A Note on Recursively Enumerable Classes of Partial Recursive Functions

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Abstract. A class F of partial recursive functions is called recursively enumerable if there exists an r.e. set $J \subseteq \mathbb{N}$ such that $F = \{\phi_i \mid i \in J\}$. We prove that every r.e. class F of partial recursive functions with infinite domains must have a *recursive witness array*, i.e. there is a computable array of finite sets $X = [X_n]_{n \in \omega}$ such that (i) for every $f \in F$ one has $f(n) \in X_n$ for infinitely many n and (ii) $X_n = \emptyset$ for infinitely many n. The result gives a powerful diagonalisation tool for proving properties of r.e. classes. We show for example that no r.e. class of partial functions with infinite domains can contain all recursive involutions or all cyclefree recursive permutations.

Keyword and phrases: Recursively enumerable classes, Rice-Shapiro theorem, recursive witness arrays, recursive permutations.

1 Introduction

In this note we obtain a powerful diagonalisation method for classes of partial recursive functions with infinite domains. We use it to give elegant proofs of some properties that show the limitation of these classes.

Let $\{\phi_i\}_{i \in \omega}$ be a common, acceptable indexing of the partial recursive functions [12]. For every partial recursive function ϕ, we refer to i as an index, or a program, for ϕ if $\phi = \phi_i$. A class F of partial recursive functions is called *recursively enumerable* (r.e.) if there exists an r.e. set $J \subseteq \mathbb{N}$ such that F consists precisely of the functions with an index in J, i.e. $F = \{\phi_i \mid i \in J\}$.

The study of recursively enumerable classes was originated by Dekker [3] and Rice [10]. The problem of characterising r.e. classes is closely connected to, but different from, that of characterising *full index sets.* Recall that a class F is called *completely recursively enumerable* (c.r.e.) if the set of all indices for the functions of F is recursively enumerable. By the well-known Rice-Shapiro theorem [7, 8, 10, 12] it follows that, if F is c.r.e., then $f \in F$ if and only if f is the extension of some finite function in F. This characterizes c.r.e. classes quite precisely.

There clearly is a much greater variety of 'arbitrary' recursively enumerable classes of partial recursive functions and, likewise, of sets. The first attempt to classify all r.e. classes was made by Dekker and Myhill [4] in the late 1950s. They also gave examples of r.e. classes whose intersection is not recursively enumerable. (Young [14] later gave examples where the intersection is even immune, i.e. does not have infinite r.e. subclasses.) The theory of r.e. classes is thus very different from the theory of r.e. sets. For an overview of known results and the theory of enumerations of r.e. classes, we refer to [8, 12].

We aim at a simple property that certifies whether a certain class of partial recursive functions is r.e. or not. For this we consider a special kind of arrays of finite sets. Let F be any class of partial functions.

Definition 1. *A computable array of finite sets* $X = [X_n]_{n\in\omega}$ *is said to be a* recursive witness array *for* F *if (i) for every* $f \in F$ *one has* $f(n) \in X_n$ *for infinitely many* n*, and (ii)* $X_n = \emptyset$ *for infinitely many* n*.*

An array of finite sets $X = [X_n]_{n\in\omega}$ is called 'computable' if there exists a recursive function $g : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \omega$, $g(n)$ equals the canonical index[1] of the finite set X_n. In a computable array $X = [X_n]_{n\in\omega}$ for some class F, the empty and non-empty sets among the X_n are easily recognized. This leads to the following property that we often use later on: *for every recursive witness array* $X = [X_n]_{n\in\omega}$ *there exists an infinite recursive set* Y *such that* $n \in Y \Rightarrow X_n = \emptyset$ (and vice versa). Given Y, all (infinitely many) indices n with $X_n \neq \emptyset$ are contained in its complement.

Every denumerable class of *total* functions $F = \{f_0, f_1, \cdots\}$ trivially has a recursive witness array: take $X_n = \{f_0(n), \cdots, f_n(n)\}$ for n even, and $X_n = \emptyset$ for n odd. This applies in particular to every r.e. class of total recursive functions. However, we are more interested in r.e. classes of functions that are *not* necessarily total.

If a class of partial functions F has a recursive witness array, then the functions in F necessarily all have an *infinite* domain. We will prove that the converse holds as well: *every non-empty r.e. class of partial recursive functions with infinite domains has a recursive witness array.*

In Section 2 we prove the theorem. In Section 3 we show that the theorem gives an elegant tool for proving properties of r.e. classes of functions when the Rice-Shapiro theorem does not apply. We show, for example, that no r.e. class of partial recursive functions with infinite domains can contain all recursive involutions or all cyclefree recursive permutations.

Terminology For non-empty r.e. sets A, any recursive function f with $Range(f)$ $= A$ called a *recursive enumerator* of A. In case A is infinite and f is both recursive and one-one, f is called a *recursive generator* of A. It is well-known that every non-empty r.e. set A has a recursive enumerator and that every infinite r.e. set A has a recursive generator [12]. A non-empty r.e. set A is called a splinter if $A = \{a, \phi(a), \phi^2(a), \cdots\}$ for some recursive function ϕ and $a \in A$. A is a one-one splinter if one can take ϕ recursive and one-one. We refer to [12] and [8] for all further recursion-theoretic preliminaries.

[1] The canonical index of a finite set is a standard encoding of its distinct elements into a single number. Thus, knowing the canonical index of a set is equivalent to knowing its individual elements.

2 Recursive Witness Arrays

We assume throughout that the indexing $\{\phi_i\}_{i\in\omega}$ is given together with a *Blum measure* Φ, i.e. an effective enumeration $\{\Phi_i\}_{i\in\omega}$ of partial computable functions which satisfies the following properties [2]:

- for all i, n: $\phi_i(n)\downarrow \Leftrightarrow \Phi_i(n)\downarrow$,
- the predicate $\Phi_i(n) \leq m$ is decidable in i, n and m.

For every i, one can think of Φ_i as a *step-counting* function for program i. Step-counting is used later on in constructions that involve dove-tailing.

Constructing Witness Arrays Let F be an arbitrary, non-empty, r.e. class of partial recursive functions with infinite domains. Let $J \subseteq \mathbb{N}$ be an r.e. set such that $F = \{\phi_i \mid i \in J\}$, and let f be a recursive function that enumerates the elements of J, possibly with repetitions. Thus $F = \{\phi_{f(0)}, \phi_{f(1)}, \cdots\}$.

The domain of every ϕ_i is recursively enumerable, by definition. If the domain of ϕ_i is infinite, then it is well-known that the domain is the range of a one-to-one recursive function [12]. This follows from a uniform dovetailing construction, given the program i. Hence, there exists a recursive function τ such that, if i is the program of a partial recursive function with infinite domain, then $\tau(i)$ is the program of a recursive generator for the domain of ϕ_i. (Compare [12], §5.2.)

For $i \in \omega$, let $\psi_i = \phi_{\tau(f(i))}$. Thus, with F as above and for any $i \in \omega$, we have $\phi_{f(i)} \in F$ and $\{\psi_i(0), \psi_i(1), \cdots\}$ is a complete, one-one recursive enumeration of its (infinite) domain.

Theorem 1. *Every non-empty r.e. class of partial recursive functions with infinite domains has a recursive witness array.*

Proof. Let $F = \{\phi_{f(0)}, \phi_{f(1)}, \cdots\}$ be any non-empty r.e. class of partial recursive functions with infinite domains. Let $\psi_i = \phi_{\tau(f(i))}$ as defined above.

For $i, n \in \omega$, define the sets $R_{i,n} \subseteq \mathbb{N}$ and $R_n \subseteq \mathbb{N}$ and the integers L_n and M_n as follows:

$$R_{i,n} = \{\psi_i(0), \cdots, \psi_i(n+1)\}$$
$$R_n = \textstyle\bigcup_{i=0}^{n} R_{i,n}$$
$$L_n = \max_{0\leq i\leq n}\{\max_{m\in R_{i,n}} \Phi_{f(i)}(m)\}$$
$$M_n = \max R_n$$

The sets $R_{i,n}$ and R_n are all finite and effectively determined and, because the ψ_i's are all one-one, the sets have at least one element greater than n. Thus, for all $n \in \omega$, M_n is effectively determined also and $M_n \geq n + 1$.

Because $R_{i,n}$ is a subset of the domain of $\phi_{f(i)}$, $\Phi_{f(i)}$ is defined for all its elements and thus L_n is effectively determined as well. Note that, for all $n \in \omega$, we have $M_n \leq M_{n+1}$ and $L_n \leq L_{n+1}$, by the definition of the numbers.

Before we can define a suitable witness array for F, we need some auxiliary numbers and sets first. To start with, define the set $Y = \{y_0, y_1, \cdots\}$ with elements $y_0, y_1, \cdots$ determined in order as follows:

$$y_0 = 0$$
$$y_{n+1} = M_{y_n} + 1$$

Because the M_n for $n \in \omega$ are effectively determined, Y is a recursive set. Also, because $M_{y_n} \geq y_n + 1$, we have $y_{n+1} \geq y_n + 2$. Hence, the 'sequence' $y_0, y_1, \cdots$ is monotone increasing. It follows that Y is infinite.

Let X'_n be the set of outputs obtained by running the programs $f(0), \cdots, f(n)$ on input n for at most L_n steps each. Now define the array of sets $X = [X_n]_{n\in\omega}$ as follows:

$$\text{if } n \in Y, \text{ then } X_n = \emptyset$$
$$\text{if } n \notin Y, \text{ then } X_n = X'_n$$

We claim that $X = [X_n]_{n\in\omega}$ is a recursive witness array for F. Clearly, every X_n is finite and effectively determined. Also, we have that $X_n = \emptyset$ for infinitely many n, namely for all n occurring in the set Y. It remains to show that for every $i \in \omega$, one has $\phi_{f(i)}(n) \in X_n$ for infinitely many n.

Consider any function $\phi_{f(i)} \in F$, for $i \in \omega$. Let k be any index such that $i \leq y_k$. (As $y_0, y_1, \cdots$ is monotone increasing, k exists.) Let

$$n_{i,k} = \max R_{i,y_k}$$

By the definition of R_{i,y_k} and R_{y_k} it follows that $y_k + 1 \leq n_{i,k} \leq M_{y_k}$. Because $y_{k+1} = M_{y_k} + 1$, we obtain $y_k < n_{i,k} < y_{k+1}$. In particular we have $n_{i,k} \notin Y$.

Now consider $X_{n_{i,k}}$. By definition this set consists of the outputs obtained by running the programs $f(0), \cdots, f(n_{i,k})$ on input $n_{i,k}$ for at most $L_{n_{i,k}}$ steps each. Because $i \leq y_k < n_{i,k}$, this set of programs includes program $f(i)$. Because $n_{i,k} \in R_{i,y_k}$ we have that $n_{i,k}$ belongs to the domain of $\phi_{f(i)}$. It follows that $\phi_{f(i)}(n_{i,k})\downarrow$, and thus $\Phi_{f(i)}(n_{i,k})\downarrow$ as well. Now observe that

$$\Phi_{f(i)}(n_{i,k}) \leq \max_{m\in R_{i,y_k}} \Phi_{f(i)}(m) \leq L_{y_k} \leq L_{n_{i,k}}$$

again because $i \leq y_k < n_{i,k}$. It follows that the definition of $X_{n_{i,k}}$ gives program $f(i)$ enough time to complete on input $n_{i,k}$. Hence, $\phi_{f(i)}(n_{i,k}) \in X_{n_{i,k}}$.

Let s be the smallest integer such that $i \leq y_s$. Then the above argument holds for every k with $k \geq s$. It follows that $\phi_{f(i)}(n_{i,k}) \in X_{n_{i,k}}$ for infinitely many k. Because $y_k < n_{i,k} < y_{k+1}$ for every $k \geq s$, the $n_{i,k}$ values are distinct for distinct values of k. This gives infinitely many n with $\phi_{f(i)}(n) \in X_n$.

This proves that $X = [X_n]_{n\in\omega}$ is a recursive witness array for F. □

The set Y constructed in the given proof is easily seen to be a *one-one recursive splinter*. This follows from the construction, but also from the generic fact that every non-empty recursive set is a one-one splinter [13].

Corollary 1. *Every non-empty r.e. class of partial recursive functions with infinite domains has infinitely many recursive witness arrays.*

Proof. Clearly, every finite modification of a recursive witness array for F gives another recursive witness array for F. □

In general, every countable class of partial functions with denumerable domains can be seen to have a witness array as intended in Definition 1. The key issue of Theorem 1 is the recursiveness of the array in the case of r.e. classes of partial recursive functions with infinite domains. In Section 3 we will see that Theorem 1 allows for elegant diagonalisation proofs.

3 Applications

By the Rice-Shapiro theorem, *no* class of partial recursive functions with infinite domains can be completely recursively enumerable. The question arises what limitations there are for the classes of this kind that are 'just' recursively enumerable. Can one say more than that any recursive enumeration of such a class cannot be 'complete' and must miss 'some' indices of 'some' of the functions in the class?

We first argue that r.e. classes of partial recursive functions with infinite domains must be 'very incomplete'. Next we apply Theorem 1 to a number of special r.e. classes of partial recursive functions and to recursive permutations, and to r.e. classes of infinite r.e. sets. In all cases we show that the enumerations cannot be very comprehensive.

Functions Let F be any non-empty recursively enumerable class of partial recursive functions with infinite domains and let J be any r.e. set of indices for F. We first show that any recursive enumeration of F must leave out *infinitely many* programs for *all* functions in F.

Theorem 2. *Let F be a non-empty recursively enumerable class of partial recursive functions with infinite domains, and let J be any r.e. set of indices for F. For every $\phi_i \in F$ with $i \in J$, there are infinitely many indices $e \notin J$ such that $\phi_e = \phi_i$.*

Proof. Let J be any r.e. set of indices for F. As J must be non-empty, there is a total recursive function f with $Range(f) = J$ (cf. [12], §5.2). It follows that $F = \{\phi_{f(0)}, \phi_{f(1)}, \cdots\}$.

Let ϕ_i be an arbitrary function of F. By the *s-m-n theorem* [12] there exists a total recursive function h such that

$$\phi_{h(e)}(n) = \textbf{if } e \in \{f(0), \cdots, f(n)\} \textbf{ then} \uparrow \textbf{else } \phi_i(n)$$

which is clearly a well-defined partial recursive function. By the recursion theorem [12] there are infinitely many indices e such that $\phi_{h(e)}(n) = \phi_e(n)$.

Consider any index e with $\phi_{h(e)}(n) = \phi_e(n)$. If $e \in J$, then e would occur in the enumeration of J for every n large enough and $\phi_e(n)$ would have finite domain. This contradicts that $e \in J$. Hence, $e \notin J$ and thus $\phi_e = \phi_i$, for every index e that satisfies $\phi_{h(e)}(n) = \phi_e(n)$. □

Recursively enumerable classes of partial recursive functions with infinite domains are known not to be very extensive. For example, Rice ([10], Corollary B) already observed that for every infinite r.e. set A, the set of all partial recursive functions that enumerate A is not recursively enumerable. We prove a much stronger statement.

Theorem 3. *Let F be a recursively enumerable class of partial recursive functions with infinite domains, and let A be any infinite r.e. set. Then there are infinitely many recursive generators of A that are not contained in F.*

Proof. Let F be a recursively enumerable class of partial recursive functions with infinite domains. We may assume w.l.o.g. that F is non-empty. By Theorem 1 we know that F has a recursive witness array $X = [X_n]_{n \in \omega}$. Let Y be any infinite recursive set such that $n \in Y \Rightarrow X_n = \emptyset$. We may assume w.l.o.g. that $0 \in Y$ (implying that $X_0 = \emptyset$).

Let A be an infinite r.e. set, f a recursive generator of A, and a an arbitrary element of A. Now define the function π_a recursively as follows.

$\pi_a(n) =$
- (1) if $n = 0$: then output a.
- (2) if $n > 0$ and $n \in Y$: then output the first element x occurring in the enumeration $\{f(0), f(1), \cdots\}$ of A that is different from $\pi_a(0), \cdots, \pi_a(n-1)$.
- (3) if $n > 0$ and $n \notin Y$: then output the first element x occurring in the enumeration $\{f(0), f(1), \cdots\}$ of A that is different from $\pi_a(0), \cdots, \pi_a(n-1)$ and not contained in X_n.

As A is infinite, $\pi_a(n)$ is clearly well-defined for every n. By design π_a is recursive and one-one. Also note that every element x of A is eventually assigned as the value of $\pi_a(n)$ for some n. If an element is not assigned in (3), then it certainly is assigned eventually in (2), due to the fact that Y is infinite. Hence, π_a is a recursive generator of A.

Note that for every n we have $\pi_a(n) \notin X_n$, by design. (For $n \in Y$ this follows because $X_n = \emptyset$, for $n \notin Y$ by clause (3) of the construction.) Thus $\pi_a \notin F$, as all functions in F must hit the witness array infinitely often and π_a does not.

By varying a over all elements of A, we get infinitely many recursive generators of A that are not contained in F. □

We conclude that Rice's Corollary B ([10]) can be strengthened considerably, as follows.

Corollary 2. *Let A be an arbitrary infinite r.e. set, and let F be any recursively enumerable class of partial recursive enumerators of A. Then there are infinitely many recursive generators of A that are not contained in F.*

Remark. Let F be the class of all partial recursive enumerators of A, for some infinite r.e. set A. Let G be a recursively enumerable class of functions with $G \supset$

F and let J be any r.e. set of indices for G. Then the indices in J corresponding to finite functions must form a non-recursive set. This follows because, if they did not, the recursive enumeration of G could be modified to one of a class of partial recursive functions with infinite domains, including F. However, by Theorem 3 such an enumeration must leave out infinitely many elements of F. Contradiction.

Recursive permutations Let F be an arbitrary non-empty r.e. class of partial recursive functions with infinite domains again. We showed that F necessarily misses infinitely many recursive generators of *every* infinite r.e. set. We now argue that this even holds for more restricted kinds of generators as well.

Let $A = \mathbb{N}$. The recursive generators for A are known as the *recursive permutations* of A. A recursive permutation consisting of 1- and 2-cycles only is called an *involution.* A recursive permutation π is called *cycle-free* if and only if for every non-empty finite set D, $\pi(D) \neq D$ ([12], Exercise 7-37).

Theorem 4. *Let F be any recursively enumerable class of partial recursive functions with infinite domains. Then:*

(i) there are infinitely many recursive involutions that are not contained in F, and

(ii) there are infinitely many cycle-free recursive permutations that are not contained in F.

Proof. We may assume w.l.o.g. that F is non-empty. By Theorem 1 it follows that F has a recursive witness array $X = [X_n]_{n \in \omega}$. Let Y be an infinite recursive set such that $n \in Y \Rightarrow X_n = \emptyset$. We may assume w.l.o.g. that $0 \in Y$ (implying that $X_0 = \emptyset$).

(i) We construct infinitely many recursive involutions that cannot be contained in F. To obtain them, we proceed as follows.

Let y be an arbitrary element of Y. Define the function π_y as follows. Set $\pi_y(0) = y$ and $\pi_y(y) = 0$, and iterate the following stages:

Stage

- (1) determine the smallest n for which $\pi_y(n)$ is undefined;
- (2) determine the smallest $j \geq n$ such that $\pi_y(j)$ is undefined, $j \notin X_n$, and $n \notin X_j$;
- (3) set $\pi_y(n)$ to j and $\pi_y(j)$ to n.

Observe that during any stage and given the choice of n in action (1), only finitely many values are excluded for j, including at most finitely many values from Y. Thus in action (2) a value for j is always found, noting that any sufficiently large element from Y would already qualify.

Hence, π_y is well-defined, and it is an involution by construction. Because π_y violates the key property of the recursive witness array X in all stages, we have that $\pi_y \notin F$. By letting y range over all elements of Y, we obtain an infinite class of involutions not contained in F.

(ii) We now show how to obtain infinitely many cycle-free recursive permutations that cannot be contained in F.

Let $a \in \mathbb{N}$ be arbitrary, with $a > 0$. Define a recursive permutation π_a as follows. In stage 0 we set $\pi_a(0) = a$. In stage n ($n \geq 1$) we define $\pi_a(n)$, using the assignments from the preceding stages. In so doing we may have to modify the value of $\pi_a(j)$ for some $j < n$. When this happens, argument j will be called *injured*. We must avoid that j gets injured too often, otherwise π_a will not settle on a definite value for argument j.

We proceed inductively as follows. Assume that π_a is defined, one-one and cycle-free on $D_{n-1} = [0, \cdots, n-1]$ at the beginning of stage n. (It holds for $n = 1$ and it will be satisfied inductively.) In stage n, we have to define $\pi_a(n)$ such that no cycle is created by the value assigned to $\pi_a(n)$.

At the beginning of stage n, let m_{n-1} be the smallest integer in D_{n-1} that does not yet occur as image in $\{\pi_a(0), \cdots, \pi_a(n-1)\}$. Note that m_{n-1} exists and is well-defined as otherwise $\pi_a(D_{n-1}) = D_{n-1}$, contradicting that π_a is cycle-free on D_{n-1} at this point. Let $M_{n-1} = \max\{\pi_a(0), \cdots, \pi_a(n-1)\}$. Clearly $m_0 = 0$ and $M_0 = a$.

We use $i_1 \rightarrow i_2 \rightarrow \cdots$ to denote any (maximal) chain of π_a insofar as it is defined, with $i_1 \in D_{n-1}$ and $\pi_a(i_1) = i_2$ and so on. By the cyclefreeness of π_a, chains cannot get stuck on a subset of D_{n-1}. Thus, chains are finite and must end in $\cdots i_k \rightarrow n'$ for some $i_k \in D_{n-1}$ and $n' \notin D_{n-1}$ (i.e. $n' \geq n$). Note that $\pi_a(n')$ is undefined, at the beginning of stage n. If there is no $i \in D_{n-1}$ with $\pi_a(i) = i_1$, then i_1 is called a root value. Hence, at the start of stage n, m_{n-1} is the smallest root value. If a chain ends as $\cdots i_k \rightarrow n'$ for some $n' \geq n$, we call i_k a head value.

Stage n ($n > 0$).

Carry out the action that applies from the following list, in order of priority:

- (1) if there is a j that is injured and has $\pi_a(j) = n$: then set $\pi_a(n)$ to an integer that is larger than M_{n-1} and any element of X_n. Change the status of j to *healed*. Set M_n accordingly. Clearly $m_n = m_{n-1}$.

- (2) if $n \notin Y$: then set $\pi_a(n)$ to an integer that is larger than M_{n-1} and any element of X_n. Set M_n accordingly. Clearly $m_n = m_{n-1}$.

- (3) if $n \in Y$: consider the chain $m_{n-1} \rightarrow \cdots j \rightarrow n'$ in the defined part of π_a with n' the first integer encountered with $n' \geq n$. By cyclefreeness, j and n' exist, with $j \in D_{n-1}$. Now carry out the action that applies from the following list (in order).
  - (3a) if $n' = n$: then *modify* $\pi_a(j)$ to $M_{n-1} + 1$, turn the status of j to *injured*, and set $\pi_a(n)$ to m_{n-1}.
  - (3b) if $n' > n$: then set $\pi_a(n)$ to m_{n-1}.

  Finally, set m_n to the new smallest root value in $D_n = [0, \cdots, n]$ and set M_n to $M_{n-1} + 1$ or M_{n-1} as appropriate, respectively.

Note in (3a) that the modification of $\pi_a(j)$ leaves π_a cyclefree on D_{n-1}. In (3b) π_a remains cyclefree as well, because the chain ending at n cannot contain m_{n-1} here. Thus, in all cases, setting $\pi_a(n)$ as shown keeps π_a one-one and cycle-free on $D_n = [0, \cdots, n]$. We argue that executing the stages for n from 1 to ∞ yields a cyclefree recursive permutation, despite the modifications of values along the way.

Observe that only j's that occur as head values can get injured. If j gets injured while $\pi_a(j)$ is set to (say) N, then $\pi_a(j)$ remains unaffected until stage N. In precisely stage N, j is healed by action (1). As $\pi_a(N)$ is assigned a value, j can never become a head value again from this point onward. Hence, values can get injured at most once, and if they get injured, they are healed again (and forever) after finitely many stages.

We now argue that π_a is not only one-one but also becomes onto, i.e. every integer will eventually occur in the image of π_a. Consider m_n for $n \to \infty$. From the construction one sees that m_n is monotonically non-increasing. Observe that the value of m_n is not changed in the actions of type (1) and (2). Whatever happens in these actions does not change the fact that all values less than $m_n (= m_{n-1})$ occur as images of π_a. Now consider what happens in action (3). If $j < m_{n-1}$ in action (3), j remains in the image of π_a even if it gets injured. But, setting $\pi_a(n)$ to m_{n-1} now leads to a chain $n \to m_{n-1} \to \cdots \to N$ (for some $N > n$) and m_{n-1} gets added to the image of π_a. Note also that by this action, n cannot ever become a head value. Thus $m_{n-1}(= \pi_a(n))$ is added permanently to the image of π_a in both (3a) and (3b). Hence, in action (3) m_{n-1} always gets updated to a value m_n with $m_n > m_{n-1}$.

Note that at the start of any stage there can be at most finitely many injured j's. If only type (1) and (2) actions would be performed in subsequent stages, especially type (1) actions will be triggered every once in a while and the number of injured j's will steadily decrease. Hence, in finitely many steps some stage n must be reached with $n \in Y$ in which action (3) is performed. Consequently, m_n is increased infinitely often as $n \to \infty$. We conclude that π_a becomes onto and thus, that it is a well-defined and cyclefree recursive permutation. For every n we have $\pi_a(n) \notin X_n$. (For $n \in Y$ this follows because $X_n = \emptyset$, for $n \notin Y$ it follows from the definition of actions (1) and (2).) Thus $\pi_a \notin F$, as all functions in F must hit the witness array infinitely often and π_a does not.

The construction does not guarantee that $\pi_a(0)$ keeps its initial value a. However, $\pi_a(0)$ can never settle at a smaller value. Hence, if we take $b = \pi_a(0)+1$, we certainly obtain a cyclefree permutation π_b different from π_a. Repeating this *ad infinitum* gives an infinite class of cyclefree recursive permutations that are not contained in F. □

An immediate conclusion is the following.

Corollary 3. *The class of recursive involutions and the class of cyclefree recursive permutations are not recursively enumerable.*

Proof. Suppose one of the classes was recursively enumerable. Then Theorem 4 immediately gives a contradiction. □

Recall that the group of all recursive permutations is not finitely generated, by virtue of the fact that the class is not recursively enumerable (cf. [12], Exercise 4-6). The following stronger statement can be made.

Corollary 4. *No finitely generated group of recursive permutations can contain all recursive involutions, and neither can it contain all cyclefree recursive permutations.*

Proof. Let G be a finitely generated group of recursive permutations. Clearly G is recursively enumerable. The result now follows from Theorem 4. □

Sets Finally, we consider an application of Theorem 1 to classes of r.e. sets. A class S of r.e. sets is called recursively enumerable if there exists an r.e. set $J \subseteq \mathbb{N}$ such that $S = \{W_i | i \in J\}$, where $W_i = Dom(\phi_i)$ $(i \in \omega)$ is the usual indexing of the r.e. sets.

The study of recursively enumerable classes of r.e. sets parallels the one on classes of partial recursive functions. Fundamental results have been obtained in terms of sets. For example, there are infinite recursively enumerable classes of r.e. sets such that the deletion of the set '$\mathbb{N}$' from them gives a class that is no longer recursively enumerable [9]. More generally, for every $m \in \mathbb{N}$, there exists an infinite recursively enumerable class of r.e. sets which has only m proper infinite recursively enumerable subclasses, indeed with $m = 0$ allowed [14, 5].

It is well-known that the class of all r.e. sets and the class of all recursive sets are both recursively enumerable ([8], Section II.5). See also [1]. On the other hand, the class of all infinite r.e. sets and the class of all infinite recursive sets are not r.e. ([8], Exercise II.5.27). A much stronger statement can be proved.

Theorem 5. *Let S be any recursively enumerable class of infinite r.e. sets. Then there is an infinite recursive set A such that neither A nor any infinite r.e. subset of A belongs to S.*

Proof. Let J be any r.e. set of indices such that $S = \{W_i \mid i \in J\}$. Consider the corresponding r.e. class of functions $F_S = \{\phi_i \mid i \in J\}$. By definition, the functions in F_S have infinite domains.

By Theorem 1, F_S has a recursive witness array $X = [X_n]_{n \in \omega}$. Let Y be any infinite recursive set with the property that $n \in Y \Rightarrow X_n = \emptyset$.

Consider any infinite r.e. set W such that $W \subseteq Y$. Suppose that $W \in S$, hence that $W = W_e$ for some $e \in J$. Considering ϕ_e we note that there can be no n such that $\phi_e(n) \in X_n$, contradicting the required property of the witness array. Hence $W \notin S$. The theorem follows by taking $A = Y$. □

4 Conclusion

In this note we considered some properties of classes of partial recursive functions that are not necessarily completely recursively enumerable and thus do not fit the criteria of the Rice-Shapiro theorem. In particular, we proved that every non-empty r.e. class of partial recursive functions with infinite domains has a recursive witness array. This property enabled us to give elegant proofs and improve on some classical results on the power of recursive enumeration for common r.e. classes of partial recursive functions, recursive permutations, and sets. It would be of interest to find similar properties that could facilitate proofs for general r.e. classes.

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