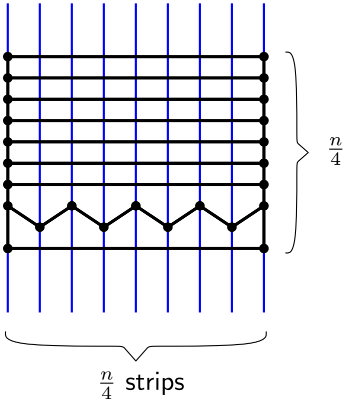


Solution with strips

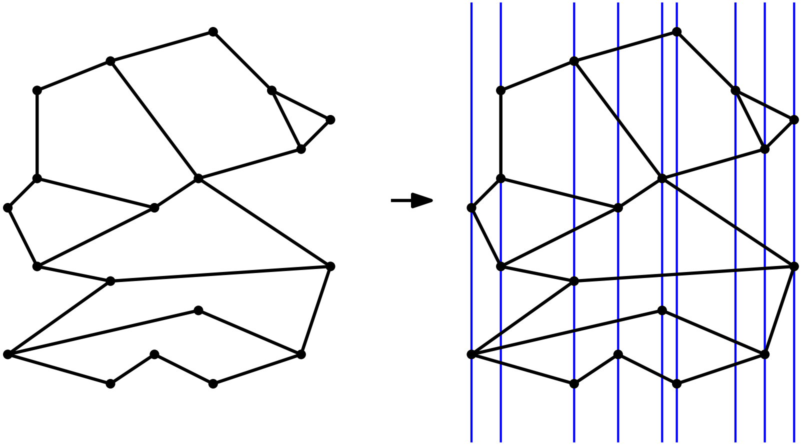
To answer a query with $q = (q_x, q_y)$, search in the main tree with q_x to find a leaf, then follow the pointer to search in the tree that is correct for the strip that contains q_x

Question: What are the storage requirements and what is the query time of this structure?

Solution with strips



Solution with strips



Different idea

The vertical strips idea gave a *refinement* of the original subdivision, but the number of faces went up from linear in n to quadratic in n

Is there a different refinement whose size remains linear, but in which we can still do point location queries easily?

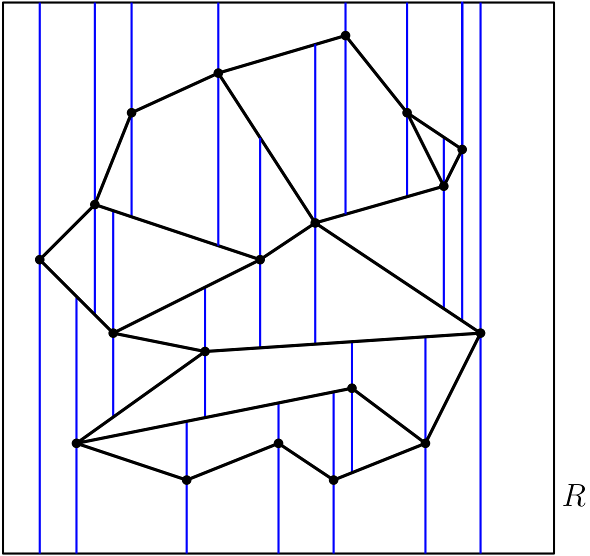
Vertical decomposition

Suppose we draw vertical extensions from every vertex up and down, *but only until the next line segment*

- Assume the input line segments are not vertical
- Assume every vertex has a distinct x-coordinate
- Assume we have a bounding box R that encloses all line segments that define the subdivision

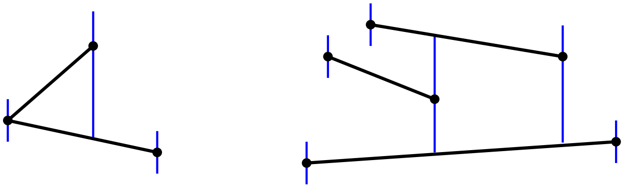
This is called the vertical decomposition or trapezoidal decomposition

Vertical decomposition



Vertical decomposition faces

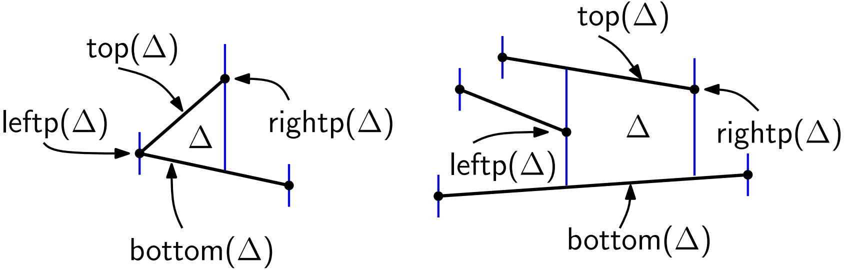
The vertical decomposition has triangular and trapezoidal faces



Vertical decomposition faces

Every face has a vertical left and/or right side that is a vertical extension, and is bounded from above and below by some line segment of the input

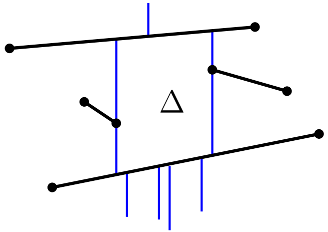
The left and right sides are defined by some endpoint of a line segment



Vertical decomposition faces

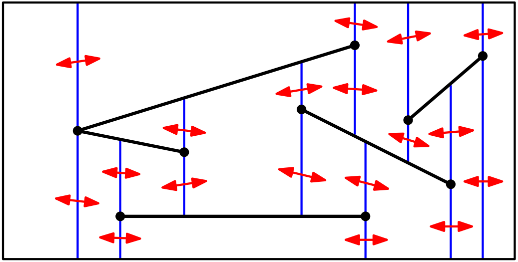
Every face is defined by no more than four line segments

For any face, we ignore vertical extensions that end on top(Δ) and bottom(Δ)



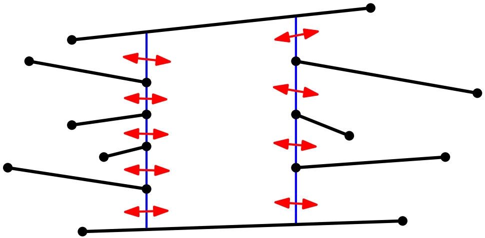
Neighbors of faces

Two trapezoids (including triangles) are *neighbors* if they share a vertical side



Each trapezoid has 1, 2, 3, or 4 neighbors

Neighbors of faces



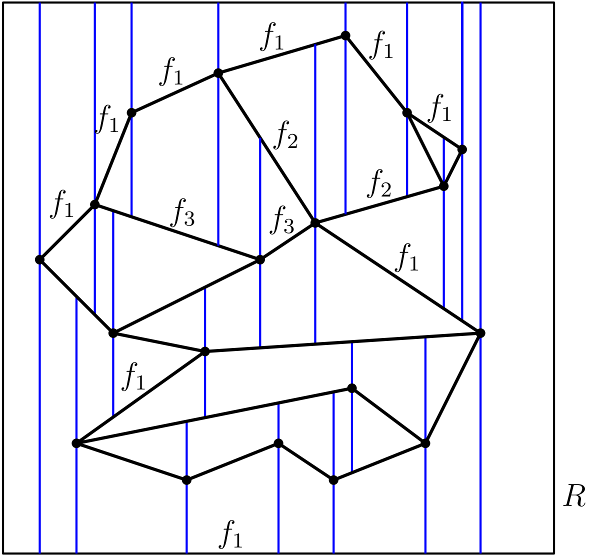
A trapezoid could have many neighbors if vertices had the same x-coordinate

Representation

We could use a DCEL to represent a vertical decomposition, but we use a more direct & convenient structure

- Every face Δ is an object; it has fields for top(Δ), bottom(Δ), leftp(Δ), and rightp(Δ) (two line segments and two vertices)
- Every face has fields to access its up to four neighbors
- Every line segment is an object and has fields for its endpoints (vertices) and the name of the face in the original subdivision directly above it
- Each vertex stores its coordinates

Representation

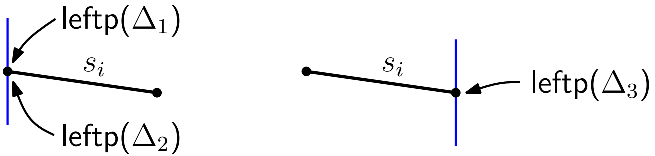


Representation

Any trapezoid Δ can find out the name of the face it is part of via bottom(Δ) and the stored name of the face

Complexity

A vertical decomposition of n non-crossing line segments inside a bounding box R, seen as a proper planar subdivision, has at most $6n+4$ vertices and at most $3n+1$ trapezoids



Point location preprocessing

The input to planar point location is a planar subdivision, for example in DCEL format

First, store with each edge the name of the face above it (our structure will find the edge below any query point)

Then extract the edges to define a set S of non-crossing line segments; ignore the DCEL otherwise

Point location solution

We will use *randomized incremental construction* to build, for a set S of non-crossing line segments,

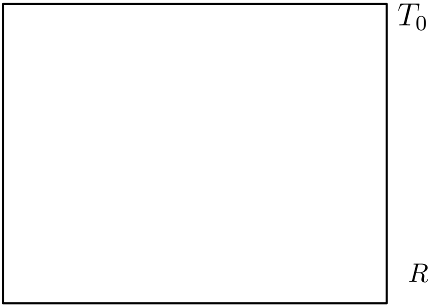
- a vertical decomposition T of S and R
- a search structure D whose leaves correspond to the trapezoids of T

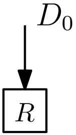
The simple idea: Start with R, then add the line segments in random order and maintain T and D

Point location solution

Let $s_1, \ldots, s_n$ be the n line segments in random order

Let T_i be the vertical decomposition of R and $s_1, \ldots, s_i$, and let D_i be the search structure obtained by inserting $s_1, \ldots, s_i$ in this order





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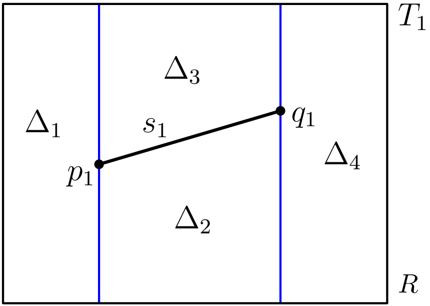
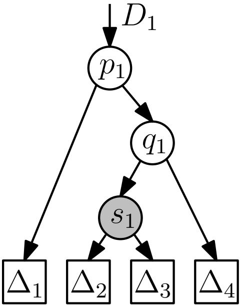




Point location solution

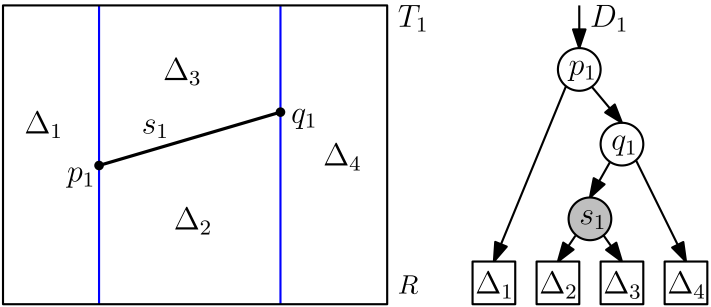
The search structure D has x-nodes, which store an endpoint, and y-nodes, which store a line segment s_j

For any query point t, we only test at an x-node: Is t left or right of the vertical line through the stored point?

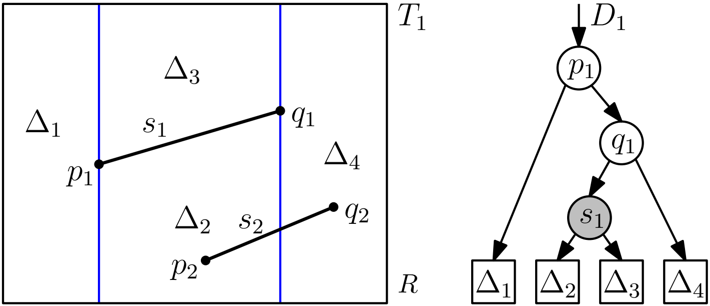
For any query point t, we only test at an y-node: Is t below or above the stored line segment?

We will guarantee that the question at a y-node is only asked if the query point t is between the vertical lines through p_j and q_j, if line segment $s_j = \overline{p_j q_j}$ is stored

Point location solution



Point location solution



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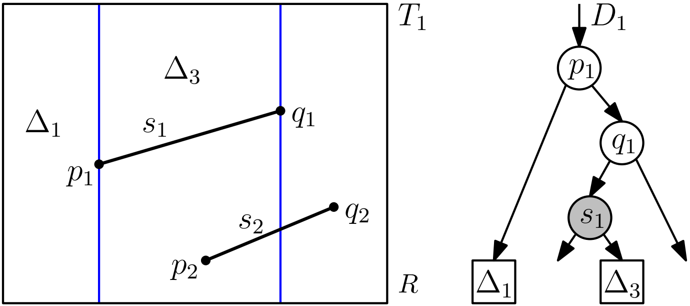

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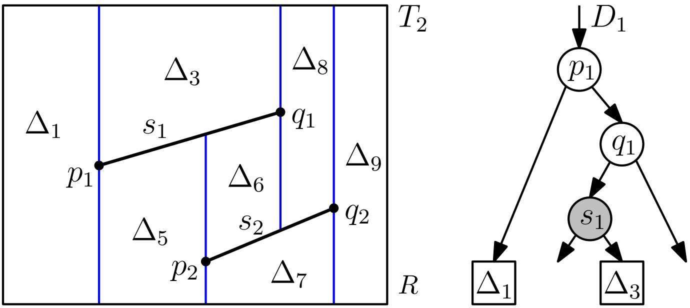
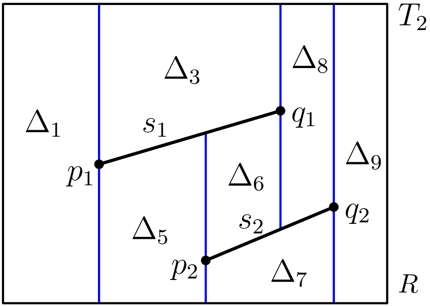
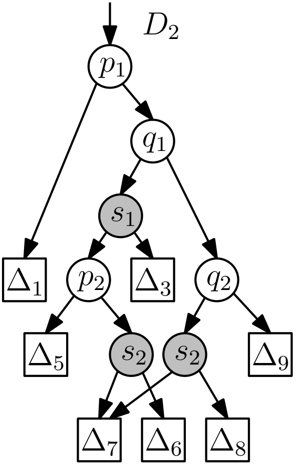


Point location solution



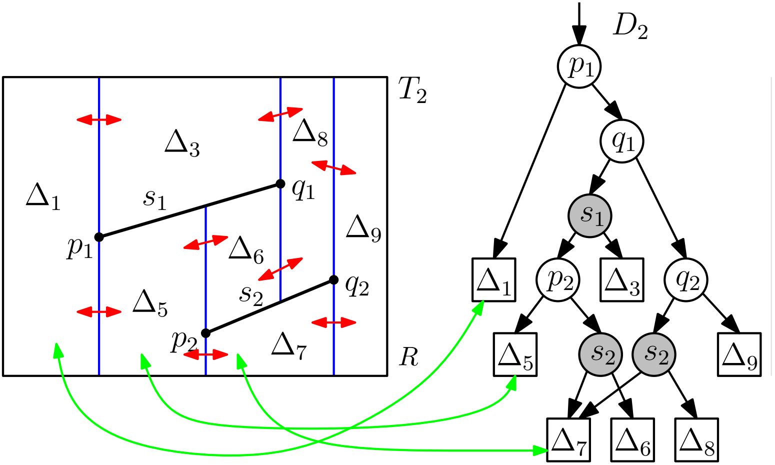


Point location solution

We want only one leaf in D to correspond to each trapezoid; this means we get a search *graph* instead of a search *tree*

It is a directed acyclic graph, or DAG, hence the name D

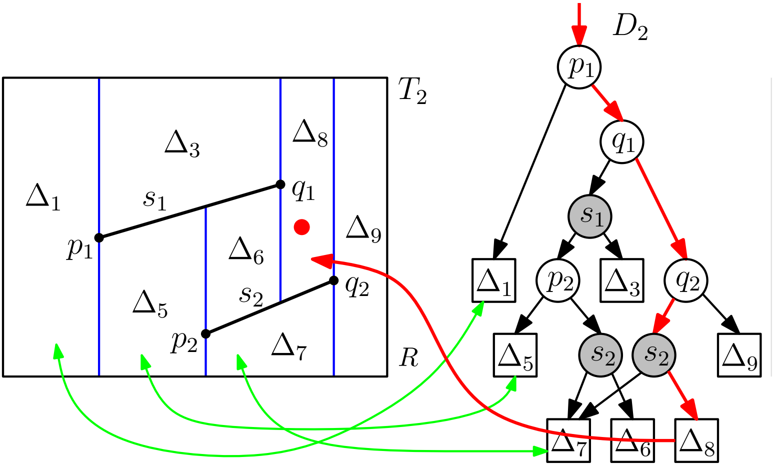
Point location solution



Point location query

A point location query is done by following a path in the search structure D to a leaf, then following its pointer to a trapezoid of T, then accessing bottom(..) of this trapezoid, and reporting the name of the face stored with it

Point location query



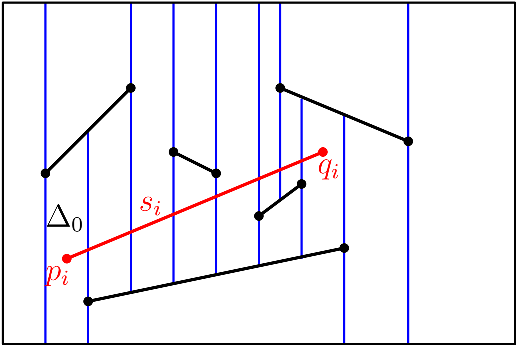
The incremental step

Suppose we have D_{i-1} and T_{i-1}, how do we add s_i?

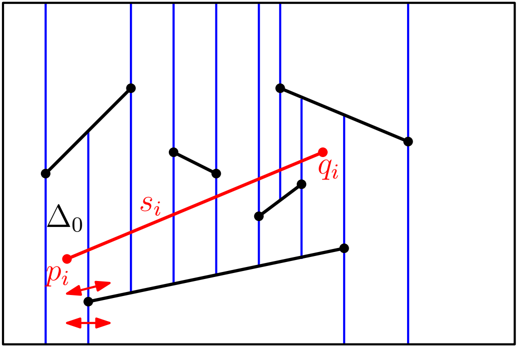
Because D_{i-1} is a *valid point location structure* for $s_1, \ldots, s_{i-1}$, we can use it to find the trapezoid of T_{i-1} that contains p_i, the left endpoint of s_i

Then we use T_{i-1} to find all other trapezoids that intersect s_i

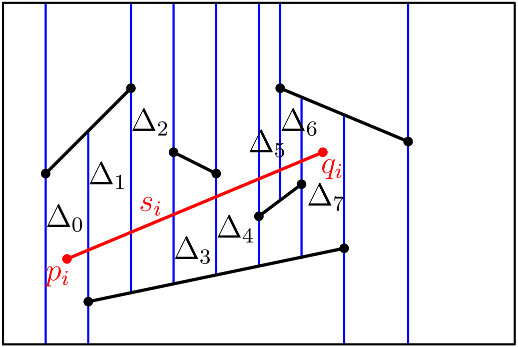
Find intersected trapezoids



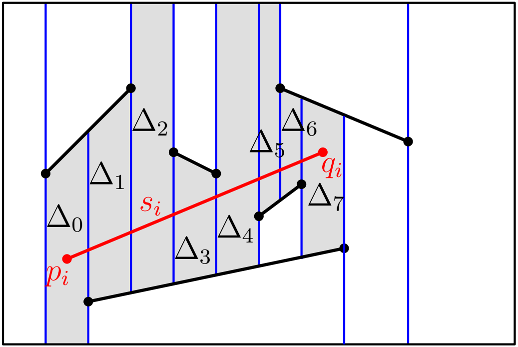
Find intersected trapezoids



Find intersected trapezoids



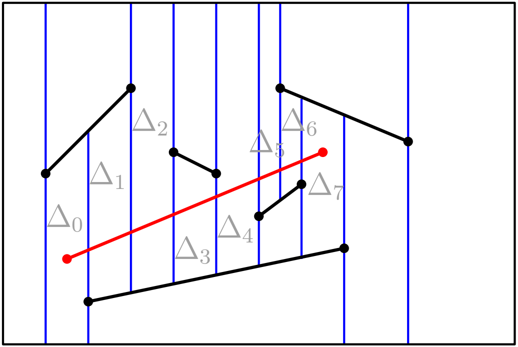
Find intersected trapezoids



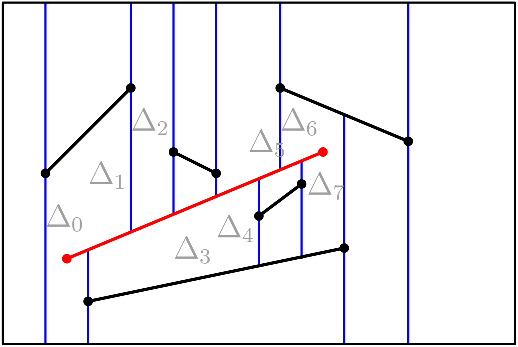
Find intersected trapezoids

After locating the trapezoid that contains p_i, we can determine all k trapezoids that intersect s_i in $O(k)$ time by traversing T_{i-1}

Updating the vertical decomposition



Updating the vertical decomposition



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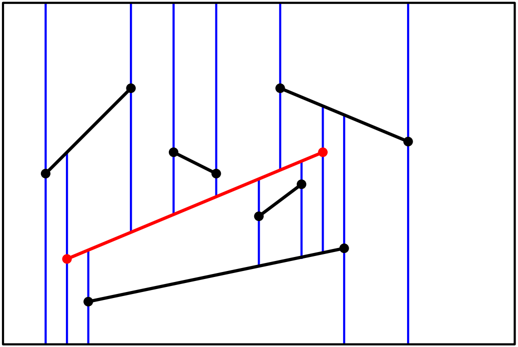


Updating the vertical decomposition

We can update the vertical decomposition in $O(k)$ time as well

Updating the search structure

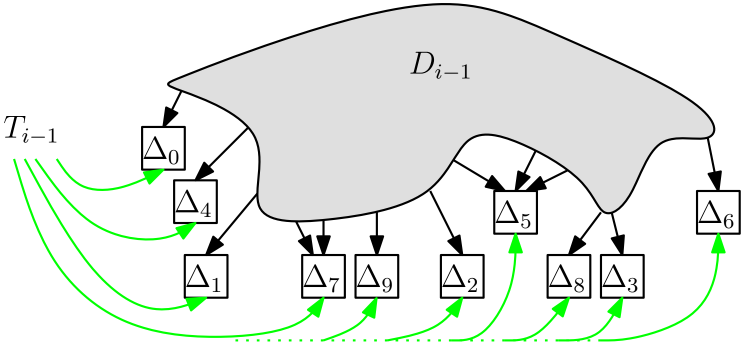
The search structure has k leaves that are no longer valid as leaves; they become internal nodes

We find these using the pointers from T_{i-1} to D_{i-1}

From the update of the vertical decomposition T_{i-1} into T_i we know what new leaves we must make for D_i

All new nodes besides the leaves are x-nodes with p_i and q_i and y-nodes with s_i

Updating the search structure



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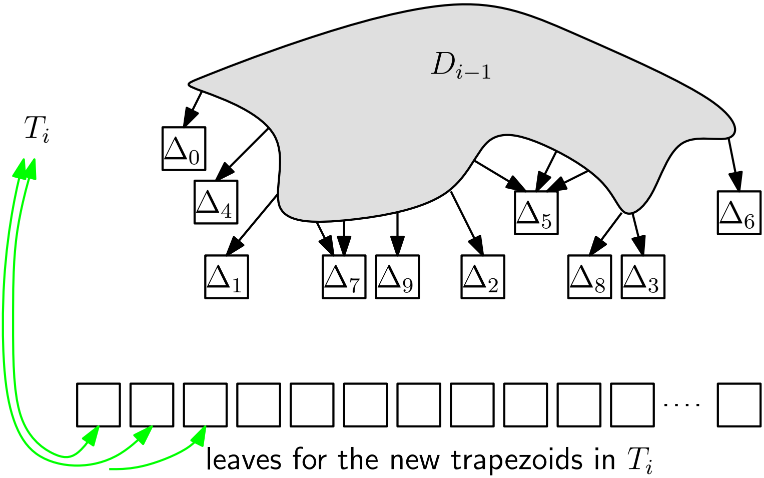

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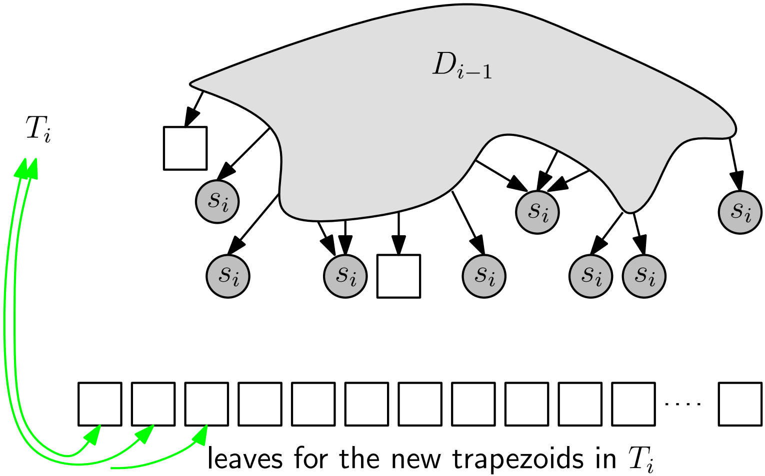

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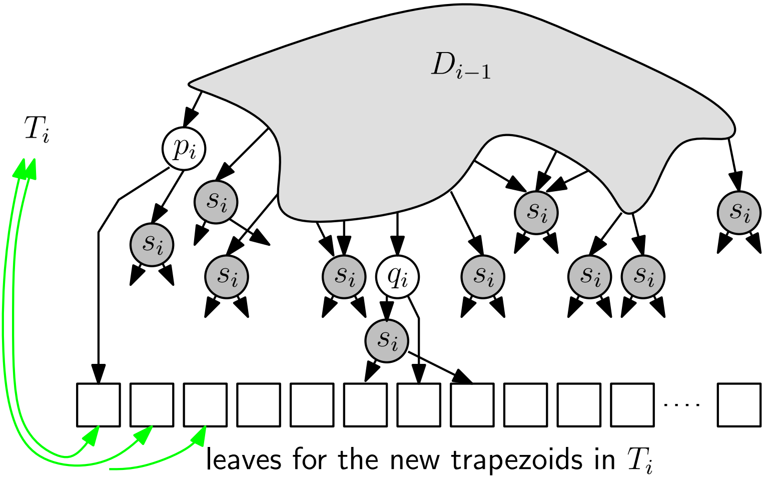


Observations

For a single update step, adding s_i and updating T_{i-1} and D_{i-1}, we observe:

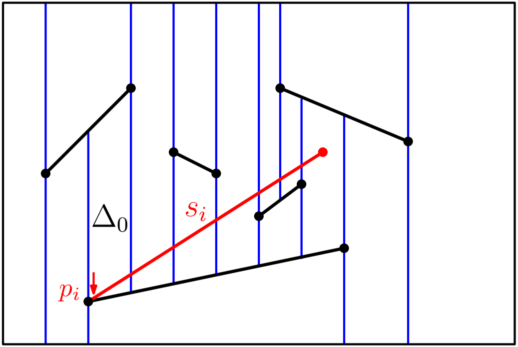
- If s_i intersects k_i trapezoids of T_{i-1}, then we will create $O(k_i)$ new trapezoids in T_i
- We find the k_i trapezoids in time linear in the search path of p_i in D_{i-1}, plus $O(k_i)$ time
- We update by replacing k_i leaves by $O(k_i)$ new internal nodes and $O(k_i)$ new leaves
- The maximum depth increase is three nodes

Questions

Question: In what case is the depth increase three nodes?

Question: We noticed that the directed acyclic graph D is binary in its out-degree, what is the maximum in-degree?

A common but special update



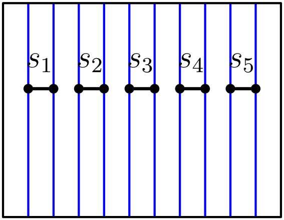
If p_i was already an existing vertex, we search in D_{i-1} with a point a fraction to the right of p_i on s_i

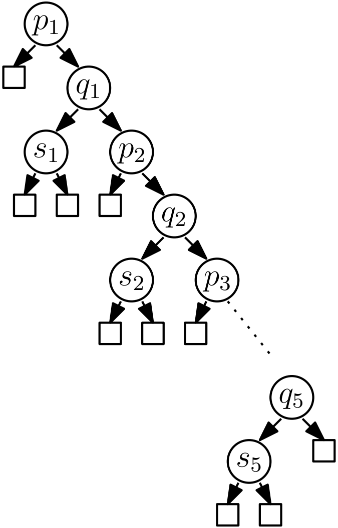
Randomized incremental construction

Randomized incremental construction, where does it matter?

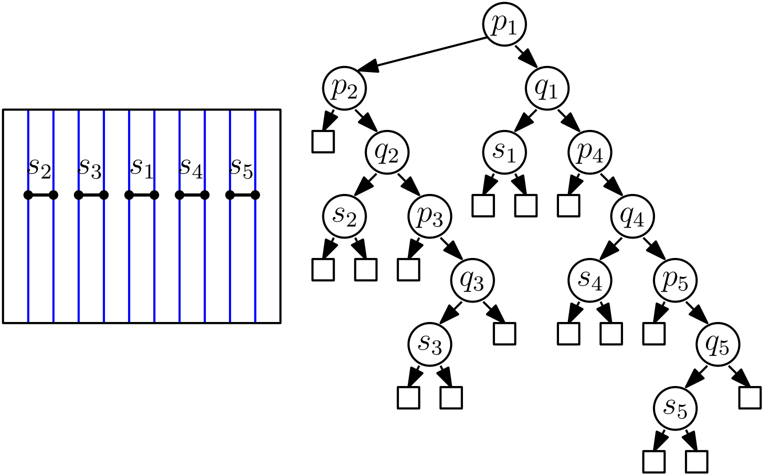
- The vertical decomposition T_i is independent of the insertion order among $s_1, \ldots, s_i$
- The search structure D_i can be different for many orders of $s_1, \ldots, s_i$
- The *number of nodes in D_i* depends on the order
- The *depth of search paths in D_i* depends on the order

Randomized incremental construction





Randomized incremental construction



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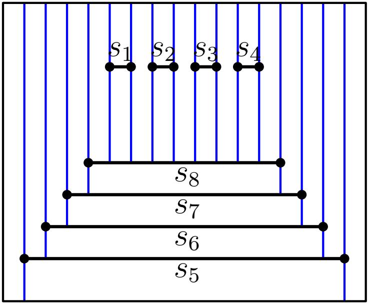
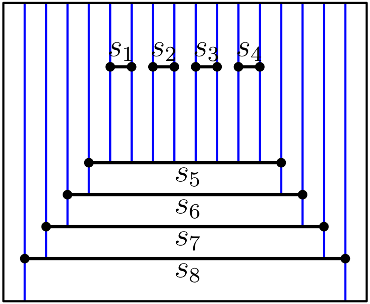




Storage of the structure

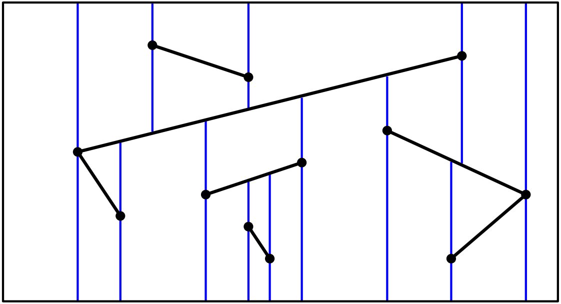
The vertical decomposition structure T always uses linear storage

The search structure D can use anything between linear and quadratic storage

We analyse the expected number of new nodes when adding s_i, using *backwards analysis* (of course)

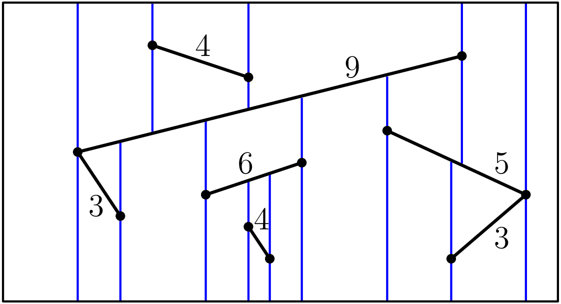
Storage of the structure

Backwards analysis in this case: Suppose we added s_i and have computed T_i and D_i. All line segments (existing in T_i) had the same probability of having been the last one added



Storage of the structure

For each of the i line segments, we can see how many trapezoids would have been created if it were the last one added



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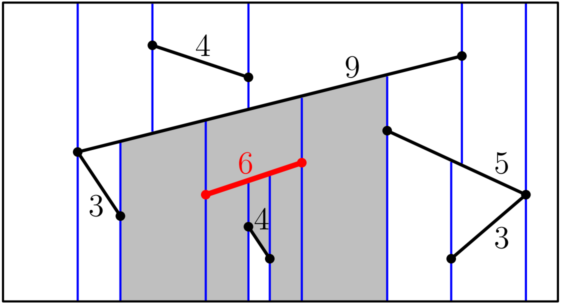

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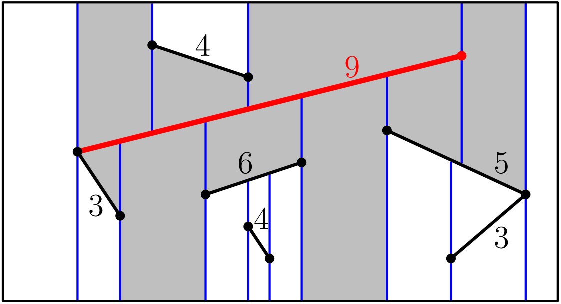


Storage of the structure

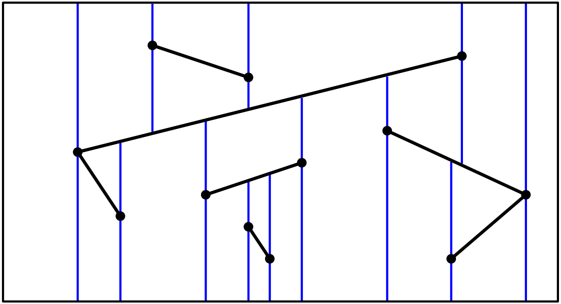
The number of created trapezoids is linear in the number of deleted trapezoids (leaves of D_{i-1}), or intersected trapezoids by s_i in T_{i-1}; this is linear in k_i

We will analyze

$$K_i = \sum_{j=1}^{i} [\text{no. of trapezoids created if } s_j \text{ were last}]$$

Storage of the structure

Consider K_i from the “trapezoid perspective”: For any trapezoid Δ, there are at most **four** line segments whose insertion would have created it (top(Δ), bottom(Δ), leftp(Δ), and rightp(Δ))



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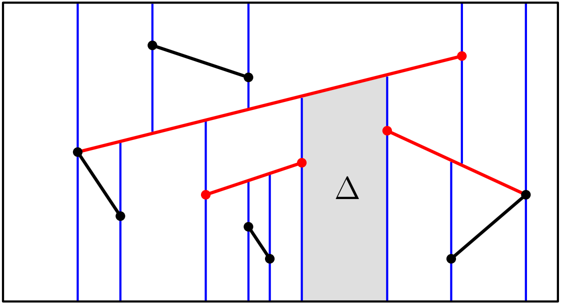

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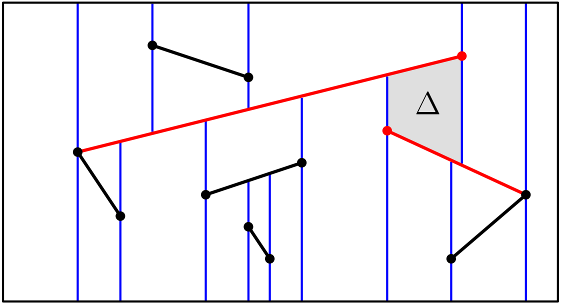


Storage of the structure

We know: There are at most $3i+1$ trapezoids in a vertical decomposition of i line segments in R

Hence,

$$K_i = \sum_{\Delta \in T_i} [\text{no. of segments that would create } \Delta]$$

$$\leq \sum_{\Delta \in T_i} 4 = 12i + 4$$

Storage of the structure

Since K_i is defined as a sum over i line segments, the *average* number of trapezoids in T_i created by s_i is at most $(12i+4)/i \leq 13$

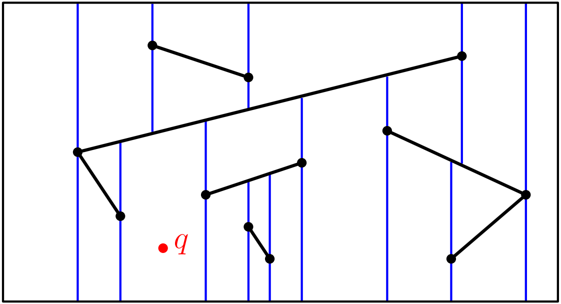
Since the expected number of new nodes is at most 13 in every step, the expected size of the structure after adding n line segments is $O(n)$

Query time of the structure

Fix any point q in the plane as a query point, we will analyze the probability that inserting s_i makes the search path to q longer

Query time of the structure

Backwards analysis: Take the situation after s_i has been added, and ask the question: How many of the i line segments made the search path to q longer?



Query time of the structure

Backwards analysis: Take the situation after s_i has been added, and ask the question: How many of the i line segments made the search path to q longer?

The search path to q only became longer if q is in a trapezoid that was *just created* by the latest insertion!

At most four line segments define the trapezoid that contains q, so the probability is $4/i$

Query time of the structure

We analyze

$$\sum_{i=1}^{n} [\text{search path became longer due to } i\text{-th addition}]$$

$$\leq \sum_{i=1}^{n} 4/i = 4 \cdot \sum_{i=1}^{n} 1/i \leq 4(1 + \log_e n)$$

So the expected query time is $O(\log n)$

Result

Theorem: Given a planar subdivision defined by a set of n non-crossing line segments in the plane, we can preprocess it for planar point location queries in $O(n \log n)$ expected time, the structure uses $O(n)$ expected storage, and the expected query time is $O(\log n)$