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Typed Transformations of Typed Abstract Syntax

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What is typed abstract syntax?

- values have types
- values can be composed
- types prevent invalid compositions of values

What is typed abstract syntax?

- descriptions of values have types
- descriptions of values can be composed
- types prevent invalid compositions of descriptions of values

$$
\begin{array}{l}
\textbf{data}\ \mathit{Expr}\ \ a\ \textbf{where} \\
\qquad \mathit{Val}\quad\ \ :: a \to \mathit{Expr}\ a \\
\qquad \mathit{Apply} :: \mathit{Expr}\ (b \to a) \to (\mathit{Expr}\ b) \to \mathit{Expr}\ a
\end{array}
$$

Where does Typed Astract Syntax arise?

- we want to implement *Embedded Domain Specific Languages*

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- we want to implement *Embedded Domain Specific Languages*
- which inherit their type system from the host language
- instead of directly building the semantics we:
  - build the typed abstract syntax tree
  - which we *analyse*, *transform* and from which we finally construct the *semantics*

Our ultimate goal is to "compile" embedded languages just as we compile normal languages.

GADTs

Generalised Algebraic Data Types enable us to encode the typing of the EDSL in the typing of the host language:

$$
\begin{array}{lll}
\textbf{data}\ \mathit{Exp}\ a\ \textbf{where} & & \\
\quad \mathit{IntVal} & :: \mathit{Int} & \to \mathit{Exp}\ \mathit{Int} \\
\quad \mathit{BoolVal} & :: \mathit{Bool} & \to \mathit{Exp}\ \mathit{Bool} \\
\quad \mathit{Add} & :: \mathit{Exp}\ \mathit{Int} \to \mathit{Exp}\ \mathit{Int} & \to \mathit{Exp}\ \mathit{Int} \\
\quad \mathit{Cons1} & :: \mathit{Exp}\ a \to \mathit{Exp}\ [a] & \to \mathit{Exp}\ [a] \\
\quad \mathit{Nil1} & :: & \mathit{Exp}\ [a] \\
\quad \mathit{LessThan} & :: \mathit{Exp}\ \mathit{Int} \to \mathit{Exp}\ \mathit{Int} & \to \mathit{Exp}\ \mathit{Bool} \\
\quad \mathit{If} & :: \mathit{Exp}\ \mathit{Bool} \to \mathit{Exp}\ a \to \mathit{Exp}\ a & \to \mathit{Exp}\ a
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\end{array}
$$

As a consequence our "compilers" do not have to perform type checking; the price we pay is that we have to maintain well-typedness during program transformations.

EDSL's may contain references

We extend $Expr$ with an argument describing the environment in which referred values are located:

$$
\begin{array}{l}
\textbf{data}\ Expr\ a\ env\ \textbf{where} \\
\qquad Var \quad :: Ref\ a\ env \rightarrow Expr\ a\ env \\
\qquad IntVal \ :: Int \qquad\quad \rightarrow Expr\ Int\ env \\
\qquad BoolVal :: Bool \qquad \rightarrow Expr\ Bool\ env \\
\qquad \ldots
\end{array}
$$

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\quad BoolVal :: Bool \quad\quad\quad\quad \rightarrow Expr\ Bool\ env \\
\quad \ldots
\end{array}
$$

$$
\begin{array}{lll}
lookup :: Ref\ \ a\ env & \rightarrow env \rightarrow a & \\
eval \quad :: Expr\ a\ env & \rightarrow env \rightarrow a & \\
eval\ (Var\ r) \quad\quad e & = lookup\ r\ e & \\
eval\ (IntVal\ i) \quad \_ & = i & \\
eval\ (BoolVal\ b)\ \_ & = b & \\
eval\ (Add\ x\ y) \quad e & = eval\ x\ e + eval\ y\ e & \\
\ldots & &
\end{array}
$$

Sidestepping: Type equality

Using a GADT we can provide the witness of the proof that two types are equal:

$$\begin{array}{l} \textbf{data}\ Equal :: \ * \ \rightarrow \ * \ \rightarrow \ * \ \textbf{where} \\ \quad Eq :: Equal\ a\ a \end{array}$$

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If a non-$\perp$ value $Eq\ a\ b$ takes part in a successful pattern match, the type checker may conclude that the types a and b are the same; otherwise the Eq could not have been produced.

Typed References

Ref-erences are labelled with the type a of the value they point to in an environment env:

$$
\begin{array}{l}
\textbf{data}\ Ref\ a\ env\ \textbf{where} \\
\quad Zero :: \qquad\qquad\qquad\qquad Ref\ a\ (env', a) \\
\quad Suc\ \ :: Ref\ a\ env' \rightarrow Ref\ a\ (env', b)
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Typed References

Environments are nested products

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References can be compared; if they are equal they return the proof that the values they refer to have the same type:

$$\begin{array}{llll} match :: Ref\ a\ env \rightarrow Ref\ b\ env \rightarrow Maybe\ (Equal\ a\ b) \\ match\ Zero & Zero & = Just\ Eq \\ match\ (Suc\ x) & (Suc\ y) & = match\ x\ y \\ match\ _ & _ & = Nothing \end{array}$$

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match\ _ \qquad\quad\ \ _ \qquad\quad = Nothing
\end{array}
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Here a and b are equal

Mutally Recursive Declarative Structures

We want to represent:

$$\begin{aligned}\textbf{let}\ x &= 1 : y \\ y &= 2 : x\end{aligned}$$

A first attempt:

$$\begin{aligned}\textbf{type}\ TwoLists = ((), &\ Expr\ [Int]\ TwoLists) \\ , &\ Expr\ [Int]\ TwoLists)\end{aligned}$$

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$$\begin{aligned}\textbf{type}\ TwoLists = (((), &\ Expr\ [Int]\ TwoLists) \\ , &\ Expr\ [Int]\ TwoLists)\end{aligned}$$

Unfortunately this is not correct Haskell.

Mutally Recursive Declarative Structures

We want to represent:

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We split the environment in two type parameters: the *used* environment and the *defined* environment:

$$\begin{aligned}&\textbf{data}\ \mathit{Env} :: (\ * \ \rightarrow\ * \ \rightarrow\ * \ ) \rightarrow\ * \ \rightarrow\ * \ \rightarrow\ * \\ &\quad \textbf{where}\ \mathit{Empty} :: \mathit{Env}\ \mathit{term}\ \mathit{used}\ () \\ &\qquad\qquad\ \mathit{Ext} \quad :: \mathit{Env}\ \mathit{term}\ \mathit{used}\ \mathit{defined} \rightarrow \mathit{term}\ a\ \mathit{used} \\ &\qquad\qquad\qquad\qquad \rightarrow \mathit{Env}\ \mathit{term}\ \mathit{used}\ (\mathit{defined}, a)\end{aligned}$$

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By choosing the two environment parameters to be the same we enforce that the environemnt is closed.

Example

The expression:

$$\begin{aligned}\textbf{let}\ x &= 1 : y\\ y &= 2 : x\end{aligned}$$

is now encoded as:

$$\begin{aligned}&\textbf{type}\ \mathit{Final} = (((), [\mathit{Int}]), [\mathit{Int}])\\ &x \quad = \mathit{Var}\ (\mathit{Suc}\ \mathit{Zero}) :: \mathit{Expr}\ [\mathit{Int}]\ \mathit{Final}\\ &y \quad = \mathit{Var}\ \quad \mathit{Zero}\ :: \mathit{Expr}\ [\mathit{Int}]\ \mathit{Final}\\ &\mathit{decls} :: \mathit{Env}\ \mathit{Expr}\ \mathit{Final}\ \mathit{Final}\\ &\mathit{decls} = \mathit{Empty}\ \text{`}\mathit{Ext}\text{`}\ \mathit{Cons}\ (\mathit{IntVal}\ 1)\ y\\ &\qquad\qquad\qquad \text{`}\mathit{Ext}\text{`}\ \mathit{Cons}\ (\mathit{Intval}\ 2)\ x\end{aligned}$$

Example

The expression:

We have nicer syntax for this

$$\begin{aligned}\textbf{let}\ x &= 1 : y\\ y &= 2 : x\end{aligned}$$

is now encoded as:

$$\begin{aligned}&\textbf{type}\ \mathit{Final} = (((), [\mathit{Int}]), [\mathit{Int}])\\ &x \quad = \mathit{Var}\ (\mathit{Suc}\ \mathit{Zero}) :: \mathit{Expr}\ [\mathit{Int}]\ \mathit{Final}\\ &y \quad = \mathit{Var}\ \quad \mathit{Zero}\ :: \mathit{Expr}\ [\mathit{Int}]\ \mathit{Final}\\ &\mathit{decls} :: \mathit{Env}\ \mathit{Expr}\ \mathit{Final}\ \mathit{Final}\\ &\mathit{decls} = \mathit{Empty}\ \text{`}\mathit{Ext}\text{`}\ \mathit{Cons}\ (\mathit{IntVal}\ 1)\ y\\ &\qquad\qquad\quad \text{`}\mathit{Ext}\text{`}\ \mathit{Cons}\ (\mathit{Intval}\ 2)\ x\end{aligned}$$

The problem: Common Subexpression Elimination

Suppose we want to transform the program:

$$a = 4;$$
$$b = (a + 4) + (a + 4);$$

into:

$$a = 4;$$
$$x = a + a;$$
$$b = x + x;$$

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into:

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$$x = a + a;$$
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In order to do so we have to build a new environment, containing the extra definition for x, and the new right hand sides for a and b. This new environment is built incrementally.

The Transformation Library

Eventually all references have to point into the final environment. We thus introduce the following types:

$$\mathbf{type}\ \mathit{FinalEnv}\ t\ \mathit{usedef} = \mathit{Env}\ t\ \mathit{usedef}\ \mathit{usedef}$$
$$\mathbf{newtype}\ T\ e\ s = T\{\mathit{unT} :: \forall x\ .\ \mathit{Ref}\ x\ e \rightarrow \mathit{Ref}\ x\ s\}$$

The Transformation Library

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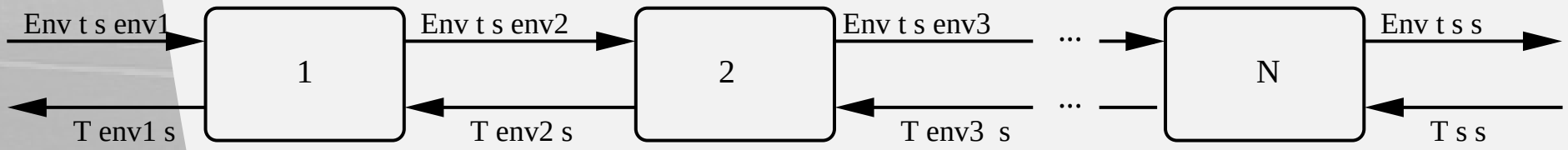
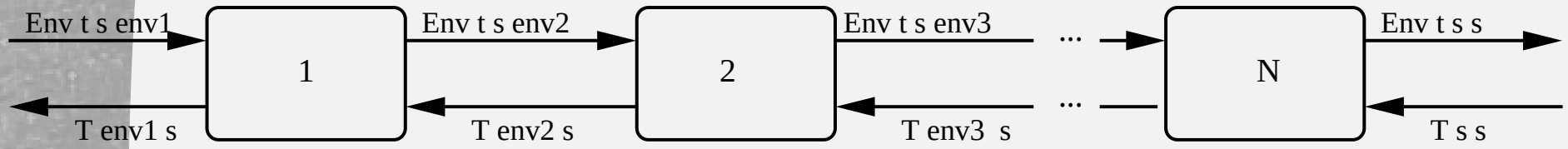
$$\textbf{type}\ FinalEnv\ t\ usedef = Env\ t\ usedef\ usedef$$
$$\textbf{newtype}\ T\ e\ s = T\{\, unT :: \forall x\ .\ Ref\ x\ e \rightarrow Ref\ x\ s \,\}$$

Whenever we add a new element to the environment under construction we have to update the already existing references. Instead we make a function available which maps them directly into the final environment:



The *Trafo* type



$$\begin{array}{l}\textbf{type}\ \mathit{Trafo}\ t\ s = \\ \quad \forall \mathit{env1}\ .\ \exists \mathit{env2}\ . \qquad T\ \mathit{env2}\ s \rightarrow \mathit{Env}\ t\ s\ \mathit{env1} \\ \qquad\qquad\qquad\qquad \rightarrow (T\ \mathit{env1}\ s, \quad \mathit{Env}\ t\ s\ \mathit{env2})\end{array}$$

$\mathit{Env}\ t\ s\ \mathit{env1}$ the environment constructed thus far

$T\ \mathit{env2}\ s$ represents the number of future additions to the environment

$\mathit{Env}\ t\ s\ \mathit{env2}$ the updated environment, in which $\mathit{env2}$ is (an extension of) $\mathit{env1}$

$T\ \mathit{env1}\ s$ the updated $T\ \mathit{env2}\ s$

Extend with an arrow like interface

We extend the type with an arrow-like interface:

$$
\begin{array}{l}
\textbf{type}\ \mathit{Trafo}\ t\ s\ a\ b = \\
\quad \forall \mathit{env1}\ .\ \exists \mathit{env2}\ . \qquad a \rightarrow T\ \mathit{env2}\ s \rightarrow \mathit{Env}\ t\ s\ \mathit{env1} \\
\qquad\qquad\qquad\qquad \rightarrow (b, \quad T\ \mathit{env1}\ s, \quad \mathit{Env}\ t\ s\ \mathit{env2})
\end{array}
$$

In our example a will e.g. be the mapping which tells us where the old variables have ended up in the new environment.

Meta-information

Since the elements in the constructed environment cannot be fully inspected (parts depend on the T which is still has to be constructed by future transformations), we maintain meta information m:

$$
\begin{array}{l}
\textbf{type}\ \mathit{Trafo}\ m\ t\ s\ a\ b = \\
\quad \forall \mathit{env1}\ .\ m\ \mathit{env1} \\
\qquad\qquad \rightarrow \exists \mathit{env2}\ . \\
\qquad\qquad\quad (\ \ m\ \mathit{env2} \\
\qquad\qquad\quad ,\qquad a \rightarrow T\ \mathit{env2}\ s \rightarrow \mathit{Env}\ t\ s\ \mathit{env1} \\
\qquad\qquad\qquad \rightarrow (b,\quad T\ \mathit{env1}\ s,\quad \mathit{Env}\ t\ s\ \mathit{env2}) \\
\qquad\qquad\quad)
\end{array}
$$

Haskelize

Since Haskell only allows existenstial constructors in combination with a data constructor we have to write:

$$
\begin{array}{l}
\textbf{data}\ \mathit{Trafo}\ m\ t\ s\ a\ b = \\
\quad \mathit{Trafo}\ (\forall \mathit{env1}\ .\ m\ \mathit{env1} \rightarrow \mathit{TrafoE}\ m\ t\ s\ a\ b\ \mathit{env1}) \\
\textbf{data}\ \mathit{TrafoE}\ m\ t\ s\ a\ b\ \mathit{env1} = \\
\quad \forall \mathit{env2}\ .\ \mathit{TrafoE}\ (m\ \mathit{env2}) \\
\qquad\qquad\qquad\qquad (\qquad a \rightarrow T\ \mathit{env2}\ s \rightarrow \mathit{Env}\ t\ s\ \mathit{env1} \\
\qquad\qquad\qquad\qquad\quad \rightarrow (b,\quad T\ \mathit{env1}\ s,\quad \mathit{Env}\ t\ s\ \mathit{env2}) \\
\qquad\qquad\qquad\qquad)
\end{array}
$$

Creating a new Ref-erence

The meta-data type has to be filled in depending on the situation:

$$\begin{array}{l} newSRef :: Trafo\ Unit\ t\ s\ (t\ a\ s)\ (Ref\ a\ s) \\ \mathbf{data}\ Unit\ s = Unit \end{array}$$

Here $t\ a\ s$ is the term we add to the environemnt, and $Ref\ a\ s$ is the reference pointing to this element in the final environment!

Creating a new *Ref*-erence

$$newSRef = Trafo\ (\lambda\_ \rightarrow TrafoE\ Unit\ extEnv)$$

$$\begin{aligned}
&extEnv :: \quad t\ a\ s \rightarrow T\ (e, a)\ s \rightarrow Env\ t\ s\ e \\
&\qquad\qquad \rightarrow (Ref\ a\ s, T\ e\ \ \ s, \quad Env\ t\ s\ (e, a)) \\
&extEnv \\
&\quad = \lambda ta\ (T\ tr)\ env \rightarrow (tr\ Zero, T\ (tr\ .\ Suc), Ext\ env\ ta)
\end{aligned}$$

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Tell the predecessors that an element was added

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Map the newly added element into the final environment

Tell the predecessors that an element was added

Creating a new Ref-erence

$$newSRef = Trafo\ (\lambda\_ \rightarrow TrafoE\ Unit\ extEnv)$$

Extend the environment with the new term ta

$$\begin{aligned}
&extEnv :: \quad t\ a\ s \rightarrow T\ (e, a)\ s \rightarrow Env\ t\ s\ e \\
&\qquad \rightarrow (Ref\ a\ s, T\ e\ \ s, \quad Env\ t\ s\ (e, a)) \\
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\end{aligned}$$

Map the newly added element into the final environment

Tell the predecessors that an element was added

Running the *Trafos*

When we are done we require that the used and the built environment are equally labelled, hence we use $FinalEnv$:

$$
\begin{aligned}
&\mathbf{data}\ Result\ m\ t\ b \\
&\quad = \forall env2\ .\ Result\ (m\ env2)\ (b\ env2) \\
&\qquad\qquad\qquad\qquad (FinalEnv\ t\ env2)
\end{aligned}
$$

Running the *Trafo*s

When we are done we require that the used and the built environment are equally labelled, hence we use $FinalEnv$:

```
data Result m t b
    = ∀env2 . Result (m env2) (b env2)
                     (FinalEnv t env2)
```

```
runTrafo :: ∀m t a b . (∀s . Trafo m t s a (b s))
                     → m () → a → Result m t b
runTrafo trafo m a =
  let Trafo trf      = trafo
      TrafoE m2 f    = trf m
  in  case f a (T id) Empty of
        (b, _, env) → Result m2 b env
```

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&\qquad\quad TrafoE\ m2\ f = trf\ m \\
&\quad \textbf{in}\ \ \textbf{case}\ f\ a\ (T\ id)\ Empty\ \textbf{of} \\
&\qquad\quad (b, \_, env) \rightarrow Result\ m2\ b\ env
\end{aligned}
$$

use equals *def*

Common Subexpression Elimination

After CSE we have a larger, closed environment:

$$\textbf{type}\ \mathit{Decls}\ \mathit{env}' = \mathit{Env}\ \mathit{Expr}\ \mathit{env}'\ \mathit{env}'$$

Common Subexpression Elimination

After CSE we have a larger, closed environment:

$$\textbf{type}\ Decls\ env' = Env\ Expr\ env'\ env'$$

We also compute a ref-transformer which maps old references in env to new references in env':

$$\begin{aligned}\textbf{data}\ TDecls\ env = \forall env'\ .\ TDecls\ &(Decls\ env') \\ &(T\ env\ env')\end{aligned}$$

Common Subexpression Elimination

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$$\textbf{type}\ Decls\ env' = Env\ Expr\ env'\ env'$$

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$$\begin{aligned}\textbf{data}\ TDecls\ env = \forall env'\ .\ TDecls\ &(Decls\ env') \\ &(T\ env\ env')\end{aligned}$$

The type of cse now becomes:

$$cse :: Decls\ env \rightarrow TDecls\ env$$

Maintain a $Memo$ table

In the meta-information we maintain a memo table, which we use to remember which expressions labelled with env have already been incorporated in the new environment:

$$
\begin{array}{l}
\textbf{newtype}\ Memo\ env\ env' \\
\quad = Memo \\
\qquad (\forall x\ .\ Expr\ x\ env \\
\qquad\qquad \rightarrow Maybe\ (Ref\ x\ env') \\
\qquad)
\end{array}
$$

$$
\begin{array}{l}
emptyMemo :: Memo\ env\ () \\
emptyMemo = Memo\ (const\ Nothing)
\end{array}
$$

Maintain a *Memo* table

In the meta-information we maintain a memo table, which we use to remember which expressions labelled with *env* have already been incorporated in the new environment:

And we construct the type of our transformations:

$$
\begin{array}{l}
\textbf{type}\ \mathit{TrafoCSE}\ \mathit{env} = \mathit{Trafo}\ (\mathit{Memo}\ \mathit{env})\ \mathit{Expr} \\
\mathit{extMemo} :: \mathit{Expr}\ a\ \mathit{env} \rightarrow \mathit{Memo}\ \mathit{env}\ \mathit{env}' \\
\qquad\qquad \rightarrow \mathit{Memo}\ \mathit{env}\ (\mathit{env}', a) \\
\mathit{extMemo}\ e\ (\mathit{Memo}\ m) \\
\qquad\qquad = \mathit{Memo}\ (\lambda s \rightarrow \textbf{case}\ \mathit{equals}\ e\ s\ \textbf{of} \\
\qquad\qquad\qquad\qquad\qquad \mathit{Just}\ \mathit{Eq} \rightarrow \mathit{Just}\ \mathit{Zero} \\
\qquad\qquad\qquad\qquad\qquad \mathit{Nothing} \rightarrow \mathit{fmap}\ \mathit{Suc}\ (m\ s) \\
\qquad\qquad\qquad\qquad\quad)
\end{array}
$$

Dealing with a single expression

$$
\begin{aligned}
&app\_cse :: Expr\ a\ env \\
&\qquad\qquad \rightarrow TrafoCSE\ env\ s\ (T\ env\ s)\ (Ref\ a\ s) \\
&app\_cse\ (Var\ r) = \mathbf{proc}\ (T\ tenv\_s) \rightarrow \\
&\qquad\qquad\qquad\qquad returnA \prec tenv\_s\ r
\end{aligned}
$$

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Where do the old variables go?

Dealing with a single expression

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&app\_cse :: Expr\ a\ env \\
&\qquad \rightarrow TrafoCSE\ env\ s\ (T\ env\ s)\ (Ref\ a\ s) \\
&app\_cse\ (Var\ r) = \mathbf{proc}\ (T\ tenv\_s) \rightarrow \\
&\qquad\qquad returnA \prec tenv\_s\ r
\end{aligned}
$$

Position of this expression

Where do the old variables go?

Dealing with a single expression

$$
\begin{aligned}
&app\_cse :: Expr\ a\ env \\
&\qquad \rightarrow TrafoCSE\ env\ s\ (T\ env\ s)\ (Ref\ a\ s) \\
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&\qquad\qquad returnA \prec tenv\_s\ r
\end{aligned}
$$

$$
\begin{aligned}
&app\_cse\ e@(LessThan\ x\ y) \\
&\quad = \mathbf{proc}\ tt \rightarrow \\
&\quad\ \ \mathbf{do}\ l \leftarrow app\_cse\ x \prec tt \\
&\qquad\quad r \leftarrow app\_cse\ y \prec tt \\
&\qquad\quad insertIfNew\ e \prec LessThan\ (Var\ l)\ (Var\ r) \\
&\quad \ldots
\end{aligned}
$$

Running the transformations

$$refTransformer :: Env\ Ref\ s\ env \rightarrow T\ env\ s$$
$$refTransformer\ refs = T\ (\lambda r \rightarrow lookupEnv\ r\ refs)$$

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$$
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\end{aligned}
$$

The result of $\mathit{cse\_env}$ is used to compute its own input.Hence we use **mdo**:

$$
\begin{aligned}
&\mathit{trafo} :: \mathit{Decls}\ \mathit{env} \rightarrow \mathit{TrafoCSE}\ \mathit{env}\ s\ ()\ (T\ \mathit{env}\ s) \\
&\mathit{trafo}\ \mathit{decls} = \mathbf{proc}\ \_ \rightarrow \\
&\qquad\qquad \mathbf{mdo}\ \mathbf{let}\ \mathit{tt} = \mathit{refTransformer}\ \mathit{refs} \\
&\qquad\qquad\qquad \mathit{refs} \leftarrow \mathit{cse\_env}\ \mathit{decls} \prec \mathit{tt} \\
&\qquad\qquad\qquad \mathit{returnA} \prec \mathit{tt}
\end{aligned}
$$

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&\qquad\qquad\qquad \mathit{refs} \leftarrow \mathit{cse\_env}\ \mathit{decls} \prec \mathit{tt} \\
&\qquad\qquad\qquad \mathit{returnA} \prec \mathit{tt}
\end{aligned}
$$

Finally we present the function cse which simply runs the trafo and extracts the result:

$$
\begin{aligned}
&\mathit{cse} :: \forall \mathit{env}\ .\ \mathit{Decls}\ \mathit{env} \rightarrow \mathit{TDecls}\ \mathit{env} \\
&\mathit{cse}\ \mathit{decls} \\
&\quad = \mathbf{case}\ \mathit{runTrafo}\ (\mathit{trafo}\ \mathit{decls})\ \mathit{emptyMemo}\ ()\ \mathbf{of} \\
&\qquad\qquad \mathit{Result}\ \_\ t\ \mathit{env} \rightarrow \mathit{TDecls}\ \mathit{env}\ t
\end{aligned}
$$

GHC problems

Unfortunately we have used lazy pattern binding on the existential type $TrafoE$:

```
data Trafo m t s a b =
   Trafo (∀env1 . m env1 → TrafoE m t s a b env1)
data TrafoE m t s a b env1 =
  ∀env2 . TrafoE (m env2)
                 (a → T env2 s → Env t s env1 →
                 (b,    T env1 s,    Env t s env2)
                 )
```

```
runTrafo :: ∀m t a b . (∀s . Trafo m t s a (b s))
                     → m () → a → Result m t b
runTrafo trafo m a =
   let Trafo trf       = trafo
       TRafoE m2 f = trf m
   in  case f a (T id) Empty of
          (b, _, env) → Result m2 b env
```

The blunt solution: *unsafeCoerce*

```
unsafeCoerce :: a → b
runTrafo :: (∀s . Trafo m t s a (b s)) → m () → a
         → Result m t b
runTrafo trafo m a = case trafo of
  Trafo trf → case trf m of
    TrafoE m2 f →
      case f a (T unsafeCoerce) Empty of
        (rb, tt, env2) →
          Result (unsafeCoerce m2)
                 rb
                 (unsafeCoerce env2)
```

Alternative: move the final s inwards

$$
\begin{array}{l}
\textbf{data}\ \mathit{Trafo2}\ m\ t\ a\ b = \\
\quad \mathit{Trafo2}\ (\forall \mathit{env1}\ .\ m\ \mathit{env1} \rightarrow \mathit{TrafoE2}\ m\ t\ a\ b\ \mathit{env1}) \\
\textbf{data}\ \mathit{TrafoE2}\ m\ t\ a\ b\ \mathit{env1} = \\
\quad \forall \mathit{env2}\ .\ \mathit{TrafoE2} \\
\qquad\qquad (m\ \mathit{env2}) \\
\qquad\qquad (\forall s\ .\quad a\ s \rightarrow T\ \mathit{env2}\ s \rightarrow \mathit{Env}\ t\ s\ \mathit{env1} \\
\qquad\qquad\quad \rightarrow (b\ s,\quad T\ \mathit{env1}\ s,\quad \mathit{Env}\ t\ s\ \mathit{env2}) \\
\qquad\qquad)
\end{array}
$$

Alternative: move the final s inwards

$$
\begin{array}{l}
\textbf{data}\ \mathit{Trafo2}\ m\ t\ a\ b = \\
\quad \mathit{Trafo2}\ (\forall \mathit{env1}\ .\ m\ \mathit{env1} \rightarrow \mathit{TrafoE2}\ m\ t\ a\ b\ \mathit{env1}) \\
\textbf{data}\ \mathit{TrafoE2}\ m\ t\ a\ b\ \mathit{env1} = \\
\quad \forall \mathit{env2}\ .\ \mathit{TrafoE2} \\
\qquad\qquad (m\ \mathit{env2}) \\
\qquad\qquad (\forall s\ .\quad a\ s \rightarrow T\ \mathit{env2}\ s \rightarrow \mathit{Env}\ t\ s\ \mathit{env1} \\
\qquad\qquad\quad \rightarrow (b\ s, \quad T\ \mathit{env1}\ s, \quad \mathit{Env}\ t\ s\ \mathit{env2}) \\
\qquad\qquad)
\end{array}
$$

Unfortunately now a and b have an s parameter,and we can no longer use the arrow notation.

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- The library enables a whole new way of dealing with embedded domain specific languages.