# Spanning tree congestion of $k$-outerplanar graphs

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# Spanning tree congestion of $k$-outerplanar graphs

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**Abstract**

In 1987, Simonson conjectured that every $k$-outerplanar graph of the maximum degree $d$ has spanning tree congestion at most $k \cdot d$ [Math. Syst. Theory 20 (1987) 235–252]. We show that his conjecture is true and the bound is tight for outerplanar graphs and $k$-outerplanar graphs of maximum degree 4. We give a precise characterization of the spanning tree congestion of outerplanar graphs, and thus show that the spanning tree congestion of outerplanar graphs can be determined in linear time.

## 1 Introduction

In this paper, we settle a conjecture posed by Simonson [17] in 1987. His conjecture claims that every $k$-outerplanar graph $G$ has spanning tree congestion at most $k \cdot \Delta(G)$, where $\Delta(G)$ is the maximum degree of $G$. We prove that Simonson's conjecture is true. Moreover, we establish tightness of the bound, by giving outerplanar graphs and $k$-outerplanar graphs of maximum degree 4 whose spanning tree congestion equals $kd$. We also show that the spanning tree congestion of outerplanar graphs can be determined in linear time.

The parameter of spanning tree congestion is defined as follows. Let $G$ be a graph and $T$ a spanning tree of $G$. The *detour* for an edge $\{u, v\} \in E(G)$ is the unique $u$–$v$ path in $T$. We define the *congestion* of $e \in E(T)$, denoted by $cng_{G,T}(e)$, as the number of edges in $G$ whose detours contain $e$. The *congestion of $G$ in $T$*, denoted by $cng_G(T)$, is the maximum congestion over all edges in $T$. The *spanning tree congestion* of $G$, denoted by $stc(G)$, is the minimum congestion over all spanning trees of $G$.

Spanning tree congestion was formally defined by Ostrovskii [14] in 2004. Prior to Ostrovskii [14], Simonson [17] studied the parameter for outerplanar graphs as a variation of cutwidth. He showed that every outerplanar graph $G$ has spanning tree congestion at most $\Delta(G) + 1$. He

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also showed that there exists some outerplanar graph $G$ such that $stc(G) = \Delta(G)$. Ostrovskii [15] investigated the parameter for planar graphs, and presented some lower and upper bounds on the parameter. Recently, Otachi, Bodlaender, and van Leeuwen have shown that the problem of determining the spanning tree congestion is NP-hard even for planar graphs [16]. The parameter has been studied intensively [17, 14, 4, 8, 10, 11, 13, 15, 16, 9, 12].

# 2 Preliminaries

Let $G$ be a connected graph. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph induced by $S$. For an edge $e \in E(G)$, we denote by $G - e$ the graph obtained from $G$ by the deletion of $e$. Similarly, for a vertex $v \in V(G)$, we denote by $G - v$ the graph obtained from $G$ by the deletion of $v$. The neighborhood of $v$ in $G$ is denoted by $N_G(v)$. The distance between $u$ and $v$ in $G$, denoted by $dist_G(u, v)$, is the length of a shortest $u$–$v$ path in $G$.

An embedding of a graph into the surface is *planar* if it has no edge crossing. A graph is *planar* if it has a planar embedding. A planar graph with a certain planar embedding is called a *plane graph*. A planar embedding of a graph is 1-*outerplanar*, if all vertices lie on the exterior face. For $k \geq 2$, a planar embedding of a graph is $k$-*outerplanar*, if the embedding obtained from the original embedding by removing all vertices on the exterior face is a $(k-1)$-outerplanar embedding. A plane graph with a $k$-outerplanar embedding is called a $k$-*outerplanar* graph. Usually 1-outerplanar graphs are called *outerplanar graphs*. The *outerplanarity* $op(G)$ of a planar graph $G$ is the minimum $k$ such that $G$ has a $k$-outerplanar embedding.

The *dual graph* $G^*$ of a plane graph $G$ is the graph that has the vertex set $\mathcal{F}(G)$, the faces of $G$, and in which two vertices $f, f' \in \mathcal{F}(G)$ are adjacent if and only if the two faces $f$ and $f'$ have a common edge of $G$. If edge $e \in E(G)$ is adjacent to faces $f$ and $f'$, then we call the edge $\{f, f'\} \in G^*$ *the dual edge of* $e$. We say that $e \in E(G)$ is an *outer edge* of $G$ if it lies on the exterior face. For each outer edge $e$ and each inner face $F$ of $G$, define the *index* $i_e(F)$ as the length of a shortest path in $G^*$ which joins the exterior face $\Omega$ with $F$ and satisfies the additional condition: its first edge is $e^*$. For each inner face $F$ of $G$, we define the *absolute index* $i(F) = \min_e i_e(F)$, where the minimum is taken over all outer edges $e$.

In this paper, we only consider simple planar graphs; that is, we consider finite undirected planar graphs without loops and parallel edges. Note that the dual of a simple plane graph may have loops or parallel edges.

# 3 Upper bound for $k$-outerplanar graphs

In this section, we present our main result. That is, we prove that Simonson's conjecture is true and the bound in the conjecture is tight in some sense. Although Simonson's proof for outerplanar graphs was somewhat involved, our proof for $k$-outerplanar graphs is very simple. The idea is to use the connection between spanning tree congestion and edge remember number.

## 3.1 Proof of the bound

For the sake of simplicity, we first extend the notion of spanning tree congestion to disconnected graphs. We define the *spanning tree congestion of a disconnected graph* to be the maximum spanning tree congestion over all its connected components.

A *maximal spanning forest* $T$ of a graph $G$ contains a spanning tree of every connected component of $G$. Let $G$ be a graph and $T$ its maximal spanning forest. A *fundamental cycle* for an edge $\{u, v\} \in E(G) \setminus E(T)$ is the cycle consists of $\{u, v\}$ and the unique $u$–$v$ path in $T$. Bodlaender [2] defined the *edge remember number* $er_G(T)$ as the maximum over all edges $e \in E(T)$ of the number of fundamental cycles that contain $e$. From the definitions, the following proposition follows.

**Proposition 3.1.** *For any graph $G$ and its maximal spanning forest $T$, $er_G(T) + 1 = cng_G(T)$.*

*Proof.* Let $G$ be a graph, $T$ its maximal spanning forest, and $e \in E(G) \setminus E(T)$. Let $C$ and $D$ be the fundamental cycle and the detour for $e$, respectively. Obviously, $C = D \cup \{e\}$. This implies one-to-one correspondence between the detours and the fundamental cycles for edges in $E(G) \setminus E(T)$. Since the congestion of an edge in $T$ counts the edge itself as well, $cng_G(T) = er_G(T) + 1$. □

To bound the treewidth of $k$-outerplanar graphs, Bodlaender [2] showed the following lemma.

**Lemma 3.2** ([2, Lemma 79])**.** *Let $G = (V, E)$ be a plane graph, $H = (V, E')$ be the graph obtained from $G$ by removing all outer edges, and $T' = (V, F')$ be a maximal spanning forest of $H$. Then there exists a maximal spanning forest $T = (V, F)$ of $G$, such that $er_G(T) \leq er_H(T') + 2$.*

Proposition 3.1 and Lemma 3.2 together imply the following corollary.

**Corollary 3.3.** *Let $G = G_0$ be a plane graph, and $G_p$ be the graph obtained from $G_{p-1}$ by removing all outer edges. Then $stc(G) \leq stc(G_p) + 2p$.*

Now, we can prove the following theorem, which settles Simonson's conjecture. Note that we prove a slightly stronger bound for the odd maximum degree case.

**Theorem 3.4.** *For any $k$-outerplanar graph $G$ with the maximum degree $d$, $stc(G) \leq kd$. Furthermore, if $d$ is odd, then $stc(G) \leq k(d - 1) + 1$.*

*Proof.* Let $G_p$ be the graph defined in Corollary 3.3. Let $v$ be a vertex on the exterior face of $G_p$ for some $p$, where $0 \leq p \leq kd/2$. For convenience, we denote by $deg_p(v)$ the degree of $v$ in the graph $G_p$. Observe that $deg_{p+1}(v) = \max\{0, deg_p(v) - 2\}$. We assume $k \geq 1$.

If $d$ is even, then $deg_{p+d/2}(v) = 0$. Hence, $G_{kd/2}$ has no edge, and thus $stc(G_{kd/2}) = 0$. By Corollary 3.3, $stc(G) \leq stc(G_{kd/2}) + kd = kd$.

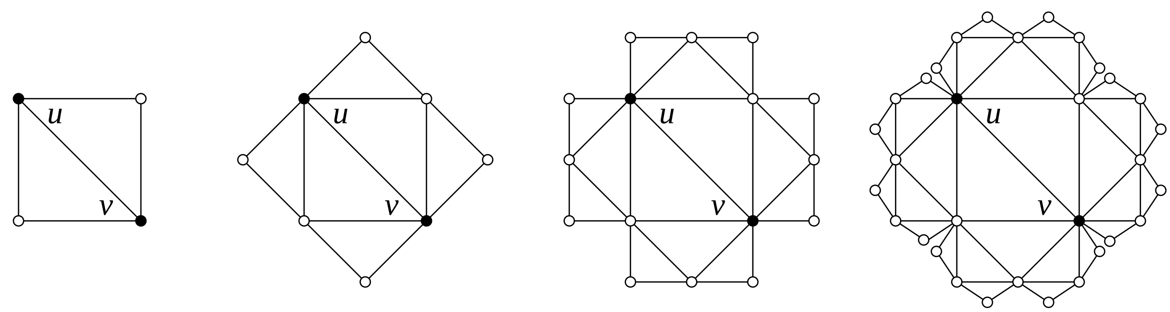
If $d$ is odd, then $deg_{p+\lfloor d/2 \rfloor}(v) \leq 1$. Thus we have $op(G_{p+\lfloor d/2 \rfloor}) < op(G_p)$, since vertices of degree one do not contribute to outerplanarity. This implies $op(G_{(k-1)\lfloor d/2 \rfloor}) \leq 1$, and thus $\Delta(G_{k\lfloor d/2 \rfloor}) \leq 1$. Therefore, $stc(G) \leq stc(G_{k\lfloor d/2 \rfloor}) + 2k\lfloor d/2 \rfloor \leq 1 + 2k\lfloor d/2 \rfloor = k(d - 1) + 1 \leq kd$. □

### 3.2 Tightness of the bound

We shall discuss the tightness of the bounds. For $k = 1$, that is, for the case of outerplanar graphs, we can show that the bounds is tight.

**Theorem 3.5.** *For any $d \geq 1$, there exists an outerplanar graph $G$ with the maximum degree $d$ such that $stc(G) = d$.*

*Proof.* Simonson [17] proved this fact for all even $d \geq 2$. Thus we can assume $d$ is odd. If $d = 1$, then the proposition is trivially true ($G$ is an edge $K_2$). For each odd $d \geq 3$, we define the outerplanar graph $H_d$ as follows (see Figure 1): the initial graph $H_3$ is a square with a diagonal edge; for $d \geq 5$, $H_d$ is obtained from $H_{d-2}$ by attaching triangles to all outer edges. Clearly, $\Delta(H_d) = d$ and $H_d$ is outerplanar. Let $u$ and $v$ be the vertices marked with black dots in Figure 1. It is easy to see that, there are $d$ edge disjoint $u$–$v$ paths in $H_d$. On the other hand, any spanning tree of $H_d$ has only one $u$–$v$ path. This implies $stc(H_d) \geq d$ [14]. □




$H_3$ $H_5$ $H_7$ $H_9$

Figure 1: Graphs $H_d$ for $d \in \{3, 5, 7, 9\}$.

Next we show the tightness of the bound for $k \geq 2$. The $k \times k$ *grid* $\Gamma_k$ is the graph that has the vertex set $\{(i, j) \mid i, j \in \{1, \ldots, k\}\}$ and in which two vertices $(i, j)$ and $(i', j')$ are adjacent if and only if $|i - i'| + |j - j'| = 1$. Clearly, $op(\Gamma_{2k}) \leq k$ and $\Delta(\Gamma_{2k}) = 4$ for $k \geq 2$. It is known that $stc(\Gamma_{2k}) = 2k$ [8, 4]. Hence, we have $stc(\Gamma_{2k}) \geq op(\Gamma_{2k}) \cdot \Delta(\Gamma_{2k})/2$. This example shows the bound is tight up to the constant factor $1/2$. Modifying this example, we can eliminate the constant factor. The *D-grid*, denoted by $D_k$, is the induced subgraph of $\Gamma_{4k}$ obtained by removing isosceles right triangles with side lengths $2k - 2$ from each corner (see Figure 2). We assume that the planar embedding of $D_k$ is induced from the natural embedding of the original grid $\Gamma_{4k}$ as depicted in Figure 2. It is easy to see that $op(D_k) \leq k$ and $\Delta(X_k) = 4$. To prove $stc(D_k) = 4k$, we need Ostrovskii's result.

Ostrovskii [15] defined the center-tail system and its congestion indicator for planar graphs. The system is designed to give a lower bound on the spanning tree congestion of planar graphs. Here, we use that system. Note that we use a simplified system, which contains only one vertex as its center. The original system can contain a set of center vertices. See [15] for the original definition.

A *center-tail system* $\mathcal{S}$ in the dual graph $G^*$ of a plane graph $G$ consists of:

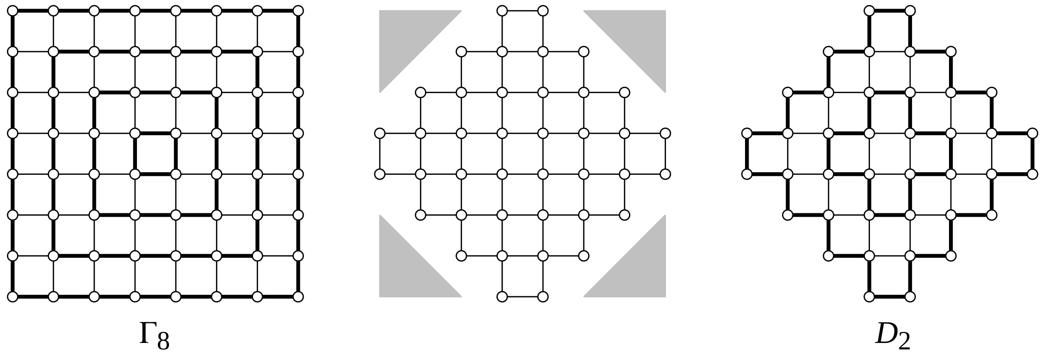



Figure 2: Grid $\Gamma_8$ and D-grid $D_2$.

1. A *center* vertex $C$ of $G^*$.

2. A set of paths in $G^*$ joining the center $C$ with the exterior face $\Omega$. Each such path is called a *tail*. The *tip of a tail* is the last vertex of the corresponding path before it reaches $\Omega$.

3. An assignment of *opposite tails* for outer edges of $G$. This means: For each outer edge $e$ of the graph $G$ one of the tails is assigned to be the opposite tail of $e$, it is denoted by $T(e)$ and its tip is denoted by $t(e)$.

The congestion indicator $\mathrm{CI}(\mathcal{S})$ of a center-tail system $\mathcal{S}$ is defined as the minimum of the following two numbers:

1. $\min_e i_e(t(e)) + 1$, where the minimum is taken over all outer edges of $G$.

2. $\min_e \min_{F \in T(e)} \min_{e' \neq e}(i_e(F) + i_{e'}(F') + 1)$, where the first minimum is taken over all outer edges of $G$; the second minimum is over vertices $F$ from the path $T(e)$ different from $t(e)$ and the exterior face, $F'$ is the vertex in $T(e)$ which follows immediately after $F$ if one moves along $T(e)$ from $F$ to $t(e)$; and the third minimum is over all outer edges different from $e$.

**Theorem 3.6** ([15])**.** *Let $\mathcal{S}$ be a center-tail system in a plane graph $G$. Then $stc(G) \geq CI(\mathcal{S})$.*

Using the Ostrovskii's center-tail system, we can show that $stc(D_k) = 4k$.

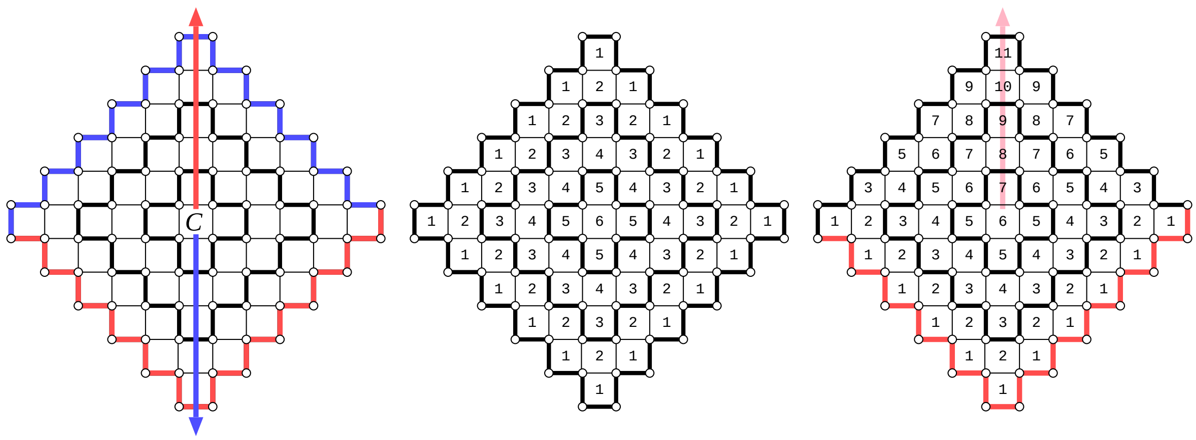
**Theorem 3.7.** *For any $k \geq 1$, there exists a $k$-outerplanar graph $G$ of maximum degree 4 such that $stc(G) = 4k$.*

*Proof.* It suffices to prove that $stc(D_k) = 4k$. We define the center-tail system for $D_k$ as follows (see Figure 3(a)):

1. The center $C$ is the face depicted in Figure 3(a).

2. We have two tails: one tail goes straight to the north; another one to the south. We call these tails the *north tail* and the *south tail*, respectively.

3. For each outer edge on the clockwise path between $(1, 2k)$ and $(4k, 2k+1)$, we assign the south tail as its opposite tail. For the remaining outer edges, we assign the north tail.



(a) Center-tail system. (b) Absolute indices. (c) Distances from the bottom.

Figure 3: Center-tail system for $D$-grids ($k = 3$).

First we evaluate the first value of the congestion-indicator of this system. Let $e$ be an outer edge. Observe that any shortest $\Omega$–$t(e)$ path with the first edge $e^*$ can be converted to a shortest $\Omega$–$t(e)$ path passing through the center $C$. It is easy to see that $i_e(C) = 2k$ for any outer edge $e$ (see Figure 3(b)). Hence, we have $i_e(t(e)) + 1 = i_e(C) + dist_{D_k^*}(C, t(e)) + 1 = 4k$, as required.

Next we evaluate the second value of the congestion-indicator. Let $e$ be an outer edge. Let $T(e) = (t_0, t_1, \ldots, t_{2k-1})$ be the opposite tail of $e$, where $t_0 = C$ and $t_{2k-1} = t(e)$. Clearly, $i_e(t_j) = 2k + j$ and $i(t_j) = 2k - j$ (see Figure 3(b) and 3(c)). Hence, for any outer edge $e'$ other than $e$,

$$i_e(t_j) + i_{e'}(t_{j+1}) + 1 \geq i_e(t_j) + i(t_{j+1}) + 1 = (2k + j) + (2k - (j + 1)) + 1 = 4k.$$

We have $stc(D_k) \geq 4k$ by Theorem 3.6, and $stc(D_k) \leq 4k$ by Theorem 3.4. □

# 4 Linear time algorithm for outerplanar graphs

It is known that $k$-outerplanar graphs have treewidth at most $3k - 1$ [2]. Thus, one expects that the problem can be solved in polynomial time for $k$-outerplanar graphs. However, using standard dynamic programming for graphs of bounded treewidth for the spanning tree congestion problem appears to give algorithms with a running time of the form $O(n^{f(k)})$, i.e., belong to XP. It is still open if there exists an algorithm for the spanning tree congestion of $k$-outerplanar graphs whose running time is of the form $O(f(k) \cdot n)$, or if this problem belongs to the class FPT. However, we present in this section a simple linear time algorithm for determining the spanning tree congestion of outerplanar graphs. That is, we prove the following theorem.

**Theorem 4.1.** *Given outerplanar graph $G$, $stc(G)$ can be determined in linear time.*

Our algorithm is based on an elegant characterization of the spanning tree congestion of outerplanar graphs, Theorem 4.3, which is a variant of an upper bound given by Ostrovskii [15].

From the definition of the spanning tree congestion of graphs, it is easy to see that the spanning tree congestion of a graph is the maximum spanning tree congestion over all its biconnected components. The biconnected components of a graph can be obtained in linear time [7]. Thus we assume graphs are biconnected in the rest of this section. We also assume that graphs are not cycles since any cycle has spanning tree congestion two. It is easy to see that if $G$ is a simple biconnected outerplanar graph and $G$ is not a cycle, then each vertex of $G^*$ has at least two neighbors. Ostrovskii [15] showed the following upper bound.

**Lemma 4.2** ([15])**.** *For any plane graph $G$, $stc(G) \le \max_{F,F'}(i(F)+i(F'))+1$, where the maximum is taken over all pairs $F, F'$ of inner faces with a common edge.*

He also showed that the bound is not tight for some planar graphs. We can prove, however, that the bound is tight for outerplanar graphs. That is, we will prove the following theorem, which itself is of interest in the graph-theoretical point of view.

**Theorem 4.3.** *For any outerplanar graph $G$, $stc(G) = \max_{F,F'}(i(F) + i(F')) + 1$, where the maximum is taken over all pairs $F, F'$ of inner faces with a common edge.*

Ostrovskii [15] showed that for any plane graph $G$, a spanning tree whose congestion meets the bound can be constructed as follow: take the dual $G^*$ of $G$; construct a breadth first search tree $T^*$ of $G^*$ rooted at the exterior face of $G$; remove edges $e$ from $G$ if whose dual edges $e^*$ appear in $T^*$; resultant graph is the desired tree. It is easy to see that these processes can be done in linear time, and $\max_{F,F'}(i(F) + i(F'))$ can be calculated by using $T^*$ in linear time also. Hence, Theorem 4.3 implies Theorem 4.1.

To prove Theorem 4.3, we use a connection between the spanning tree congestion and the tree spanners of planar graphs. Let $G$ be a graph and $T$ a spanning tree of $G$. If $dist_T(u, v) \le t$ for any $\{u, v\} \in E(G)$, then $T$ is a *tree $t$-spanner* of $G$ [3]. We denote by $tsp(G)$ the minimum number $t$ such that $G$ has a tree $t$-spanner. Since a cut in $G$ corresponds to a cycle in $G^*$, the next relation holds.

**Lemma 4.4** ([5])**.** *For any connected plane graph $G$, $stc(G) = tsp(G^*) + 1$.*

We define the *weak dual $G^w$* of a plane graph $G$ as $G^* - \Omega$, where $\Omega$ is the exterior face of $G$. It is well known that if $G$ is a biconnected outerplanar graph, then $G^w$ is a tree (see [6, 1]). Now, we are ready to prove Theorem 4.3.

*Theorem 4.3.* Let $G^*$ be a dual of $G$ and $\Omega \in V(G^*)$ the exterior face vertex of $G^*$. Thus $G^w = G^* - \Omega$. Let $\{F, F'\}$ be an edge in $G^*$ such that $F$ and $F'$ maximize $i(F) + i(F')$. It suffices to show that $tsp(G^*) \ge i(F) + i(F')$ by Lemma 4.4. Note that $i(R) = dist_{G^*}(R, \Omega)$ for any $R \in V(G^*)$. Let $T^*$ be a spanning tree of $G^*$ such that $T^*$ is a tree $tsp(G^*)$-spanner of $G^*$. We have two cases.

*[Case 1]* $\{F, F'\} \notin E(T^*)$: Since $G^w$ is a tree, there is no $F$–$F'$ path in $G^w - \{F, F'\}$, and thus the unique $F$–$F'$ path in $T^*$ contains the exterior vertex $\Omega$. Hence, we have

$$tsp(G^*) \ge dist_{T^*}(F, F') = dist_{T^*}(F, \Omega) + dist_{T^*}(F', \Omega) \ge dist_{G^*}(F, \Omega) + dist_{G^*}(F', \Omega).$$

*[Case 2]* $\{F, F'\} \in E(T^*)$: Let $T_1^*$ and $T_2^*$ be the components of $T^* - \{F, F'\}$. Without loss of generality, we assume that $F, \Omega \in T_1^*$ and $F' \in T_2^*$. We also assume that the trees $T^*$ and $G^w$ are rooted at $F$. Let $T_\Omega^*$ be the subtree of $T_1^*$ rooted at $\Omega$. We have the following claim (see Figure 4).

**Claim 4.5.** *There exist vertices $R \in V(T_\Omega^*)$ and $R' \in V(T_2^*)$ such that $\{R, R'\} \in E(G^*)$.*

*Claim 4.5.* Suppose that the claim does not hold. Recall that any vertex has at least two neighbors in $G^*$. Thus any leaf of the tree $G^w$ is adjacent to $\Omega$ in $G^*$. Let $R'$ be a leaf of $T_2^*$. From the assumption, $R'$ is not adjacent to any vertex of $T_\Omega^*$ in $G^*$, and thus $R'$ is not a leaf of $G^w$. Hence, $R'$ has at least two neighbors in $G^w$. Let $N_{T^*}(R') = \{p(R')\}$ and $R \in N_{G^w}(R') \setminus \{p(R')\}$.

Since $R \notin V(T_\Omega^*)$, the unique $R$–$R'$ path $P$ in $T^*$ does not contain $\Omega$. This implies that $P$ is the unique $R$–$R'$ path in $G^w$ as well. Since $R'$ is a leaf of $T^*$, $P$ contains $p(R')$. This implies that $P$ is not a single edge $\{R, R'\}$, since $R \neq p(R')$. Therefore, the path $P \subseteq G^w$ and the edge $\{R, R'\} \in E(G^w)$ together form a cycle in $G^w$. This contradicts that $G^w$ is a tree. □

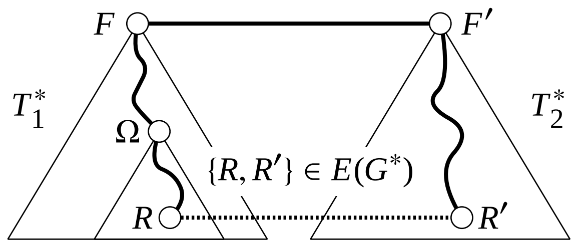



Figure 4: Illustration of Claim 4.5.

Let $R$ and $R'$ be the vertices in the above claim. Clearly, $\{R, R'\} \notin E(T^*)$. Since $dist_{T^*}(F, F') = dist_{G^*}(R', R) = 1$, we have

$$\begin{aligned} dist_{T^*}(R, R') &= dist_{T^*}(R, \Omega) + dist_{T^*}(\Omega, F) + dist_{T^*}(F, F') + dist_{T^*}(F', R') \\ &= dist_{T^*}(\Omega, F) + dist_{T^*}(F', R') + dist_{G^*}(R', R) + dist_{T^*}(R, \Omega) \\ &\geq dist_{G^*}(\Omega, F) + dist_{G^*}(F', \Omega). \end{aligned}$$

This implies that $tsp(G^*) \geq dist_{G^*}(\Omega, F) + dist_{G^*}(F', \Omega)$, as required. □

# 5 Conclusions

In this paper, we obtained an upper bound of $k \cdot d$ for the spanning tree congestion of $k$-outerplanar graphs with maximum degree $d$, and gave examples where this bound is tight, for outerplanar graphs, and $k$-outerplanar graphs of maximum degree four. We also have obtained a characterization of the spanning tree congestion of outerplanar graphs, which enabled a linear time algorithm for the problem on outerplanar graphs.

An interesting open problem is the complexity of determining the spanning tree congestion of $k$-outerplanar graphs. Taking the outerplanarity $op(G) = k$ as parameter, does this problem belong to the class FPT?

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