Enhanced Qualitative Probabilistic Networks for Resolving Trade-offs

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Abstract

Qualitative probabilistic networks were designed to overcome, to at least some extent, the quantification problem known to probabilistic networks. Qualitative networks abstract from the numerical probabilities of their quantitative counterparts by using signs to summarise the probabilistic influences between their variables. One of the major drawbacks of these qualitative abstractions, however, is the coarse level of representation detail that does not provide for indicating strengths of influences. As a result, the trade-offs modelled in a network often remain unresolved upon inference. We present an enhanced formalism of qualitative probabilistic networks to provide for a finer level of representation detail. An enhanced qualitative probabilistic network differs from a basic qualitative network in that it distinguishes between strong and weak influences. Now, if a strong influence is combined, upon inference, with a conflicting weak influence, the sign of the net influence may be readily determined. Enhanced qualitative networks are purely qualitative in nature, as basic qualitative networks are, yet allow for resolving more trade-offs upon inference.

1 Introduction

The formalism of *probabilistic networks* introduced in the 1980s [20], is an intuitively appealing formalism for capturing knowledge of complex problem domains along with the uncertainties involved. Associated with the formalism are powerful algorithms for reasoning with uncertainty in a mathematically correct way; these algorithms allow for causal reasoning, diagnostic reasoning as well as case-specific reasoning. Applications of probabilistic networks can be found in areas such as (medical) diagnosis and prognosis, planning, monitoring, vision, and information retrieval (see, for example, [1, 2, 3, 4, 15, 25]).

A probabilistic network basically is a concise representation of a joint probability distribution on a set of statistical variables. It consists of an acyclic directed graph encoding the relevant variables from a domain of application along with their probabilistic interrelationships. Associated with each variable is a set of conditional probability distributions describing the relationship of the variable with its predecessors in the graph. The first task in constructing a probabilistic network is to identify the important domain variables, their values, and their interdependencies. This knowledge is then modelled in a directed graph, referred to as the network's qualitative part. The final task is to obtain the probabilities that constitute the network's quantitative part. As (conditional) probabilities are required for each variable in the graph, their numbers can be quite large, even for small applications. While the construction of the qualitative part of a probabilistic network is generally considered feasible, its quantification is a far harder task. Probabilistic information available from literature or data is often insufficient or unusable, and domain experts have to be relied upon to assess the required probabilities [12]. Unfortunately, experts are often uncomfortable with having to provide probabilities. Moreover, the problems of bias encountered when directly eliciting probabilities from experts are widely known [14]. The usually large number of probabilities required for a probabilistic network, as a consequence, tends to pose a major obstacle to their application [12, 13].

To mitigate the quantification bottleneck to at least some extent, *qualitative probabilistic networks* have been introduced [26]. Qualitative networks in essence are qualitative abstractions of probabilistic networks. Like a probabilistic network, a qualitative network encodes variables and the probabilistic relationships between them in a directed graph. However, while the relationships between the represented variables are quantified by conditional probabilities in a probabilistic network, these relationships are summarised in its qualitative abstraction by qualitative signs capturing stochastic dominance. For reasoning with a qualitative probabilistic network, an efficient algorithm is available, based on the idea of propagating and combining these signs [9].

Qualitative probabilistic networks, by their nature, have a coarse level of representation detail. Influential relationships between variables can be modelled as positive, negative, zero or ambiguous, but no indication of their strengths can be provided as in a quantified network. One of the major drawbacks of this coarse level of representation detail is the ease with which the ambiguous '?'-sign arises upon inference. Ambiguous signs typically arise from trade-offs. A qualitative network models a trade-off if two nodes in the network's digraph are connected by multiple parallel reasoning chains with conflicting signs. In the absence of a notion of strength of influences, qualitative networks do not provide for resolving such trade-offs. Inference with a qualitative network for a real-life domain of application, as a consequence, often introduces ambiguous signs. Moreover, once an ambiguous sign has been generated, it will spread throughout major parts of the network. Although not incorrect, ambiguous signs provide no information whatsoever about the influence of one variable on another and are therefore not very useful in practice.

Ambiguous results from inference can be averted by enhancing the formalism of qualitative probabilistic networks to provide for a finer level of representation detail. Roughly speaking, the finer the level of detail, the more trade-offs can be resolved during inference. The finer levels of detail, however, typically come at the price of a higher computational complexity of inference. The problem of trade-off resolution for qualitative networks has been addressed by various researchers. S. Parsons has introduced, for example, the concept of categorical influence, which is either an influence that serves to increase a probability to 1, or an influence that decreases a probability to 0, and thus serves to resolve any trade-off in which it is involved [19]. Parsons has also studied the use of the less well-defined order-of-magnitude reasoning [19]. C.-L. Liu and M.P. Wellman have designed two methods for resolving trade-offs based upon the idea of reverting to numerical probabilities whenever necessary [17]. While only some trade-offs can be resolved by the use of categorical influences, the methods of Liu and Wellman provide for resolving any trade-off, but require a fully quantified probabilistic network.

To provide for trade-off resolution without resorting to numerical probabilities, we have designed an intuitively appealing formalism of *enhanced qualitative networks*. An enhanced qualitative probabilistic network differs from a basic qualitative network in that it introduces a notion of relative strength by distinguishing between strong and weak influences. If a trade-off is modelled in an enhanced network and the positive influence, for example, is known to be stronger than the conflicting negative one, we may upon inference conclude the net influence to be positive. Trade-off resolution during inference thus builds upon the idea that strong influences dominate over conflicting weak influences. To provide for inference with an enhanced network, we have generalised the sign-propagation algorithm for basic qualitative networks to deal with strong and weak influences. This generalisation is rather straightforward and derives from the observation that the properties upon which the basic sign-propagation algorithm is based are also provided for in an enhanced network. The new inference algorithm takes into account that the effect of one variable on another diminishes as variables are further apart in the network's graph; it also takes into account that a variable may affect another variable along multiple pathways with differing strengths. To maintain the correct strengths of indirect influences, the algorithm has to do some bookkeeping, as a result of which it is intuitively less appealing than the inference algorithm for basic qualitative networks. The distinction between strong and weak influences in an enhanced network, however, is very intuitive and domain experts should have no problems providing and interpreting the associated signs.

The paper is organised as follows. In Section 2, we provide some preliminaries from the fields of probabilistic networks and qualitative networks to introduce our notational conventions. In Section 3, we present our new formalism of enhanced qualitative probabilistic networks. In Section 4, we detail various properties of enhanced networks, on which we build two alternative sign-propagation algorithms. Section 5 provides an example of inference with an enhanced qualitative probabilistic network and discusses some complexity issues concerning the different propagation algorithms. Related work is reviewed in Section 6. The paper is rounded off with our conclusions and directions for future research in Section 7.

2 Preliminaries

Before introducing qualitative probabilistic networks, we briefly review their quantitative counterparts.

2.1 Probabilistic networks

A probabilistic network basically is a concise representation of a joint probability distribution on a set of statistical variables [20]. A probabilistic network $B = (G, \Pr)$ encodes, in an acyclic directed graph $G = (V(G), A(G))$, the relevant variables from a domain of application along with their probabilistic interrelationships. Each node $A \in V(G)$ represents a statistical variable that can take one of a finite set of values. We assume a total order '$>$' on the values of a variable. Variables will be indicated by capital letters from the beginning of the alphabet. We will restrict ourselves to binary-valued variables, where we write a to denote $A =$ *true* and $\bar{a}$ to denote $A =$ *false*, with $a > \bar{a}$. As there is a one-to-one correspondence between variables and nodes, we will use the terms 'node' and 'variable' interchangeably.

The probabilistic relationships between the represented variables are captured by the digraph's set of arcs $A(G)$. Informally speaking, we take an arc $A \to B$ in G to represent an influential relationship between the variables A and B, designating B as the effect of cause A. Given an arc $A \to B$, node A is called a (immediate) predecessor of node B and node B is called a successor of node A. We write $\pi(A)$ to denote the set of all predecessors of node A in G, and $\pi^*(A)$ to denote the set of its ancestors; similarly, $\sigma(A)$ is used to denote the set of all successors of node A and $\sigma^*(A)$ to denote its descendants. Two variables A and B are said to be connected by a (simple) trail in G iff they are connected by a (simple) path in the underlying undirected graph of G. Absence of an arc between two variables in the digraph of a probabilistic network means that the variables do not influence each other directly and, hence, are (conditionally) independent. More formally, probabilistic independence can be read from the digraph by means of the d-separation criterion, which builds on the concept of blocking. We say that a trail between two variables is *blocked* by the available evidence if it includes either an observed variable with at least one outgoing arc, or an unobserved variable with two incoming arcs and no observed descendants. A trail that is not blocked is called *active*. Two variables are now said

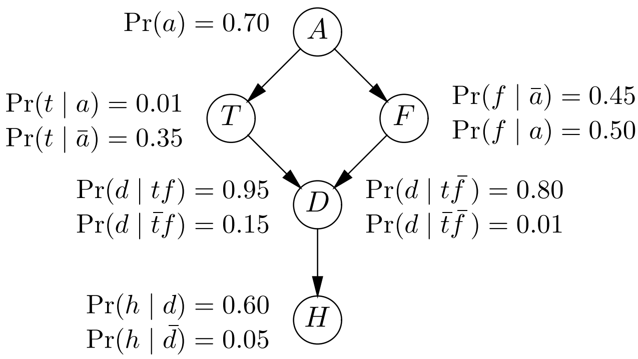


Figure 1: The *Antibiotics* network.

to be *d-separated* if all trails between them are blocked. If two variables are d-separated then they are considered conditionally independent given the available evidence [20].

Associated with each variable $A \in V(G)$ in the network's digraph G is a set of conditional probability distributions $\Pr(A \mid \pi(A))$ that describe the strengths of the various dependences between A and its (immediate) predecessors. These (conditional) probabilities with each other provide all information necessary for uniquely defining a joint probability distribution on the network's variables: the probabilistic network $B = (G, \Pr)$ defines the distribution $\Pr$ on $V(G)$ with

$$\Pr(V(G)) = \prod_{A \in V(G)} \Pr(A \mid \pi(A))$$

that respects the independences portrayed by the digraph G. Since a probabilistic network thus captures a unique joint probability distribution, it provides for computing any prior or posterior probability over its variables. To this end, various algorithms are available [16, 20].

We introduce a small probabilistic network that will serve as our running example throughout the paper.

Example 2.1 We consider the probabilistic network shown in Fig. 1. The network represents a fragment of fictitious medical knowledge pertaining to the effects of administering antibiotics on a patient. Node A represents whether or not a patient has been taking antibiotics. Node T models whether or not the patient is suffering from typhoid fever, node D represents the presence or absence of diarrhoea in the patient, and node H represents whether or not the patient is dehydrated. Node F, to conclude, describes whether or not the composition of the bacterial flora in the patient's intestines has changed. Typhoid fever and a change in bacterial flora are modelled as the possible causes of diarrhoea. Diarrhoea, in turn, can cause dehydration. Antibiotics can cure typhoid fever by killing the bacteria that cause the infection. As a result, the probability of a patient contracting diarrhoea decreases. However, antibiotics can also change the composition of the intestinal bacterial flora, thereby increasing the risk of diarrhoea. □

2.2 Qualitative probabilistic networks

Qualitative probabilistic networks bear a strong resemblance to their quantitative counterparts. A qualitative probabilistic network $Q = (G, \Delta)$ also comprises an acyclic digraph $G = (V(G), A(G))$ modelling variables and the probabilistic relationships between them. Moreover, the set of arcs $A(G)$

of this digraph again models probabilistic independence. Instead of conditional probability distributions, however, a qualitative probabilistic network associates with its digraph a set Δ of qualitative influences and qualitative synergies.

A *qualitative influence* between two variables expresses how the values of one variable influence the probabilities of the values of the other variable; the direction of the shift in probabilities is indicated by the *sign* of the influence. A *positive qualitative influence* of a variable A on a variable B, for example, expresses that observing a higher value for A makes a higher value for B more likely, regardless of any other influences on B [26].

Definition 2.2 *Let $G = (V(G), A(G))$ be an acyclic digraph and let $\Pr$ be a joint probability distribution on $V(G)$ that respects the independences in G. Let A, B be variables in G with $A \rightarrow B \in A(G)$. Then, variable A* positively influences *variable B along arc $A \rightarrow B$, written $S^{+}(A, B)$, iff*

$$\Pr(b \mid ax) - \Pr(b \mid \bar{a}x) \geq 0$$

for any combination of values x for the set $\pi(B) \setminus \{A\}$ of predecessors of B other than A.

A *negative qualitative influence*, denoted by S^{-}, and a *zero qualitative influence*, denoted by S^{0}, are defined analogously, replacing $\geq$ in the above formula by $\leq$ and $=$, respectively. If the influence of variable A on variable B is not monotonic or if it is unknown, we say that it is *ambiguous*, denoted $S^{?}(A, B)$.

With each arc in the digraph of a qualitative probabilistic network, a qualitative influence is associated. Variables, however, not only influence each other directly along arcs, they can also exert indirect influences on one another. The definition of qualitative influence trivially extends to indirect influences, that is, influences along active trails. We denote an indirect influence of sign δ along an active trail t from variable A to variable B by $\hat{S}^{\delta}(A, B, t)$. From here on, the term *trail* will be used to refer to either a simple trail, basically consisting of a *concatenation* of arcs, or to a subgraph containing all simple trails between two variables. The latter type of trail is said to consist of a *composition* of simple trails. The set of all variables on a trail t will be denoted $V(t)$.

The set of influences of a qualitative probabilistic network exhibits various convenient properties that constitute the basis for an efficient algorithm for qualitative probabilistic inference [26]. The property of *symmetry* guarantees that, if a network includes the influence $S^{\delta}(A, B)$, then it also includes $S^{\delta}(B, A)$ with the same sign $\delta \in \{+, -, 0, ?\}$. The property of *transitivity* asserts that qualitative influences along an active trail without head-to-head nodes, that is, without nodes with two incoming arcs on the trail, combine into an indirect influence whose sign is determined by the $\otimes$-operator from Table 1. The property of *composition* asserts that multiple qualitative influences between two variables along parallel active trails combine into a composite influence whose sign is determined by the $\oplus$-operator. From Table 1, we observe that combining non-ambiguous qualitative influences with the $\oplus$-operator can yield an ambiguous result. Such an ambiguity, in fact, results whenever influences with opposite signs are combined. We say that the *trade-off* that is reflected by the conflicting influences cannot be *resolved*. Note that, in contrast with the $\oplus$-operator, the $\otimes$-operator cannot introduce ambiguities upon combining signs. The operators in Table 1 adhere to the standard algebraic properties of commutativity, associativity, and distributivity of $\otimes$ over $\oplus$.

In addition to influences, a qualitative probabilistic network includes synergies that model the interactions between triples of variables. An *additive synergy*, for example, captures the joint influence of two variables on a common successor [26]. A positive additive synergy of two variables A and B on a variable C, more specifically, expresses that the joint influence of A and B on C is greater than their separate influences.

Table 1: The $\otimes$- and $\oplus$-operators.

$\otimes$	$+$	$-$	0	$?$
$+$	$+$	$-$	0	$?$
$-$	$-$	$+$	0	$?$
0	0	0	0	0
$?$	$?$	$?$	0	$?$

$\oplus$	$+$	$-$	0	$?$
$+$	$+$	$?$	$+$	$?$
$-$	$?$	$-$	$-$	$?$
0	$+$	$-$	0	$?$
$?$	$?$	$?$	$?$	$?$

Definition 2.3 *Let $G = (V(G), A(G))$ be an acyclic digraph and let* $\Pr$ *be a joint probability distribution on $V(G)$ that respects the independences in G. Let A, B, C be variables in G with $A \to C, B \to C \in A(G)$. Then, variables A and B exhibit a* positive additive synergy *on C iff*

$$\Pr(c \mid abx) + \Pr(c \mid \bar{a}\bar{b}x) - \Pr(c \mid a\bar{b}x) - \Pr(c \mid \bar{a}bx) \geq 0$$

for any combination of values x for the set $\pi(C) \setminus \{A, B\}$ of predecessors of C other than A and B.

Positive, zero, and *ambiguous additive synergies* are defined analogously.

If two variables A and B have a common successor C, then observation of a value for variable C serves to activate the trail $A \to C \leftarrow B$. The observation thus induces a dependence between A and B. This dependence can be represented by a qualitative influence of A on B, or vice versa. Such an induced influence is commonly known as an *intercausal influence*. The sign of the intercausal influence is captured by the sign of the *product synergy* associated with the variables involved and the observation. A product synergy thus expresses how the value of one variable influences the probabilities of the values of another variable in view of a given value for a third variable [10]. A negative product synergy of A and B on C with value c, for example, expresses that, given c, a high value for A renders a high value for B less likely.

Definition 2.4 *Let $G = (V(G), A(G))$ be an acyclic digraph and let* $\Pr$ *be a joint probability distribution on $V(G)$ that respects the independences in G. Let A, B, C be variables in G with $A \to C, B \to C \in A(G)$. Then, variables A and B exhibit a* negative product synergy *on variable C with value c, denoted $X^-(\{A, B\}, c)$, iff*

$$\Pr(c \mid abx) \cdot \Pr(c \mid \bar{a}\bar{b}x) - \Pr(c \mid a\bar{b}x) \cdot \Pr(c \mid \bar{a}bx) \leq 0$$

for any combination of values x for the set $\pi(C) \setminus \{A, B\}$ of predecessors of C other than A and B.

Positive, zero, and *ambiguous product synergies* again are defined analogously.

With each triple of variables $\{A, B, C\} \in V(G)$ with $A \to C$, $B \to C \in A(G)$, an additive synergy and two product synergies are associated. Note that a product synergy is defined for every possible value of C. These qualitative synergies are again trivially extended to trails and also exhibit symmetry, transitivity and composition properties. For details, we refer the reader to [21, 26].

Example 2.5 We consider the qualitative abstraction of the *Antibiotics* network from Fig. 1. From the conditional probabilities specified for the variable T, we have that

$$\Pr(t \mid a) - \Pr(t \mid \bar{a}) = 0.01 - 0.35 = -0.34 \leq 0$$

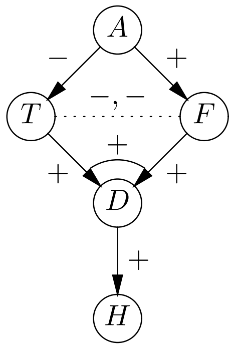


Figure 2: The qualitative *Antibiotics* network.

and therefore that $S^-(A,T)$. We similarly find that $S^+(A,F)$ and $S^+(D,H)$. From the conditional probabilities specified for the variable D, we have that

$$\Pr(d \mid tf) - \Pr(d \mid \bar{t}f) = 0.95 - 0.15 = 0.80 \geq 0, \text{ and}$$

$$\Pr(d \mid t\bar{f}) - \Pr(d \mid \bar{t}\bar{f}) = 0.80 - 0.01 = 0.79 \geq 0$$

and therefore that $S^+(T,D)$. We similarly find that $S^+(F,D)$. The variables T and F exert a positive additive synergy on variable D since

$$\begin{aligned}\Pr(d \mid tf) + \Pr(d \mid \bar{t}\bar{f}) &- \Pr(d \mid \bar{t}f) - \Pr(d \mid t\bar{f}) = \\ &= 0.95 + 0.01 - 0.15 - 0.80 = 0.01 \geq 0\end{aligned}$$

Either value for D, in addition, induces a negative intercausal influence between the variables T and F. For example, we have that

$$\begin{aligned}\Pr(\bar{d} \mid tf) \cdot \Pr(\bar{d} \mid \bar{t}\bar{f}) &- \Pr(\bar{d} \mid \bar{t}f) \cdot \Pr(\bar{d} \mid t\bar{f}) = \\ &= 0.05 \cdot 0.99 - 0.85 \cdot 0.20 = -0.12 \leq 0\end{aligned}$$

The resulting qualitative probabilistic network is depicted in Fig. 2. The figure shows the signs of the qualitative influences along the arcs; the additive synergy is indicated over the curve over node D, and the negative product synergies are displayed over the dotted edge. □

We would like to note that, although in the previous example we have computed the qualitative probabilistic relationships between the variables from the probabilities of the original quantified network, in real-life applications these relationships are elicited directly from domain experts.

For reasoning with a qualitative probabilistic network, an efficient algorithm is available from M.J. Druzdzel and M. Henrion [9]; this algorithm, termed the *sign-propagation algorithm*, is summarised in pseudocode in Fig. 3. The basic idea of the algorithm is to trace the effect of observing a variable's value on the probabilities of the values of all other variables in the network by message-passing between neighbouring nodes. In essence, the algorithm computes the sign of the net influence along all active trails between the newly observed variable and the other variables in the network, building upon the properties of symmetry, transitivity and composition of influences. For each variable, it summarises the net influence in a *node-sign* that indicates the direction of the shift in the variable's probability distribution that is occasioned by the new observation.

The sign-propagation algorithm takes for its input a qualitative probabilistic network Q, a set *Observed* of previously observed variables, the variable O for which an observation has become available,

procedure PropagateObservation(Q, O, *sign*, *Observed*):

for each $V_i \in V(G)$
do sign[V_i] $\leftarrow$ '0';
PropagateSign($\emptyset$,O,O, *sign*).

procedure PropagateSign(*trail*, *from*, *to*, *messagesign*):

sign[*to*] $\leftarrow$ sign[*to*] $\oplus$ *messagesign*;
trail $\leftarrow$ *trail* $\cup \{$ *to* $\}$;
for each active neighbour V_i of *to*
do *linksign* $\leftarrow$ sign of (induced) influence between *to* and V_i;
    messagesign $\leftarrow$ sign[*to*] $\otimes$ *linksign*;
    if $V_i \notin$ *trail* **and** sign[V_i] $\neq$ sign[V_i] $\oplus$ *messagesign*
    then PropagateSign(*trail*, *to*, V_i, *messagesign*).

Figure 3: The sign-propagation algorithm for qualitative probabilistic inference.

and the sign *sign* of the new observation, that is, either a '$+$' for the value *true* or a '$-$' for the value *false*. Prior to the propagation of the new observation, for all variables V_i the node-sign sign[V_i] is set to '0'. For the newly observed variable O the appropriate sign is now entered into the network. The observed variable updates its node-sign to the sign-sum of its original sign and the entered sign. It thereupon notifies all its active neighbours that its sign has changed, by passing to each of them a message containing an appropriate sign. This sign is the sign-product of the variable's current node-sign and the sign *linksign* of the influence associated with the arc or induced intercausal link it traverses. Each message further records its origin in the variable *trail*; this information is used to prevent messages being passed on to nodes that have already been visited on the same trail. Upon receiving a message, a variable *to* updates its node-sign to the sign-sum of its current node-sign sign[*to*] and the sign *messagesign* from the message it has just received. The variable then sends a copy of the message to all *its* neighbours that need to reconsider their node-signs. In doing so, the variable changes the sign in each copy to the appropriate sign and adds itself to *trail* as the origin of the copy. Note that as this process is repeated throughout the network, the trails along which messages have been passed are recorded. Also note that as messages travel simple trails only, it is sufficient to just record the nodes on these trails.

During sign-propagation, variables are only visited if they need a change of node-sign. A node-sign can change at most twice, once from '0' to '$+$', '$-$' or '?', and then only from '$+$' or '$-$' to '?'. From this observation we have that no variable is ever visited more than twice upon inference. The algorithm is therefore guaranteed to halt. For a proof of the algorithm's correctness we refer the reader to [9].

We illustrate the sign-propagation algorithm by means of our running example.

Example 2.6 We consider once again the qualitative *Antibiotics* network from Fig. 2. Suppose that a patient is taking antibiotics. This observation is entered into the network by updating the node-sign of the variable A to '$+$'. Variable A thereupon propagates a message, with sign $+ \otimes - = -$, towards T. Variable T updates its node-sign to '$-$' and sends a message with sign $- \otimes + = -$ to D. Variable D updates its sign to '$-$' and sends a message with sign $- \otimes + = -$ to H. Variable H updates its node-sign to '$-$'; it sends no messages as it has no neighbours that need to update their sign. Variable D does not pass on a sign to F, since the trail from T via D to F is not active.

Variable A also sends a message, with sign $+ \otimes + = +$, to F. Variable F updates its node-sign accordingly and passes a message with sign $+ \otimes + = +$ to D. Variable D thus receives the additional sign '$+$'. This sign is combined with the previously updated node-sign '$-$', which results in the ambiguous node-sign $- \oplus + = ?$ for D. Note that the ambiguous sign arises from the trade-off represented for variable D. D now sends a message with sign $? \otimes + = ?$ to H, which updates its sign to $? \oplus - = ?$. Note that, had the network contained additional variables beyond the variables D and/or H, then these variables would have all ended up with the node-sign '?' after inference. □

3 The enhanced formalism

Qualitative probabilistic networks capture the knowledge from a problem domain at a coarse level of representation detail. Qualitative influences between variables, for example, are captured by simple signs without any indication of their strengths. As a consequence, any trade-off encountered during inference will remain unresolved. In this section, we present a new formalism for qualitative probabilistic networks with a finer level of representation detail that will allow for resolving trade-offs to at least some extent. In this new formalism, we enhance qualitative probabilistic networks by associating an indication of relative strength with their influences. Now, if upon encountering a trade-off during inference, the positive influence is known to be stronger than the conflicting negative one, for example, we may conclude the combined influence to be positive, thereby effectively resolving the trade-off.

In an *enhanced qualitative probabilistic network*, we distinguish between strong and weak influences. Intuitively, a strong influence of a variable A on a variable B is an influence that is stronger than any weak influence in the network, that is, the property

$$|\Pr(b \mid ax) - \Pr(b \mid \bar{a}x)| \geq |\Pr(d \mid cy) - \Pr(d \mid \bar{c}y)|$$

holds for all variables C and D with a weak influence between them, for any combination of values x and y for the sets X and Y of relevant predecessors. The basic idea now is to partition the set of all positive influences into two disjoint sets in such a way that any influence from the one subset is stronger than any influence from the other subset. To this end, we introduce a *cut-off value* α that serves to partition the set of qualitative influences into a set of influences that capture an absolute difference in probabilities larger than α and a set of influences that model an absolute difference smaller than α. An influence from the former subset will be termed a *strong* influence; an influence from the latter subset will be termed a *weak* influence.

Definition 3.1 *Let $G = (V(G), A(G))$ be an acyclic digraph and let $\Pr$ be a joint probability distribution on $V(G)$ that respects the independences in G. Let A, B be variables in G with $A \rightarrow B \in A(G)$. Let $\alpha \in [0,1]$ be a cut-off value. The influence of variable A on variable B along arc $A \rightarrow B$ is* strongly positive*, denoted $S^{++}(A,B)$, iff it is positive and*

$$|\Pr(b \mid ax) - \Pr(b \mid \bar{a}x)| \geq \alpha$$

for any combination of values x for the set $\pi(B) \setminus \{A\}$ of predecessors of B other than A. The influence of variable A on variable B along the arc is weakly positive*, denoted $S^{+}(A,B)$, iff it is positive and*

$$|\Pr(b \mid ax) - \Pr(b \mid \bar{a}x)| \leq \alpha$$

for any combination of values x.

If, for an influence of a variable A on a variable B, we have $\Pr(b \mid ax) - \Pr(b \mid \bar{a}x) = \alpha$ for all x, then we take the influence to be strongly positive. *Strongly negative qualitative influences*, denoted S^{--}, and *weakly negative qualitative influences*, denoted S^{-}, are defined analogously; *zero qualitative influences* and *ambiguous qualitative influences* are defined as for basic qualitative probabilistic networks. A product synergy is defined to be *strongly negative* if it induces a strongly negative intercausal influence. *Weakly negative*, *Strongly positive*, and *weakly positive product synergies* are defined analogously; *zero product synergies* and *ambiguous product synergies* again are defined as for basic qualitative networks. For additive synergies, the distinction between weak and strong is slightly more complicated. Since additive synergies are not used during sign-propagation and therefore do not contribute to the resolution of trade-offs, we will not consider them any further in this paper.

Upon abstracting a quantified probabilistic network to an enhanced qualitative probabilistic network, the cut-off value α needs to be chosen explicitly. This cut-off value will typically vary from application to application, but it is always possible to choose such a cut-off value, since the values $\alpha = 0$ or $\alpha = 1$ yield a trivial partitioning of the set of influences. In real-life applications of enhanced qualitative probabilistic networks, however, the cut-off value need *not* be established explicitly. The partitioning into strong and weak influences is then elicited directly from the domain experts involved in the construction of the network.

Example 3.2 We consider once again the *Antibiotics* network from Example 2.1. Suppose that we choose for our cut-off value $\alpha = 0.30$. For the influence of variable A on variable T, we now find that

$$\Pr(t \mid a) - \Pr(t \mid \bar{a}) \leq 0, \text{ and } |\Pr(t \mid a) - \Pr(t \mid \bar{a})| = 0.34 \geq \alpha$$

We therefore conclude that $S^{--}(A, T)$; we similarly find that $S^{+}(A, F)$ and $S^{++}(D, H)$. For the influence of variable T on variable D, we find that $\Pr(d \mid tF) - \Pr(d \mid \bar{t}F) \geq 0$, regardless of the value of F, as well as

$$\Pr(d \mid tf) - \Pr(d \mid \bar{t}f) = 0.80, \text{ and } \Pr(d \mid t\bar{f}) - \Pr(d \mid \bar{t}\bar{f}) = 0.79$$

which both exceed the level of the cut-off value α. We therefore have that $S^{++}(T, D)$; we similarly find that $S^{+}(F, D)$. The signs of the product synergies exhibited by the variables T and F on variable D, in the presence of a value for D, equal the signs of the corresponding intercausal influences. The intercausal influences are defined in terms of differences between $\Pr(f \mid tx)$ and $\Pr(f \mid \bar{t}x)$, where x represents different combinations of values for the variables D and A. These probabilities can be found from the network in Example 2.1 by applying Bayes' rule; we list them here for ease of reference:

	$\Pr(f \mid tx)$	$\Pr(f \mid \bar{t}x)$
$x = da$	0.54	0.94
$x = d\bar{a}$	0.52	0.92
$x = \bar{d}a$	0.20	0.49
$x = \bar{d}\bar{a}$	0.17	0.41

For the sign of the intercausal influence of variable T on variable F given the value d for D, we now have that

$$\Pr(f \mid tda) - \Pr(f \mid \bar{t}da) = -0.40 \leq -\alpha, \text{ and}$$

$$\Pr(f \mid td\bar{a}) - \Pr(f \mid \bar{t}d\bar{a}) = -0.40 \leq -\alpha$$

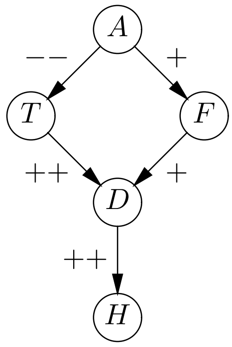


Figure 4: The enhanced *Antibiotics* network.

We conclude that the intercausal influence, and therefore its corresponding product synergy, are strongly negative: $X^{--}(\{T,F\},d)$; we similarly find that $X^{-}(\{T,F\},\bar{d})$.

The resulting enhanced qualitative probabilistic network, showing just the qualitative influences involved, is depicted in Fig. 4. □

We would like to note that, in our enhanced formalism, the semantics of the sign of an influence has slightly changed. While in a basic qualitative probabilistic network, the sign of an influence represents the sign of differences in probability only, in an enhanced qualitative network a sign in addition captures the relative magnitude of the differences.

4 Enabling inference in an enhanced network

For inference with a basic qualitative probabilistic network, an efficient algorithm is available. We recall from Section 2 that this algorithm builds on the idea of propagating signs throughout a network and combining them with the $\otimes$- and $\oplus$-operators. We further recall that the algorithm thereby exploits the properties of symmetry, transitivity, and parallel composition of influences. In this section we generalise the idea of sign-propagation to inference with an enhanced qualitative probabilistic network by taking into account the strength of influences. We first introduce, in Section 4.1, a structure for bookkeeping the strengths of influences during inference. In Section 4.2, we then address the property of symmetry, followed by a discussion and enhancement of the $\otimes$- and $\oplus$-operators to provide for the properties of transitivity and parallel composition of strong and weak influences, in Sections 4.3 and 4.4.

4.1 Multiplication-index lists for bookkeeping

In the previous section, we defined the strength of a direct qualitative influence relative to a cut-off value α. Upon inference, the strengths of indirect influences need to be computed. As we will demonstrate in Sections 4.3 and 4.4, the strengths of these indirect influences can be described in terms of a *polynomial* expression in α. To capture such a polynomial, we list its exponents together with an indication of whether its terms are added or subtracted. We will call such a list a *multiplication-index list*.

Definition 4.1 *Consider a non-empty list* $i = i_1, \ldots, i_n$ *such that each index* $i_j \in \mathbb{Z}$, $j = 1, \ldots, n$. *The* multiplication-index list i *represents the polynomial*

$$\sum_{i_j \geq 0} \alpha^{i_j} - \sum_{i_j < 0} \alpha^{|i_j|}$$

From this definition we have that a multiplication-index list can represent any polynomial in α that has a non-negative constant coefficient and all other coefficients in $\mathbb{Z}$. The polynomials we want to be able to describe adhere to these properties. We now define some properties of multiplication-index lists.

Definition 4.2 *Let* $i = i_1, \ldots, i_n$, $n \geq 1$, *be a multiplication-index list. Then*

- i *is called* positive *if for each* j, $j = 1, \ldots, n$, $i_j \geq 0$; i *is called* strictly positive *if* $i_j > 0$ *for all* j;
- i *is called* simple *if* $n \leq 2$, *or if* $n > 2$ *and for each* $i_j > 0$, $j = 1, \ldots, n$, *there exists* no i_k, $k = 1, \ldots, n$, *such that* $i_j = -i_k$.

(Strictly) negative index list are defined analogously. We take the definitions to apply to sublists as well.

The reason for defining simple multiplication-index lists is motivated by the following. Upon inference in an enhanced network, multiplication-index lists are constructed by combining such lists. After their combination, the resulting list can often be simplified. For example, the list $i = 1, 2, -1, -3$, representing the polynomial $\alpha + \alpha^2 - \alpha - \alpha^3$, can be simplified to $i = 2, -3$. That is, two complementing indices can be removed as long as this does not result in the entire elimination of the multiplication index. For example, the list $i = 1, -1$ represents the constant 0 and not $\alpha^0 = 1$. A list of the form $i = n, -n$ should therefore be represented in the given form; the actual value of n, however, is irrelevant and any non-zero integer value could be used without changing the list's meaning. Duplicates in a multiplication-index list cannot be removed, since for example $i = 1, 1$ represents the polynomial $\alpha + \alpha$ which equals $2 \cdot \alpha$ and not simply α. From here on, we assume that lists of multiplication indices are simplified as much as possible.

Upon comparing and combining signs during inference, we need to compare and combine multiplication-index lists. For this purpose, we define a number of operations on these lists. In their definition we use the standard list operators concat, for concatenating lists, and $\mathsf{map}(f, i)$, for applying a function f to every element of a list i.

Definition 4.3 *Let* $i = i_1, \ldots, i_n$, $n \geq 1$, *and* $j = j_1, \ldots, j_m$, $m \geq 1$ *be two* simple *multiplication-index lists. Let* k *be the length of the strictly negative sublist of* i. *Without loss of generality, assume* $i_1 \leq \ldots \leq i_n$ *and* $j_1 \leq \ldots \leq j_m$. *Then*

- $-i = \mathsf{map}(-, i)$;
- $i + j = \mathsf{concat}\left(\mathsf{concat}_{l=1,\ldots,k}\left(\mathsf{map}(-, \mathsf{map}(+\, i_k, j^-)), \mathsf{map}(+\, i_k, \mathsf{map}(-, j^+))\right)\right.,$ $\left.\mathsf{concat}_{l=k+1,\ldots,n}\left(\mathsf{map}(-\, i_k, j^-), \mathsf{map}(+\, i_k, j^+)\right)\right)$, *where* j^- *is the strictly negative sublist of list* j *and* j^+ *is its positive sublist;*
- $i \leq j$ *if* $n - 2 \cdot k \geq m$ *and* $i_l \leq j_l$ *for all* $l = 2 \cdot k + 1, \ldots, m$.

The summation of two lists of multiplication indices i and j results in the multiplication-index list representing a polynomial in α, which is the product of the two polynomials represented by i and j. Upon multiplying the two polynomials, we multiply all combinations of terms from the two expressions and, hence, sum all combinations of powers of the α-terms in the two expressions. In constructing the resulting multiplication-index list, we subsequently have to add in the minus-signs that indicate subtraction of terms. It is the latter step that makes the definition of $i + j$ above look so complex. With respect to the definition of $i \leq j$, at this stage we suffice it to say that it is defined so as to suit our purposes, as will become clear in Section 4.4.2.

The multiplication-index list we introduce is used to augment signs in an enhanced network during inference only. The list is used solely for the purpose of computation and we do not intend to output signs augmented with these indices to the user. The following definition describes the meaning of an influence with an augmented sign more formally.

Definition 4.4 *Let $G = (V(G), A(G))$ be an acyclic digraph in which the variables A and B are connected by an active trail t. Let $\Pr$ be a joint probability distribution on $V(G)$ that respects the independences in G. Then, the qualitative influence of variable A on variable B along trail t is* strongly positive with multiplication-index list $i_1, \ldots, i_k$, $k \geq 1$, *written $\hat{S}^{++^{i_1,\ldots,i_k}}(A, B, t)$, if it is positive, and*

$$|\Pr(b \mid ax) - \Pr(b \mid \bar{a}x)| \geq \sum_{i_j \geq 0} \alpha^{i_j} - \sum_{i_j < 0} \alpha^{|i_j|}$$

for every combination of values x for the subset $X = (\bigcup_{C \in V(t) \setminus \{A\}} \pi(C) \setminus V(t))$ of relevant ancestors of B. The qualitative influence of A on B along t is weakly positive with multiplication-index list $i_1, \ldots, i_k$, *written $\hat{S}^{+^{i_1,\ldots,i_k}}(A, B, t)$, if it is positive, and*

$$|\Pr(b \mid ax) - \Pr(b \mid \bar{a}x)| \leq \sum_{i_j \geq 0} \alpha^{i_j} - \sum_{i_j < 0} \alpha^{|i_j|}$$

for every combination of values x for X.

Strongly and weakly negative influences with a multiplication-index list are again defined analogously. Zero and ambiguous influences are once more defined as in basic qualitative probabilistic networks and are not augmented with multiplication indices.

From the definition above we note that a *strong* sign, augmented with a multiplication-index list of the form $n, -n$, represents an influence with a strength anywhere between zero and one. An equivalent representation of such an influence is given by a weak sign with the single multiplication index 0. So although, as remarked earlier, we cannot simplify a multiplication-index list of the form $n, -n$, we can, for example, replace the sign $++^{n,-n}$ by the sign $+^0$ without changing its meaning.

Upon initiating inference, the signs of the influences associated with the arcs of the digraph of an enhanced network are now interpreted as having a single multiplication index equal to 1. Note that a multiplication-index list that contains only a single index i represents an influence-strength relative to α^i; in this case we necessarily have that $i \in \mathbb{N}$.

4.2 The property of symmetry

In a basic qualitative probabilistic network, the property of symmetry guarantees that, if a variable A exerts an influence on a variable B, then variable B exerts an influence *of the same sign* on variable A. As a result, signs can be propagated during inference over an arc in both directions. In an enhanced qualitative network, as in a basic qualitative network, an influence and its reverse are both positive,

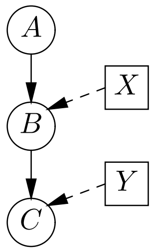


Figure 5: A fragment of a network.

both negative, both zero, or both ambiguous. The symmetry property, however, does not hold with regard to the strength of an influence: the reverse of a strongly positive qualitative influence, for example, may be a weakly positive influence, and vice versa. There are two ways of ensuring, in an enhanced network, that during inference signs can be propagated in both directions of an arc:

- specify the signs of all influences against the direction of an arc explicitly (note that these signs will have to be elicited from the domain experts involved in the network's construction);
- alternatively, use positive and negative signs of ambiguous strength, that is, signs whose strength is unknown and may be anywhere between zero and one.

Using the latter option, the symmetric counterpart of a positive influence, for example, would be an ambiguously positive influence, which can be represented by $+^0$. However, upon using signs of unknown strength, much useful information is lost and we therefore opt for explicitly specifying the signs of influences against the arc directions. With respect to intercausal influences we note that since they can be regarded as a qualitative influence, the above observations also hold with respect to the signs of such influences.

4.3 The property of transitivity

For propagating qualitative signs along active trails in an enhanced qualitative probabilistic network, we have to enhance the $\otimes$-operator that is defined for this purpose for basic qualitative networks, to apply to strong and weak influences. We recall that the $\otimes$-operator provides for multiplying signs of influences. In a basic qualitative probabilistic network, an influence in essence captures a difference between two probabilities. Combining two influences with the property of transitivity then amounts to determining the sign of the product of two such differences. In our formalism of enhanced qualitative probabilistic networks, however, we have associated an explicit notion of strength with influences. It will be evident that these strengths need to be taken into consideration when multiplying signs with the $\otimes$-operator.

To address the sign-product of two signs in an enhanced qualitative probabilistic network, we consider the network fragment shown in Fig. 5. The fragment includes an (active) trail that is composed of the variables A, B, C, and two qualitative influences between them. In addition, X denotes the set of all predecessors of B other than A, and Y is the set of all predecessors of C other than B. The following lemma now indicates that the strength of the indirect influence of A on C along the given trail equals the product of the strengths of the influences of A on B and of B on C.

Lemma 4.5 *Let $G = (V(G), A(G))$ be an acyclic digraph where $A, B, C \in V(G)$ and $A \rightarrow B, B \rightarrow C$ is the only active trail between the variables A and C. Let $\Pr$ be a joint probability* *distribution on $V(G)$ that respects the independences in G. Then,*

$$\begin{aligned}\Pr(c \mid axy) \quad - \quad & \Pr(c \mid \bar{a}xy) = \\ = \quad & (\Pr(c \mid by) - \Pr(c \mid \bar{b}y)) \cdot (\Pr(b \mid ax) - \Pr(b \mid \bar{a}x))\end{aligned}$$

for any combination of values x for the set of variables $X = \pi(B) \setminus \{A\}$ and any combination of values y for the set $Y = \pi(C) \setminus \{B\}$.

Proof: We observe that, in G, variable C is independent of the variables A and X, given B and Y; in addition, variable B is independent of variable Y, given A and X. By conditioning on B we now find

$$\begin{aligned}\Pr(c \mid axy) \quad - \quad & \Pr(c \mid \bar{a}xy) = \\ = \quad & \Pr(c \mid abxy) \cdot \Pr(b \mid axy) + \Pr(c \mid a\bar{b}xy) \cdot \Pr(\bar{b} \mid axy) \\ & - \Pr(c \mid \bar{a}bxy) \cdot \Pr(b \mid \bar{a}xy) - \Pr(c \mid \bar{a}\bar{b}xy) \cdot \Pr(\bar{b} \mid \bar{a}xy) \\ = \quad & (\Pr(c \mid by) - \Pr(c \mid \bar{b}y)) \cdot \Pr(b \mid ax) + \Pr(c \mid \bar{b}y) \\ & -(\Pr(c \mid by) - \Pr(c \mid \bar{b}y)) \cdot \Pr(b \mid \bar{a}x) - \Pr(c \mid \bar{b}y) \\ = \quad & (\Pr(c \mid by) - \Pr(c \mid \bar{b}y)) \cdot (\Pr(b \mid ax) - \Pr(b \mid \bar{a}x))\end{aligned}$$

□

Similar lemmas hold for the strengths of the influences along any other possible active trail between the variables A and C that can be obtained by reversing one or both arcs in Fig. 5 without introducing a head-to-head node on the trail. The lemma can further be easily extended to apply to the situation where A and B, and B and C, respectively, are connected by indirect active trails rather than direct arcs. We would like to note that the existence of additional (parallel) active trails between the variables A and C is handled by the $\oplus$-operator, and is therefore disregarded here.

The differences $\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy)$ for the various combinations of values xy serve to indicate the strength of the indirect influence of variable A on variable C. We informally investigate these differences using the property stated in Lemma 4.5. Suppose that the qualitative influences of A on B and of B on C both are strongly positive, that is, we have $S^{++}(A, B)$ and $S^{++}(B, C)$. Let α be the cut-off value used for distinguishing between strong and weak influences. From the expression stated in the lemma, we now find that

$$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \geq \alpha \cdot \alpha = \alpha^2$$

for any combination of values xy for the set of variables $X \cup Y$. Since $\alpha \leq 1$, we have that $\alpha^2 \leq \alpha$. Upon multiplying the signs of two strong direct influences, therefore, a sign results that indicates an indirect influence that is not necessarily stronger than a weak direct influence. Similar observations apply to strongly negative influences. Now suppose that both qualitative influences in the network fragment from Fig. 5 are weakly positive, that is, we have $S^{+}(A, B)$ and $S^{+}(B, C)$. For the indirect influence of variable A on variable C, we then find that

$$0 \leq \Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \leq \alpha \cdot \alpha = \alpha^2$$

for any combination of values xy. Similar observations apply to weakly negative influences. While the indirect influence resulting from the product of two strong influences cannot be compared to a weak direct influence, we have from the above observation that this indirect influence is always at least as strong as an indirect influence that results from the product of two weak influences. Finally, suppose that one qualitative influence in the network fragment from Fig. 5 is weakly positive and that the other is strongly positive, for example, $S^{+}(A, B)$ and $S^{++}(B, C)$. We then find for the indirect influence of variable A on variable C that

$$0 = 0 \cdot \alpha \leq \Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \leq \alpha \cdot 1 = \alpha$$

for any combination of values xy. Similar observations apply to other combinations of weak and strong influences. We thus have that the strength of an indirect influence resulting from the product of a strong and a weak influence *is* comparable to the strength of a weak direct influence.

From the previous observations, we conclude that to provide for comparing indirect qualitative influences along different trails with respect to their strengths, as required for trade-off resolution, we have to retain the length of the trail over which signs have been multiplied.

4.3.1 Enhancing the $\otimes$-operator

We employ the *multiplication-index list*, formally defined in Section 4.1, to retain information about the length of the trail over which signs have been multiplied. Table 2 now defines the enhanced $\otimes$-operator. From the table, it is readily seen that the '$+$', '$-$', '0', and '?' signs in essence combine just as in a basic qualitative probabilistic network; the only difference is in the handling of the multiplication indices. The enhanced $\otimes$-operator shapes the transitivity property for qualitative influences in an enhanced network. The following lemma shows that the operator correctly captures the sign of the transitive combination of two weakly positive influences.

Lemma 4.6 *Let $Q = (G, \Delta)$ be an enhanced qualitative probabilistic network. Let A, B, and C be variables in G for which there exist an active trail t_1 from A to B and an active trail t_2 from B to C such that their concatenation $t_1 \circ t_2$ is an active trail from A to C. Let i and j be simple multiplication-index lists. Then,*

$$\hat{S}^{+^i}(A, B, t_1) \;\wedge\; \hat{S}^{+^j}(B, C, t_2) \implies \hat{S}^{+^{i+j}}(A, C, t_1 \circ t_2)$$

Proof: Let $\Pr$ be a joint probability distribution on $V(G)$ that respects the independences in G, and let α be the cut-off value used for distinguishing between strong and weak influences. We will start by assuming that the multiplication-index lists i and j each consist of a single index. Then, the weakly positive influence $\hat{S}^{+^i}(A, B, t_1)$ of variable A on variable B expresses that

$$0 \leq \Pr(b \mid ax) - \Pr(b \mid \bar{a}x) \leq \alpha^i$$

for every combination of values x for the set $X = \bigcup_{C \in V(t_1) \setminus \{A\}} \pi(C) \setminus V(t_1)$ of relevant ancestors of B. Similarly, the weakly positive qualitative influence $\hat{S}^{+^j}(B, C, t_2)$ of variable B on variable C expresses that

$$0 \leq \Pr(c \mid by) - \Pr(c \mid \bar{b}y) \leq \alpha^j$$

for every combination of values y for the set Y of relevant ancestors of C. For the indirect influence of variable A on variable C, we thus find that

$$0 \leq \Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \leq \alpha^i \cdot \alpha^j = \alpha^{i+j}$$

for every combination of values xy for the set $X \cup Y$. More in general, we observe that the strength of the resulting influence lies between 0 and the product of the polynomial expressions in α captured by the multiplication-index lists i and j, respectively. We therefore conclude that $\hat{S}^{+^{i+j}}(A, C, t_1 \circ t_2)$. $\square$

Table 2: The enhanced $\otimes$-operator.

$\otimes$	$++^j$	$+^j$	0	$-^j$	$--^j$	$?$
$++^i$	$++^{i+j}$	$+^j$	0	$-^j$	$--^{i+j}$	$?$
$+^i$	$+^i$	$+^{i+j}$	0	$-^{i+j}$	$-^i$	$?$
0	0	0	0	0	0	0
$-^i$	$-^i$	$-^{i+j}$	0	$+^{i+j}$	$+^i$	$?$
$--^i$	$--^{i+j}$	$-^j$	0	$+^j$	$++^{i+j}$	$?$
$?$	$?$	$?$	0	$?$	$?$	$?$

From the above lemma and the appropriate entry in Table 2, we conclude that for two weakly positive influences the enhanced $\otimes$-operator indeed correctly captures the sign of their transitive combination. Similar observations hold for the transitive combination of any two weak influences or two strong influences, be they positive or negative.

The following lemma shows that the operator in Table 2 correctly captures the sign of the transitive combination of a weakly positive and a strongly positive influence.

Lemma 4.7 *Let Q, A, B, C, t_1, t_2, $t_1 \circ t_2$, i and j be as in the previous lemma. Then,*

$$\hat{S}^{+^i}(A,B,t_1) \;\wedge\; \hat{S}^{++^j}(B,C,t_2) \;\Longrightarrow\; \hat{S}^{+^i}(A,C,t_1 \circ t_2)$$

Proof: Let $\Pr$ be a joint probability distribution on $V(G)$ that respects the independences in G, and let α be the cut-off value used for distinguishing between strong and weak influences. We will start by assuming that the multiplication-index lists i and j each consist of a single index. Then, the weakly positive influence $\hat{S}^{+^i}(A,B,t_1)$ of variable A on variable B expresses that

$$0 \leq \Pr(b \mid ax) - \Pr(b \mid \bar{a}x) \leq \alpha^i$$

for every combination of values x for the set $X = \bigcup_{C \in V(t_1)\setminus\{A\}} \pi(C) \setminus V(t_1)$ of relevant ancestors of B. The strongly positive qualitative influence $\hat{S}^{++^j}(B,C,t_2)$ of variable B on variable C further expresses that

$$\alpha^j \leq \Pr(c \mid by) - \Pr(c \mid \bar{b}y) \leq 1$$

for every combination of values y for the set Y of relevant ancestors of C. For the indirect influence of variable A on variable C, we thus find that

$$0 \leq \Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \leq \alpha^i \cdot 1$$

for every combination of values xy for the set $X \cup Y$. More in general, we observe that the strength of the resulting influence lies between 0 and 1 times the polynomial expression in α captured by multiplication-index list i. We therefore conclude that $\hat{S}^{+^i}(A,C,t_1 \circ t_2)$. $\square$

From the above lemma and the appropriate entry in Table 2, we conclude that for a weakly positive and a strongly positive influence the enhanced $\otimes$-operator indeed correctly captures the sign of their transitive combination. Similar observations hold for the transitive combination of any weak influence with any strong influence, be they positive or negative. The proofs for the signs of all other transitive combinations of influences stated in Table 2, are analogous to the proofs of Lemmas 4.6 and 4.7.

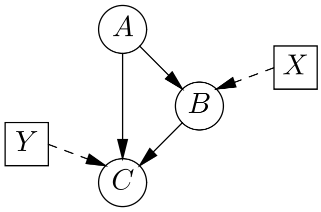


Figure 6: Another network fragment.

4.4 The property of parallel composition

For combining multiple qualitative influences between two variables along parallel active trails in an enhanced qualitative probabilistic network, we have to enhance the $\oplus$-operator that is defined for this purpose for basic qualitative networks, to apply to strong and weak influences. We recall that the $\oplus$-operator provides for summing signs of influences. We further recall that, upon adding the signs of two conflicting influences during inference with a basic qualitative network, the represented trade-off cannot be resolved and an ambiguous influence results. In our formalism of enhanced qualitative probabilistic networks, we have associated an explicit notion of strength with influences. These strengths can now be taken into consideration when summing the signs of influences and can be used to resolve trade-offs. For example, if a trade-off is encountered during inference, and the negative influence is known to be stronger than the conflicting positive one, then we may conclude that the combined influence is negative, thereby forestalling ambiguous results.

When addressing the property of transitivity in the previous section, we have argued that the product of two influences may yield an indirect influence that is weaker than the influences it is built from. We will now see that the sum of two influences, in contrast, may result in a stronger influence. To address the sign-sum of two signs in an enhanced qualitative probabilistic network, we consider the network fragment shown in Fig. 6. The fragment includes two active trails between the variables A and C, one of which captures a direct influence of A on C and the other one an indirect influence through B. In addition, the set X denotes the set of all predecessors of B other than A, and Y is the set of predecessors of C other than A and B. The following lemma now relates the strength of the net influence of variable A on variable C to the strengths of the influences it is built from.

Lemma 4.8 *Let $G = (V(G), A(G))$ be an acyclic digraph where $A, B, C \in V(G)$ and $A \rightarrow B, B \rightarrow C$ and $A \rightarrow C$ are the only active trails between the variables A and C. Let $\Pr$ be a joint probability distribution on $V(G)$ that respects the independences in G. Then,*

$$\begin{aligned} \Pr(c \mid axy) \quad - \quad & \Pr(c \mid \bar{a}xy) = \\ = \quad & (\Pr(c \mid aby) - \Pr(c \mid a\bar{b}y)) \cdot \Pr(b \mid ax) + \Pr(c \mid a\bar{b}y) \\ - \quad & (\Pr(c \mid \bar{a}by) - \Pr(c \mid \bar{a}\bar{b}y)) \cdot \Pr(b \mid \bar{a}x) - \Pr(c \mid \bar{a}\bar{b}y) \end{aligned}$$

for any combination of values x for the set X of all predecessors of B other than A and any combination of values y for the set Y of all predecessors of C other than A and B.

Proof: The proof of the property stated in the lemma is similar to that of Lemma 4.5. □

Similar lemmas hold for the strengths of the net influences of A on C along other combinations of multiple parallel trails that can be obtained by reversing one or more arcs in Fig. 6, as long as both trails remain active. A similar lemma can also be formulated for situations where one or more of the arcs in Fig. 6 are replaced by active trails. We would like to note that the existence of additional parallel trails between the variables A and C is handled by repeated application of the composition property, and is therefore disregarded here.

The differences $\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy)$ for the various combinations of values xy serve to indicate the sum of the strengths of the direct influence and the indirect influence of the variable A on the variable C. If all the arcs in the network fragment from Fig. 6 are associated with a weakly positive influence, for example, we find that

$$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \leq \alpha + \alpha^2$$

Building upon Lemma 4.8, we will prove this property shortly. From the inequality, we observe that the parallel composition of two weak direct influences of the same sign may result in an indirect influence that is stronger than a weak direct influence. Its relation to a strong influence is unknown, however. So, although the sign of the resulting influence is known unambiguously, its strength is not readily expressible as a simple power of α. Alternatively, if all the arcs in the network fragment from Fig. 6 are associated with a strongly positive influence, we find that

$$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \geq \alpha + \alpha^2$$

We now observe that the parallel composition of two strong direct influences of the same sign results in an indirect influence that is slightly stronger than a strong direct influence.

From these observations we have that the parallel composition of two or more influences may result in an influence for which the strength cannot be expressed by a single power of the cut-off value α. In the remainder of this section, we present two ways of capturing the strengths of such influences, by defining two different $\oplus$-operators for summing signs. The first operator works on signs with single multiplication indices only, and in essence discards the higher-order terms of the polynomial expression in α, leaving a single α-term whose power is taken to be the single multiplication index for the resulting sign; if these higher-order terms cannot be discarded without introducing a possible error, a sign of unknown strength is yielded. This first operator is called the *enhanced operator* $\oplus_e$ and is described in Section 4.4.1. Obviously, application of the $\oplus_e$-operator can result in loss of available information when adding two signs upon inference. To prevent such loss of information, at least to some extent, we augment signs with a multiplication-index list carrying information about the entire expression in α that describes their strengths. To provide for summing such signs, we define the *rich enhanced operator* $\oplus_r$, which is discussed in Section 4.4.2.

4.4.1 The enhanced $\oplus_e$-operator

In the foregoing, we have argued that the composition of two influences along parallel active trails may yield an influence whose sign is known but whose strength is not easily expressed as a simple power of the cut-off value used. In this section, we take such influences to be of unknown strength — where unknown implies anywhere between zero and one — and use the positive and negative signs $+^0$ and $-^0$, respectively, to denote them. We note that a *positive* qualitative influence of *unknown strength* of a variable A on a variable C, written $S^{+^0}(A, C)$, is equivalent to a positive qualitative influence in a basic qualitative probabilistic network; a similar observation applies to a *negative* qualitative influence of *unknown strength*. We further note that, while trade-offs in a basic qualitative probabilistic network result upon inference in an ambiguous sign indicating loss of information at the level of signs, trade-offs may now result upon inference in loss of information at the level of just the strengths of signs.

Table 3: The enhanced $\oplus_e$-operator.

$\oplus_e$	$++^j$	$+^j$	0	$-^j$	$--^j$	$?$
$++^i$	$++^m$	$++^i$	$++^i$	$a)$	$?$	$?$
$+^i$	$++^j$	$+^0$	$+^i$	$?$	$d)$	$?$
0	$++^j$	$+^j$	0	$-^j$	$--^j$	$?$
$-^i$	$b)$	$?$	$-^i$	$-^0$	$--^j$	$?$
$--^i$	$?$	$c)$	$--^i$	$--^i$	$--^m$	$?$
$?$	$?$	$?$	$?$	$?$	$?$	$?$

where $m = min(i, j)$, and

$a)$ $+^0$, if $i \le j$; $?$, otherwise

$b)$ $+^0$, if $j \le i$; $?$, otherwise

$c)$ $-^0$, if $i \le j$; $?$, otherwise

$d)$ $-^0$, if $j \le i$; $?$, otherwise

Table 3 defines the enhanced $\oplus_e$-operator for summing signs of influences along multiple parallel trails. From the table, it is readily seen that the '$+$', '$-$', '0', and '?' signs in essence combine just as in a basic qualitative probabilistic network; the difference is again in the handling of the multiplication indices introduced in Section 4.1. The enhanced $\oplus_e$-operator shapes the composition property for influences in an enhanced qualitative network. The following four lemmas show, for four different situations, that the operator correctly captures the sign of a combination of two parallel influences; the proofs for the other combinations of influences are quite similar. The first lemma pertains to the situation where two weakly positive influences along parallel trails are combined.

Lemma 4.9 *Let $Q = (G, \Delta)$ be an enhanced qualitative probabilistic network. Let A, C be variables in G and let t_1 and t_2 be parallel active trails in G from A to C, where $t_1 \parallel t_2$ is their trail composition. Let i and j be single multiplication indices. Then,*

$$\hat{S}^{+^i}(A, C, t_1) \wedge \hat{S}^{+^j}(A, C, t_2) \implies \hat{S}^{+^0}(A, C, t_1 \parallel t_2)$$

Proof: Let Pr be a joint probability distribution on $V(G)$ that respects the independences in G. Let α be the cut-off value used for distinguishing between strong and weak influences. For ease of exposition, we assume that the trail t_1 consists of a single arc and that the trail t_2 consists of the arcs $A \to B$, $B \to C$ for some variable B, as in the network fragment of Fig. 6. Additional trails between A and C can be handled by repeated application of the composition property, and are therefore disregarded here. We recall that with each arc is associated an influence with multiplication index 1. We further

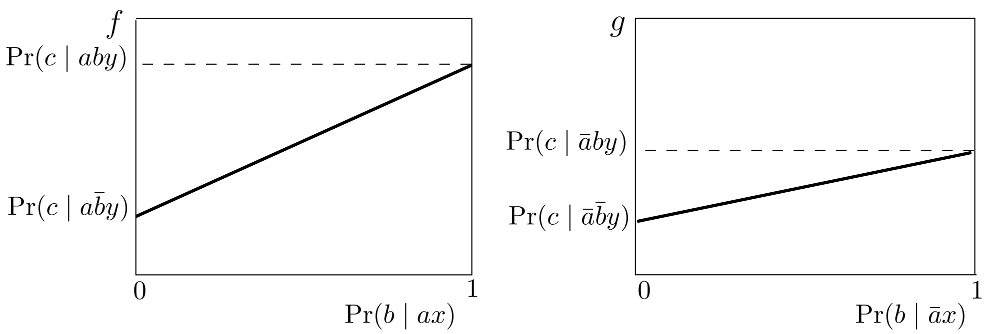


Figure 7: Possible functions $f(\Pr(b \mid ax))$ and $g(\Pr(b \mid \bar{a}x))$.

recall that Lemma 4.8 gives the net influence of variable A on variable C along the trail composition $t_1 \parallel t_2$. We now write the equation from Lemma 4.8 as the difference between two functions f and g:

$$
\begin{aligned}
\Pr(c \mid axy) \quad - \quad & \Pr(c \mid \bar{a}xy) = \\
= \quad & \big[(\Pr(c \mid aby) - \Pr(c \mid a\bar{b}y)) \cdot \Pr(b \mid ax) + \Pr(c \mid a\bar{b}y)\big] \\
& - \big[(\Pr(c \mid \bar{a}by) - \Pr(c \mid \bar{a}\bar{b}y)) \cdot \Pr(b \mid \bar{a}x) + \Pr(c \mid \bar{a}\bar{b}y)\big] \\
= \quad & f(\,\Pr(b \mid ax)\,) - g(\,\Pr(b \mid \bar{a}x)\,)
\end{aligned}
$$

for all value combinations x and y for the set X of predecessors of B other than A and the set Y of predecessors of C other than A and B, respectively. We note that the functions f and g are both linear in their respective parameter.

We now assume that the positive influence along trail t_2 is composed of two separate positive influences. From the influence of variable B on variable C being positive, we have that the functions f and g are both linearly increasing, as depicted in Fig. 7; the fact that in the figure the gradient of the function f is larger than the gradient of the function g is an arbitrary choice. From the positive direct influence of variable A on variable C we further have that $f(0) \geq g(0)$ and $f(1) \geq g(1)$. We therefore have that the functions f and G do not intersect. If the two influences along trail t_2 are both negative, then the functions f and g are decreasing and similar observations apply.

To determine the sign of the composite influence of variable A on variable C, we have to consider the sign of the difference between the functions f and g. We observe that, although the functions f and g are expressed in terms of different parameters, these parameters cannot be varied independently as their difference is restricted by the sign of the qualitative influence of variable A on variable B. Under this constraint, we are allowed to compare the function values of f and g for different parameters. For ease of comparison, we have depicted for this purpose the two functions f and g in a single graph, in Fig. 8.

Since the positive indirect influence along trail t_2 is composed of two positive influences, we have three possible situations:

(1) $S^{+^1}(A,B)$ and $S^{++^1}(B,C)$, or

(2) $S^{++^1}(A,B)$ and $S^{+^1}(B,C)$, or

(3) $S^{+^1}(A,B)$ and $S^{+^1}(B,C)$.

Here, we only consider the latter situation; the proofs for the other two situations are quite similar. As the direct influence of the variable A on the variable B is weakly positive, we have that $0 \leq \Pr(b \mid ax) - \Pr(b \mid \bar{a}x) \leq \alpha$. Therefore, when investigating the difference between the two functions f and g, we have to satisfy the following constraints:

- the parameter $\Pr(b \mid ax)$ for the function f should be greater than or equal to the parameter $\Pr(b \mid \bar{a}x)$ for the function g;
- the difference between the two parameters may not be greater than α.

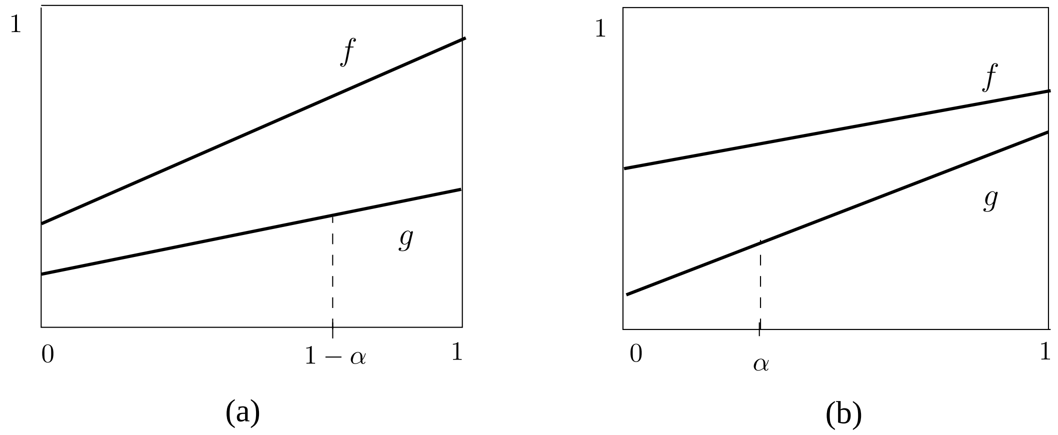


Figure 8: The functions $f(\,\Pr(b \mid ax)\,)$ and $g(\,\Pr(b \mid \bar{a}x)\,)$ depicted in a single graph, with (a) $gradient(\,f\,) > gradient(\,g\,)$, and (b) $gradient(\,g\,) > gradient(\,f\,)$.

We now show that under these constraints the difference $f(\Pr(b \mid ax)) - g(\Pr(b \mid \bar{a}x))$ is greater than or equal to zero. To this end, we consider the graph from Fig. 8(a); similar observations hold for the graph from Fig. 8(b). Under the given constraints, we have that the minimal difference between $f(\Pr(b \mid ax))$ and $g(\Pr(b \mid \bar{a}x))$ is attained for $f(0)$ and $g(0)$. We find that

$$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \geq f(0) - g(0) = \Pr(c \mid a\bar{b}y) - \Pr(c \mid \bar{a}\bar{b}y)$$

The minimal difference is positive as a result of the direct influence of A on C being positive. The sign of the composite influence of variable A on variable C is therefore positive. The maximal difference between $f(\Pr(b \mid ax))$ and $g(\Pr(b \mid \bar{a}x))$ is attained for $f(1)$ and $g(1-\alpha)$. Once again exploiting the information that the signs of the direct influences are all weakly positive, this difference equals:

$$\begin{aligned}
\Pr(c \mid axy) \quad - \quad \Pr(c \mid \bar{a}xy) \;&\leq\; f(1) - g(1-\alpha) \\
&= \Pr(c \mid aby) - \Pr(c \mid \bar{a}\bar{b}y) - \big(\Pr(c \mid \bar{a}by) - \Pr(c \mid \bar{a}\bar{b}y)\big)\cdot(1-\alpha) \\
&= \Pr(c \mid aby) - \Pr(c \mid \bar{a}by) + \alpha\cdot\big(\Pr(c \mid \bar{a}by) - \Pr(c \mid \bar{a}\bar{b}y)\big) \\
&\leq \alpha + \alpha\cdot\alpha = \alpha + \alpha^2
\end{aligned}$$

We conclude that the composite influence of variable A on variable C is positive and of unknown strength, that is, we conclude that the composite influence equals $\hat{S}^{+^0}(A, C, t_1 \parallel t_2)$. □

From the above lemma and the appropriate entry in Table 3, we conclude that for two weakly positive influences the enhanced $\oplus_e$-operator correctly captures the sign of their composition. Similar observations hold for the composition of two weakly negative signs.

The next lemma addresses the situation where two strongly positive influences along parallel trails are combined into a composite influence.

Lemma 4.10 *Let Q, A, C, t_1, t_2, $t_1 \parallel t_2$, i and j be as in the previous lemma. Then,*

$$\hat{S}^{++^i}(A, C, t_1) \;\wedge\; \hat{S}^{++^j}(A, C, t_2) \implies \hat{S}^{++^{min(i,j)}}(A, C, t_1 \parallel t_2)$$

Proof: The proof proceeds in a similar fashion as the proof of Lemma 4.9. We again assume that the positive influence along trail t_2 is composed of two separate positive influences, with similar observations applying when both influences are negative. Since the indirect influence of variable A on variable C along trail t_2 is strongly positive, it must be composed of two strongly positive direct influences. We thus have that

$$S^{++^1}(A,B) \text{ and } S^{++^1}(B,C)$$

and therefore that $S^{++^2}(A,C,t_2)$. We now investigate the difference between the two functions f and g defined in the proof of Lemma 4.9. Since the influence of the variable A on the variable B is strongly positive, we have that the difference between the two parameters for f and g should be at least α. To establish the minimum difference between $f(\Pr(b \mid ax))$ and $g(\Pr(b \mid \bar{a}x))$, we once again consider the graph from Fig. 8(a); similar observations again hold for the graph from Fig. 8(b). Under the constraint mentioned above, it is readily seen that the minimal difference between $f(\Pr(b \mid ax))$ and $g(\Pr(b \mid \bar{a}x))$ is attained for $f(\alpha)$ and $g(0)$. We find that

$$\begin{aligned}\Pr(c \mid axy) \;-\; \Pr(c \mid \bar{a}xy) \;&\geq\; f(\alpha) - g(0) \\ &= (\Pr(c \mid aby) - \Pr(c \mid a\bar{b}y)) \cdot \alpha + \Pr(c \mid a\bar{b}y) - \Pr(c \mid \bar{a}\bar{b}y) \\ &\geq \alpha^2 + \alpha \geq \alpha^{min(1,2)}\end{aligned}$$

We conclude that the composite influence of variable A on variable C is strongly positive with multiplication index 1, which is the minimum of 1 and 2. The composite influence thus equals $\hat{S}^{++^{min(1,2)}}(A,C,t_1 \parallel t_2)$. □

From the above lemma and the appropriate entry in Table 3, we conclude that for two strongly positive influences the enhanced $\oplus_e$-operator correctly captures the sign of their composition. Similar observations hold for the composition of two strongly negative signs.

The next lemma addresses the combination of a strongly positive and a weakly positive influence.

Lemma 4.11 *Let Q, A, C, t_1, t_2, $t_1 \parallel t_2$, i and j be as in the previous lemma. Then,*

$$\hat{S}^{++^i}(A,C,t_1) \;\wedge\; \hat{S}^{+^j}(A,C,t_2) \implies \hat{S}^{++^i}(A,C,t_1 \parallel t_2)$$

Proof: We distinguish between two different cases:

(I) the trail t_1 consists of a single arc and the trail t_2 consists of the arcs $A \rightarrow B$, $B \rightarrow C$ for some variable B;

(II) the trail t_1 consists of the arcs $A \rightarrow B$, $B \rightarrow C$ and the trail t_2 consists of the single arc.

For each of these cases, the proof proceeds in a similar fashion as the proof of Lemma 4.9. First we address case (I). As before, we assume that the indirect weakly positive influence of variable A on variable C along trail t_2 is composed of two separate weakly positive influences; the proofs for the other possible situations again are analogous. To establish the minimal difference between the functions f and g defined in the proof of Lemma 4.9, we once again consider the graph from Fig. 8(a). Since the influence of variable A on variable B is weakly positive, the difference between the two parameters for f and g should be at most α. Under this constraint, the minimal difference between $f(\Pr(b \mid ax))$ and $g(\Pr(b \mid \bar{a}x))$ is attained for $f(0)$ and $g(0)$. We thus find that

$$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \geq f(0) - g(0) = \Pr(c \mid a\bar{b}y) - \Pr(c \mid \bar{a}\bar{b}y)$$

Since the direct influence of variable A on variable C is strongly positive, we have that $\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \geq \alpha$. We conclude that the composite influence of variable A on variable C is strongly positive with multiplication index 1, that is, we conclude that the composite influence equals $\hat{S}^{++^1}(A, C, t_1 \parallel t_2)$ in case (I).

We now consider the minimal difference between the two functions f and g in case (II). We again assume that the indirect positive influence of A on C along trail t_1 is composed of two separate positive influences, with similar observations applying when both influences are negative. Since the indirect influence of A on C now is strongly positive, we have from Table 2 that the two separate influences from A to B and from B to C must both be strongly positive. We thus have that

$$S^{++^1}(A, B) \text{ and } S^{++^1}(B, C)$$

and, therefore, that $\hat{S}^{++^2}(A, C, t_1)$. Since the influence of variable B on variable C is positive, we have that the two functions f and g are both linearly increasing. Since the influence of A on B is strongly positive, we further have that parameter $\Pr(b \mid ax)$ for the function f should be greater than the parameter $\Pr(b \mid \bar{a}x)$ for the function g, with a difference of at least α. To establish the minimum difference between $f(\Pr(b \mid ax))$ and $g(\Pr(b \mid \bar{a}x))$, we again consider the graph from Fig. 8(a), with similar observations applying for the graph from Fig. 8(b). Under the constraints mentioned above, we observe that the minimal difference between $f(\Pr(b \mid ax))$ and $g(\Pr(b \mid \bar{a}x))$ is attained for $f(\alpha)$ and $g(0)$. We thus find that

$$\begin{aligned} \Pr(c \mid axy) \quad - \quad & \Pr(c \mid \bar{a}xy) \ \geq \ f(\alpha) - g(0) \\ = \quad & (\Pr(c \mid aby) - \Pr(c \mid a\bar{b}y)) \cdot \alpha + \Pr(c \mid a\bar{b}y) - \Pr(c \mid \bar{a}\bar{b}y)) \end{aligned}$$

Since the direct influence of A on C is weakly positive, we have that $0 \leq \Pr(c \mid a\bar{b}y) - \Pr(c \mid \bar{a}\bar{b}y) \leq \alpha$. We conclude that the composite influence of variable A on variable C is strongly positive with multiplication index 2, that is, we conclude that the composite influence equals $\hat{S}^{++^2}(A, C, t_1 \parallel t_2)$ in case (II). □

From the above lemma and the appropriate entry in Table 3, we deduce that for a weakly and a strongly positive influence the enhanced $\oplus_e$-operator correctly captures the sign of their composition. Similar observations hold for the composition of a strongly negative and a weakly negative influence.

The main reason for enhancing qualitative probabilistic networks with a notion of strength has been to provide for resolving trade-offs upon inference. Trade-off resolution in essence amounts to associating an unambiguous sign with the composite influence that is built from two or more conflicting influences along parallel active trails. The next lemma provides for the combination of conflicting influences and describes the type of trade-off that can now typically be resolved upon inference.

Lemma 4.12 *Let Q, A, C, t_1, t_2, $t_1 \parallel t_2$, i and j be as in the previous lemma. Then, if $i \leq j$,*

$$\hat{S}^{++^i}(A, C, t_1) \ \wedge \ \hat{S}^{-^j}(A, C, t_2) \implies \hat{S}^{+^0}(A, C, t_1 \parallel t_2)$$

Proof: Let $\Pr$ be a joint probability distribution on $V(G)$ that respects the independences in G and let α be the cut-off value used for distinguishing between strong and weak influences. As in the proof of Lemma 4.9, we construct two functions f and g with

$$\begin{aligned} \Pr(c \mid axy) \quad - \quad & \Pr(c \mid \bar{a}xy) = \\ = \quad & \big[(\Pr(c \mid aby) - \Pr(c \mid a\bar{b}y)) \cdot \Pr(b \mid ax) + \Pr(c \mid a\bar{b}y)\big] \\ & - \big[(\Pr(c \mid \bar{a}by) - \Pr(c \mid \bar{a}\bar{b}y)) \cdot \Pr(b \mid \bar{a}x) + \Pr(c \mid \bar{a}\bar{b}y)\big] \\ = \quad & f(\Pr(b \mid ax)) - g(\Pr(b \mid \bar{a}x)) \end{aligned}$$

Depending on the sign of the influence of variable B on variable C, we have that the functions f and g are either both linearly increasing, or linearly decreasing functions. We assume that the two functions are increasing, which implies that the influence of variable B on variable C is positive. We further assume that the gradient of the function f is larger than the gradient of the function g, as depicted in the graph from Fig. 8(a). Similar observations apply to the graph from Fig. 8(b), and to decreasing functions.

We now distinguish between the two cases (I) and (II) from the proof of Lemma 4.11:

(I) the trail t_1 consists of a single arc and the trail t_2 consists of the arcs $A \to B$, $B \to C$ for some variable B;

(II) the trail t_1 consists of the arcs $A \to B$, $B \to C$ and the trail t_2 consists of the single arc.

First we address case (I), with a strongly positive direct influence of variable A on variable C. From our assumptions we have that the indirect negative influence along trail t_2 is composed of a negative influence of A on B and a positive influence of B on C. More specifically, we have one of the following three situations:

(1) $S^{-^1}(A, B)$ and $S^{+^1}(B, C)$, or

(2) $S^{--^1}(A, B)$ and $S^{+^1}(B, C)$, or

(3) $S^{-^1}(A, B)$ and $S^{++^1}(B, C)$.

The indirect influence of variable A on variable C along trail t_2 has associated the sign $-^2$ in situation (1) and the sign $-^1$ in the situations (2) and (3).

To establish the sign of the composite influence of A on C, we first establish the minimal difference between the functions f and g. We begin by considering the situations (1) and (3) described above. Since the influence of variable A on variable B is weakly negative, we have that the parameters $\Pr(b \mid ax)$ and $\Pr(b \mid \bar{a}x)$ for the functions f and g, respectively, have to satisfy the following constraints:

- the parameter $\Pr(b \mid ax)$ for function f is smaller than or equal to the parameter $\Pr(b \mid \bar{a}x)$ for function g;
- the difference between the two parameters is at most α.

From Fig. 8(a), we observe that under these constraints the minimal difference between f and g is attained for $f(0)$ and $g(\alpha)$. The minimal difference thus is

$$\begin{aligned}\Pr(c \mid axy) \quad - \quad & \Pr(c \mid \bar{a}xy) \;\geq\; f(0) - g(\alpha) \\ = \quad & \Pr(c \mid a\bar{b}y) - \Pr(c \mid \bar{a}\bar{b}y) - \big(\Pr(c \mid \bar{a}by) - \Pr(c \mid \bar{a}\bar{b}y)\big)\cdot\alpha\end{aligned}$$

The difference between the first two terms is α or more, due to the strongly positive direct influence of A on C. The difference between the last two terms is captured by the influence of B on C, which is weakly positive in situation (1) and strongly positive in situation (3). In situation (1) we now have that

$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \geq \alpha - \alpha \cdot \alpha$; for situation (3) we find that $\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \geq \alpha - 1 \cdot \alpha = 0$.

We now consider the situation (2) described above. The strongly negative influence of variable A on variable B imposes the following constraints on the parameters for f and g:

- the parameter $\Pr(b \mid ax)$ for function f is smaller than the parameter $\Pr(b \mid \bar{a}x)$ for function g;
- the difference between the two parameters is at least α.

From Fig. 8(a), we observe that under these constraints the minimal difference between f and g is attained for $f(0)$ and $g(1)$:

$$\begin{aligned}\Pr(c \mid axy) \;-\; \Pr(c \mid \bar{a}xy) \;&\geq\; f(0) - g(1)\\ &=\; \Pr(c \mid a\bar{b}y) - \Pr(c \mid \bar{a}\bar{b}y) - (\Pr(c \mid \bar{a}by) - \Pr(c \mid \bar{a}\bar{b}y))\end{aligned}$$

We therefore have that $\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \geq \alpha - \alpha = 0$.

For all three situations (1), (2), and (3), the maximum difference between the functions f and g is attained for $f(1-\alpha)$ and $g(1)$. The maximum difference thus is

$$\begin{aligned}\Pr(c \mid axy) \;-\; \Pr(c \mid \bar{a}xy) \;&\leq\; f(1-\alpha) - g(1)\\ &=\; \Pr(c \mid aby) - \Pr(c \mid \bar{a}by) - (\Pr(c \mid aby) - \Pr(c \mid a\bar{b}y)) \cdot \alpha\end{aligned}$$

We find that the maximum difference is at most 1 in situations (1) and (2), and $1-\alpha$ in situation (3). We conclude that in case (I), the composite influence of variable A on variable C is positive but of unknown strength, that is, $\hat{S}^{+^0}(A, C, t_1 \parallel t_2)$.

We now address case (II). Since the indirect influence along trail t_1 is strongly positive, it must be composed of two strong direct influences. Recall that we assume that the influence of variable B on variable C is positive, hence both the strong influences are positive, that is,

$S^{++^1}(A, B)$ and $S^{++^1}(B, C)$,

resulting in the indirect influence $S^{++^2}(A, C)$. As the lemma addresses only situations where the multiplication index of the strong sign is at most that of the weak sign, we now assume that the weakly negative direct influence of variable A on variable C has a multiplication index of (at least) 2. The above observations result in the following constraints:

- function f lies below function g, that is, $f(0) \leq g(0)$ and $f(1) \leq g(1)$;
- the parameter $\Pr(b \mid ax)$ for function f is greater than the parameter $\Pr(b \mid \bar{a}x)$ for function g, with a difference of at least α;
- the functions f and g are both linearly increasing functions.

We again assume the gradient of f to be larger than that of g, with similar observations holding for the opposite case.

To establish the sign of the composite influence of variable A on variable B, we once again investigate the minimal and maximal differences between the functions f and g. Under the constraints above, we find that the minimal difference between f and g is attained for $f(1)$ and $g(0)$, and thus equals

$$\begin{aligned}\Pr(c \mid axy) \;\; - \;\; \Pr(c \mid \bar{a}xy) \;&\geq\; f(1) - g(0) \\ &= \Pr(c \mid aby) - \Pr(c \mid a\bar{b}y) + \Pr(c \mid a\bar{b}y) - \Pr(c \mid \bar{a}\bar{b}y)\end{aligned}$$

From the strongly positive influence of variable B on variable C, we have that $\Pr(c \mid aby) - \Pr(c \mid a\bar{b}y) \geq \alpha$; from the weakly negative influence of variable A on variable C we have that $0 \geq \Pr(c \mid a\bar{b}y) - \Pr(c \mid \bar{a}\bar{b}y) \geq -\alpha^2$. The minimal difference therefore equals $\alpha - \alpha^2$. Similarly, the maximal difference between the functions f and g is attained for $f(\alpha)$ and $g(0)$, and equals α^2. We conclude that for case (II), the composite influence of variable A on variable C is positive, but of unknown strength, that is, $S^{+^0}(A, C, t_1 \parallel t_2)$.

To summarise, in all possible situations where the multiplication index i of the strong sign is less than or equal to the multiplication index j of the weak sign, the composite influence of variable A on variable C is positive, but of unknown strength, that is, $S^{+^0}(A, C, t_1 \parallel t_2)$. Note that if $i > j$, then we cannot guarantee that the composite influence is positive. □

From the above lemma and the appropriate entry in Table 3, we observe that for a strongly positive influence with multiplication index i and a weakly negative influence with multiplication index j, $i \leq j$, the enhanced $\oplus_e$-operator correctly captures the sign of their composition. Similar observations apply to other combinations of strong and weak conflicting influences. We conclude that, under certain conditions, the composition of conflicting strong and weak influences using the enhanced $\oplus_e$-operator leads to an unambiguous result at the level of the basic sign of the composite influence. The enhanced $\oplus_e$-operator thus indeed serves to resolve trade-offs upon inference.

4.4.2 The rich enhanced operator $\oplus_r$

We recall that the composition of two influences along parallel active trails may yield an influence whose strength cannot be expressed as a simple power of the cut-off value α. In the previous section, we took such influences to be of unknown strength, which was captured by the signs $+^0$ and $-^0$. From the proofs in the previous section, however, we have that the strength of a composite influence can be expressed as a polynomial expression in terms of the cut-off value α. By discarding the higher-order terms in this polynomial, or — if discarding introduces possible errors — by settling for a sign with unknown strength, clearly some information loss occurs. To retain this information, the entire polynomial should be incorporated into the multiplication index of the sign of the composite influence. For this purpose, we augment a sign with a *list* of multiplication indices such as defined in Section 4.1.

Table 4 now defines the *rich* enhanced $\oplus_r$-operator for combining signs with a multiplication-index list. From the table, it is readily seen that the '$+$', '$-$', '0', and '?' signs combine as in a basic qualitative probabilistic network; the only difference is in the handling of the multiplication indices. When comparing the table to Table 3 for the enhanced $\oplus_e$-operator in the previous section, we note that stronger results can now be provided for the sum of two weak signs having the same basic sign. In addition, we can also give stronger results upon adding two strong signs, and for the sum of a strong and a weak sign. The following lemmas show, for the situations in which the enhanced $\oplus_r$-operator differs from the enhanced $\oplus_e$-operator, that the operator defined in Table 4 correctly captures the sign of a combination of two parallel influences; the proofs for the other combinations of influences are

Table 4: The rich enhanced $\oplus_r$-operator.

$\oplus_r$	$++^j$	$+^j$	0	$-^j$	$--^j$	$?$
$++^i$	$++^{i,j}$	$++^i$	$++^i$	$a)$	$?$	$?$
$+^i$	$++^j$	$+^{i,j}$	$+^i$	$?$	$d)$	$?$
0	$++^j$	$+^j$	0	$-^j$	$--^j$	$?$
$-^i$	$b)$	$?$	$-^i$	$-^{i,j}$	$--^j$	$?$
$--^i$	$?$	$c)$	$--^i$	$--^i$	$--^{i,j}$	$?$
$?$	$?$	$?$	$?$	$?$	$?$	$?$

where

$a)$ $++^{i,-j}$, if $i \leq j$; ?, otherwise

$b)$ $++^{-i,j}$, if $j \leq i$; ?, otherwise

$c)$ $--^{i,-j}$, if $i \leq j$; ?, otherwise

$d)$ $--^{-i,j}$, if $j \leq i$; ?, otherwise

quite similar. The first lemma pertains to the situation where two weakly positive influences along parallel trails are combined.

Lemma 4.13 *Let $Q = (G, \Delta)$ be an enhanced qualitative probabilistic network. Let A, C be variables in G and let t_1 and t_2 be parallel active trails in G from A to C, where $t_1 \parallel t_2$ is their trail composition. Let i and j be simple multiplication-index lists. Then,*

$$\hat{S}^{+^i}(A, C, t_1) \ \wedge \ \hat{S}^{+^j}(A, C, t_2) \implies \hat{S}^{+^{i,j}}(A, C, t_1 \parallel t_2).$$

Proof: Following the proof of Lemma 4.9 we find, for relevant value combinations xy, that

$$0 \ \leq \ \Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \ \leq \ \alpha^i + \alpha^j,$$

if we assume i and j to be single multiplication indices. More in general, we find that the strength of the composite influence lies between zero and the sum of the two polynomials in α, captured by the multiplication-index lists i and j, respectively.

We conclude that the composite influence of variable A on variable C is weakly positive with multiplication-index list i, j, that is, we conclude that the composite influence equals $\hat{S}^{+^{i,j}}(A, C, t_1 \parallel t_2)$. □

From the above lemma and the appropriate entry in Table 4, we conclude that for two weakly positive influences the enhanced $\oplus_r$-operator correctly captures the sign of their composition. Similar observations hold for the composition of two weakly negative signs. From the proofs of Lemmas 4.9 and 4.13, the difference between the enhanced $\oplus_e$-operator from the previous section and the rich enhanced $\oplus_r$-operator, for combining weak influences having the same basic sign, becomes apparent: whereas with the enhanced $\oplus_e$-operator the polynomial expression in α was summarised in a sign with unknown strength, the powers of each α term are now retained in the multiplication-index list of the resulting sign. The same observations hold when we combine the signs of two strong influences, positive or negative, with the same basic sign: instead of a composite sign with a single multiplication index equal to the lowest-order term from the polynomial expression in α, the power of each term is now retained in the multiplication-index list.

The next lemma provides for the important trade-off situation where two influences with conflicting signs are combined upon inference.

Lemma 4.14 *Let Q, A, C, t_1, t_2, $t_1 \parallel t_2$, i and j be as in the previous lemma. Then, if $i \leq j$,*

$$\hat{S}^{++^i}(A, C, t_1) \ \wedge \ \hat{S}^{-^j}(A, C, t_2) \ \Rightarrow \ \hat{S}^{++^{i,-j}}(A, C, t_1 \parallel t_2)$$

Proof: Following the proof of Lemma 4.12, we find for case (I), in all situations, as well as for case (II), that for relevant value combinations xy

$$\Pr(c \mid axy) - \Pr(c \mid \bar{a}xy) \geq \alpha^i - \alpha^j$$

if we assume that i and j are single multiplication indices. More in general, we find that the strength of the resulting influence is at least the difference between the two polynomials in α captured by the multiplication-index lists i and j, respectively; this difference is guaranteed to be positive if $i \leq j$ (see Proposition 4.15 below).

We conclude that the composite influence of variable A on variable C is strongly positive with multiplication-index list $i, -j$, that is, $\hat{S}^{++^{i,-j}}(A, C, t_1 \parallel t_2)$. □

From the above lemma and the appropriate entry in Table 4, we conclude that for a strongly positive influence with multiplication-index list i and a weakly negative influence with multiplication-index list j, $i \leq j$, the rich enhanced $\oplus_r$-operator correctly captures the sign of their composition. Similar observations apply to other combinations of strong and weak conflicting influences.

From Table 4 it is obvious that only in those situations where the $\oplus_r$-operator is applied to a strong sign augmented with multiplication-index list i and a weak sign with multiplication-index list j, the property $i \leq j$ is required to conclude a non-ambiguous result. The following proposition shows that in these situations the definition of $\leq$ on lists of multiplication indices ensures that the *strength* of the concluded sign is non-negative.

Proposition 4.15 *Let i and j be two simple multiplication-index lists where i is associated with a strong sign and j is associated with a conflicting weak sign. Let $[\alpha]^i$ denote the polynomial in α represented by list i. If $i \leq j$, as defined in Definition 4.3, then $[\alpha]^i - [\alpha]^j \geq 0$.*

Proof: We assume that the signs associated with the direct influences along arcs in an enhanced network are augmented with a single positive multiplication index, and this index equals 1. We now observe that multiplication-index list j is always positive: examining Tables 2 and 4 we see that a weak sign can only be the result of applying one of the enhanced operators $\otimes$ or $\oplus_r$ to at least one weak sign; in those situations, the multiplication-index list for the resulting sign is inherited from the multiplication-index list(s) of the weak sign(s) to which the operator was applied. In none of these cases do we introduce negative multiplication indices, so weak signs always have positive multiplication indices only.

We now consider multiplication-index list i. Let $i = i_1, \ldots, i_n$, $n \geq 1$, and assume that $i_1 \leq \ldots \leq i_n$. Let k be the length of the strictly negative sublist of i. From Definition 4.3 we have that $k \leq \frac{1}{2} \cdot n$. In addition, if $k \neq 0$, then the definition ensures that each negative index in i is compensated for by a positive index, that is, $|i_l| \geq i_{2k-l+1}$ for all $1 \leq l \leq k$. Therefore, we have that both $[\alpha]^{i_1,\ldots,i_{2k}} \geq 0$ and $[\alpha]^{i_{2k+1},\ldots,i_n} \geq 0$.

Finally, let $j = j_1, \ldots, j_m$, $m \geq 1$, and assume that $j_1 \leq \ldots \leq j_m$. From Definition 4.3 we have that $i \leq j$ if $n - 2 \cdot k \geq m$ and $i_l \leq j_l$ for all $l = 2 \cdot k + 1, \ldots, m$. Since both j and $i_{2k+1}, \ldots i_n$ contain only non-negative indices, we conclude that if $i \leq j$ then $[\alpha]^{i_{2k+1},\ldots,i_m} - [\alpha]^{j_1,\ldots,j_m} \geq 0$. As a result, $[\alpha]^{i_1,\ldots,i_{2k}} + [\alpha]^{i_{2k+1},\ldots,i_m} + [\alpha]^{i_m,\ldots,i_n} - [\alpha]^{j_1,\ldots,j_m} \geq 0$, that is, $[\alpha]^i - [\alpha]^j \geq 0$. □

We provide some examples to illustrate the application of the rich enhanced $\oplus_r$-operator to conflicting signs.

Example 4.16 For each of the following examples we assume the lists of multiplication indices to be simple. We now consider the following sign-sums:

1)	$--^2 \oplus_r +^2$	5)	$--^{1,2} \oplus_r +^{1,1}$
2)	$++^2 \oplus_r -^1$	6)	$++^{-2,1} \oplus_r -^2$
3)	$--^{1,2} \oplus_r +^{1,3}$	7)	$--^{-2,2} \oplus_r +^3$
4)	$++^1 \oplus_r -^{1,1}$	8)	$++^{-4,-2,1,3,3} \oplus_r -^3$

1) Both multiplication-index lists are positive and of equal length. Obviously, $2 \leq 2$, therefore $--^2 \oplus_r +^2 = --^{2,-2}$.

2) Again we have positive multiplication-index lists of equal length. However, $1 \not\leq 2$, therefore the weak negative sign may in fact be stronger than the strong positive sign and we can only conclude $++^2 \oplus_r -^1 = ?$.

3) We now have two positive multiplication-index lists, both of length 2. From $1 \leq 1$ and $2 \leq 3$ we conclude that $i = 1, 2 \leq j = 1, 3$ and therefore $--^{1,2} \oplus_r +^{1,3} = --^{1,2,-1,-3}$. The multiplication-index list of the resulting sign can be simplified to $2, -3$.

4) The two positive multiplication-index lists are of different lengths. More specifically, the number of indices for the strong sign is smaller than that for the weak sign. We can therefore not conclude that the strongly positive sign is definitely stronger than the weak negative sign. We conclude that $++^1 \oplus_r -^{1,1} = ?$.

5) Again we consider two positive multiplication-index lists of equal length. From $1 \leq 1$, but $2 \not\leq 1$ we conclude that $i = 1, 2 \not\leq j = 1, 1$ and therefore $--^{1,2} \oplus_r +^{1,1} = ?$.

6) The multiplication-index list i of the strong sign contains a sublist of length 1 with negative indices. This negative index is compensated for by the only positive index, leaving a sublist of length 0 to compare with the multiplication-index list of the weak sign. We can therefore not guarantee that the strong sign is actually stronger than the weak sign and we conclude that $++^{-2,1} \oplus_r -^2 = \ ?$.

7) Recall that multiplication-index lists of the form $-n, n$ cannot be simplified. From the same argument as given above, we conclude that if the rich enhanced $\oplus_r$-operator is applied to a sign with such an multiplication-index list and to a conflicting weak sign, then the result is always ambiguous. Therefore, $--^{-2,2} \oplus_r +^3 = \ ?$. Note that replacing the sign $--^{-2,2}$ by the equivalent sign $-^0$, does not change the result.

8) The multiplication-index list i of the strong sign again contains a sublist with negative indices; its length is 2. After discarding these indices together with the 1 to compensate for the -2 and a 3 to compensate for the -4, a positive sublist of length 1 remains. This sublist is compared to the weak sign's multiplication-index list, which has the same length. Since $3 \leq 3$, we may now conclude that the strong sign is guaranteed to be at least as strong as the weak sign and therefore $++^{-4,-2,1,3,3} \oplus_r -^3 = ++^{-4,-2,1,3,3,-3}$, which is simplified to $++^{-4,-2,1,3}$. □

Again we observe that whereas the strength of an influence resulting from the combination of a strong and a weak influence with the enhanced $\oplus_e$-operator from the previous section is summarised to be unknown, we now retain more information by representing the powers of all α terms in the list of multiplication indices.

4.5 Algebraic properties

We conclude our discussion of the enhanced operators with reviewing some basic algebraic properties. The addition of multiplication-index lists has not changed the $\otimes$-operator's commutative or associative properties. With respect to the basic enhanced $\oplus_e$-operator and the rich enhanced $\oplus_r$-operator we observe that, although the operators for combining signs of parallel influences are still commutative, they are no longer associative. The latter is illustrated by the following example:

$$\begin{array}{ccccccccc} (++^i \oplus_v +^i) & \oplus_v & -^i & = & ++^i \oplus_v -^i & = & \begin{cases} +^0 & \text{if } v = e \\ ++^{i,-i} & \text{if } v = r \end{cases} \\ ++^i & \oplus_v & (+^i \oplus_v -^i) & = & ++^i \oplus_v \; ? & = & ? \quad \text{if } v = e \text{ or } v = r \end{array}$$

We note that both combinations lead to correct results, regardless of the $\oplus$-operator used, the first just being more informative than the second. Heuristics, such as, for example, separately adding all positive and all negative signs before combining them, can be designed to prevent unnecessary ambiguous results due to order of combination.

In addition, we observe that the enhanced $\otimes$-operator distributes over neither the $\oplus_e$-operator nor the $\oplus_r$-operator. Compare, for example, the following:

$$\begin{array}{ccccccc} (++^i \oplus_v +^i) & \otimes & -^i & = & ++^i \otimes -^i & = & -^i \quad \text{if } v = e \text{ or } v = r \\ (++^i \otimes -^i) & \oplus_v & (+^i \otimes -^i) & = & -^i \oplus_v -^{2i} & = & \begin{cases} -^0 & \text{if } v = e \\ -^{i,2i} & \text{if } v = r \end{cases} \end{array}$$

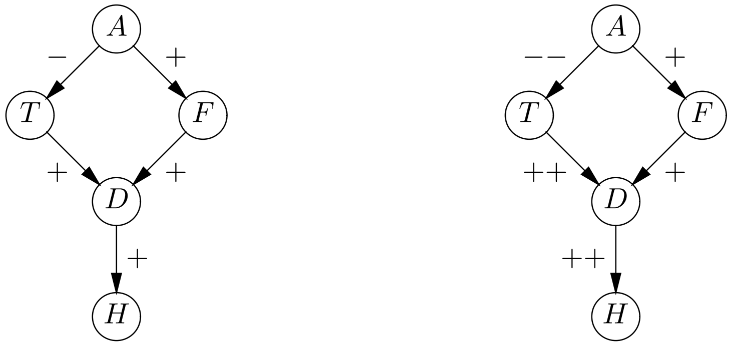
Again, we note that all these results are correct, but they strongly differ in level of informativeness with respect to the strength of the resulting sign.

5 Probabilistic inference revisited

In Section 3 we introduced the formalism of enhanced qualitative probabilistic networks. In Section 4, we enhanced the standard $\otimes$- and $\oplus$-operators for combining signs of influences upon inference and have addressed propagation of signs against the direction of arcs. Building upon the new, enhanced operators, the basic sign-propagation algorithm for probabilistic inference with a qualitative network is generalised straightforwardly to apply to enhanced networks: instead of the standard $\otimes$- and $\oplus$-operators, the enhanced operators are used for propagating and combining signs. In this section we illustrate the application of the resulting algorithm, for both versions of the enhanced $\oplus$-operator, by means of our running example which is reproduced in Fig. 9. In addition, we discuss some complexity issues concerning the different versions of the sign-propagation algorithm.

5.1 Inference using the enhanced operators

The idea behind the sign-propagation algorithm is basically to establish the net influence between an observed variable and all other variables in a qualitative probabilistic network, and multiply the sign of this net influence with the sign of the observation to return the effect of the observation on all variables. For ease of implementation, the algorithm starts by sending the sign of observation, a '$+$' or a '$-$', to the observed variable, thereby already incorporating the effect of the observation in all messages that are subsequently sent. Due to the algebraic properties of the basic $\otimes$- and $\oplus$-operators, the actual implementation does not affect the results.



(a) (b)

Figure 9: The (a) qualitative *Antibiotics* network and (b) its enhanced version.

In an enhanced qualitative network, the enhanced operators doe not adhere to the algebraic properties that ensure that the order in which signs are combined does not affect the result of their combination. As a consequence, multiplying the sign of a net influence with the sign of the observation may lead to a different result that that obtained by directly incorporating the sign of observation in the messages sent by the observed variable. To disturb the computation of the signs of net influences as least as possible, we propose entering an observation using an “identity” sign with respect to strength. More specifically, we require a sign s such that for arbitrary sign t, the result t^* of $s \otimes t$ is exactly as strong as t. We note from Table 2 that the signs $++^0$ and $--^0$ are suitable for this purpose, since they basically represent 1 and -1, respectively. We now present an example that illustrates sign-propagation with the enhanced $\otimes$- and $\oplus$-operators.

Example 5.1 We consider once again the qualitative *Antibiotics* network, which is reproduced in Fig. 9(a). Recall that entering the sign ‘$+$’ for variable A results upon inference with the basic sign-propagation algorithm in the ambiguous sign $- \oplus + =$ ‘?’ for variable D, which in turn causes an ambiguous sign for variable H.

Now, consider the enhanced *Antibiotics* network reproduced in Fig. 9(b); the signs specified are taken to hold in the direction of the corresponding arcs. We recall that initially all influences associated with the arcs in the network’s digraph have signs with a multiplication-index of 1. We once again apply the sign-propagation algorithm, now using our enhanced operators.

We enter the sign $++^0$ for variable A, reflecting a positive observation for A. Variable A propagates a message with sign $++^0 \otimes --^1 = --^1$ towards variable T. Variable T updates its node-sign to $--^1$ and sends a message with sign $--^1 \otimes ++^1 = --^2$ to variable D. Variable D updates its node-sign to $--^2$ and only sends a message with sign $--^2 \otimes ++^1 = --^3$ to variable H. Variable H updates its sign accordingly and sends no messages.

Variable A also sends a message, with sign $++^0 \otimes +^1 = +^1$, to variable F. Variable F updates its sign and passes a message with sign $+^1 \otimes +^1 = +^2$ to variable D. Variable D receives the additional sign $+^2$. Variable D will now combine the signs it has received from the two parallel trails originating in A. The result of this combination depends on the enhanced operator used. More specifically, if the sign-propagation algorithm employs the enhanced $\oplus_e$-operator then variable D updates its sign to $--^2 \oplus_e +^2 = -^0$, and then computes a message with sign $-^0 \otimes ++^1 = -^0$ for variable H. On the other hand, if the rich enhanced $\oplus_r$-operator is applied, then variable D updates its sign to $--^2 \oplus_r +^2 = --^{2,-2}$, and computes for variable H a message with sign $--^{2,-2} \otimes ++^1 = --^{3,-3}$. Variable H, however, does not need a sign update as its current sign is already correct, regardless of the operator used ($--^3 \oplus_e -^0 = --^3$ and $--^3 \oplus_r --^{3,-3} = --^3$). The variables D and H therefore send no further messages and the algorithm halts. □

From the above example, it seems at first glance that the results from sign-propagation with the rich enhanced $\oplus_r$-operator are similar to the results from using the enhanced $\oplus_e$-operator, with only the node-sign for node D differing. This illusion, however, is caused by the specific example network used. With the $\oplus_r$-operator, the node-sign of variable D is of the form $--^{i,-i}$ due to the fact that the two trails with conflicting influences have the same length; recall that this node-sign captures the same information as the negative sign $-^0$ of unknown strength returned by the $\oplus_e$-operator, and therefore the results of inference hardly differ for the two operators. If the conflicting trails have different lengths, however, then the difference between the two $\oplus$-operators becomes more important: for those situations in which the algorithm using the enhanced $\oplus_e$-operator leads to a node-sign $-^0$, the algorithm using the rich enhanced $\oplus_r$-operator will result in a node-sign $--^{i,-j}$, with $i \neq j$; this latter sign captures more information than the ambiguous negative sign and may aid in resolving even more trade-offs. Note that contrary to purely ambiguous signs, signs of unknown strength do not necessarily spread throughout a network once they occur.

We conclude that, while in the basic framework of qualitative networks trade-offs cannot be resolved upon inference and result in an ambiguous net influence, enhanced qualitative probabilistic networks allow for resolving at least some trade-offs.

5.2 Complexity of probabilistic inference

For quantitative probabilistic networks, in general, exact computation of probabilities is NP-hard [5]. The algorithms for probabilistic inference in a probabilistic network, however, are known to behave polynomially under certain restrictions concerning the topology of the network's digraph. In general, the sparser the digraph, the better most algorithms perform.

The basic sign-propagation algorithm for inference in a basic qualitative network has a worst-case runtime complexity that is polynomial in the number of nodes of the network's digraph, regardless of the digraph's topology. In a singly connected digraph, each pair of nodes is connected by a single simple trail. Upon sign-propagation, therefore, each variable A is visited at most once to receive the single sign which is the sign-product of the sign of observation and the signs associated with the arcs on the trail between A and the observed variable. In a multiply connected graph, two nodes can be connected by more than one simple trail. As a result, a variable should be visited as many times as the number of active simple trails between that variable and the observed variable to receive the sign of influence along each of those trails. To limit this possibly exponential number of visits to a variable, the basic propagation algorithm exploits the fact that node signs can only change twice; a variable needs therefore only be visited if it requires a change in node-sign. The fact that a node-sign can only change twice is apparent from the diagram in Fig. 10. The diagram displays the possible transitions from one node-sign to another using the basic $\oplus$-operator from Table 1. We observe that node-signs remain unchanged after at most two updates. As variables therefore need to be visited at most twice, and each visited variable inspects and constructs a message for at most all other variables, we have that the basic propagation algorithm halts after a number of operations that is polynomial in the number of nodes in the network's digraph.

The basic formalism of qualitative probabilistic networks does not allow for resolving trade-offs, as combining two conflicting influences with the basic $\oplus$-operator immediately results in an ambiguous node-sign. From the example in the previous section, we have that the enhanced $\oplus$-operators do provide for resolving some trade-offs, using the additional information carried by the enhanced signs and their multiplication-index lists. The possibility of resolving trade-offs, however, comes at the expense of efficiency of sign-propagation. This is not surprising, since qualitative trade-off resolution is also known to be NP-hard [17]. The main difference between sign-propagation in a basic qualitative probabilistic network and sign-propagation in an enhanced network is that, in multiply connected digraphs, the limit of two visits to each variable no longer applies. Although a variable's basic enhanced node-sign can change at most three times, the multiplication indices associated with the sign may need to change each time the variable is inspected. Therefore, the runtime of the algorithm on an enhanced network using the rich enhanced $\oplus_r$-operator is of the order of the number of active simple trails emanating from the observed variable in the network's digraph; for dense graphs this can be exponential in the number of nodes.

The diagram in Fig. 11 shows the possible transitions from one node-sign to another using the rich enhanced $\oplus_r$-operator from Table 4. In the diagram we observe several self-loops indicating that, although the basic enhanced node-sign remains the same, its multiplication-index list i needs to be updated with the multiplication-index list j of the sign added. As a result of these self-loops, in worst case, a variable's node-sign will need as many updates as the number of simple trails between that variable and the observed variable, which means that the variable is possibly visited an exponential number of times. The difference between using the enhanced $\oplus_e$-operator and the rich enhanced $\oplus_r$-operator is that the multiplication-index list of a sign cannot grow: the list for the resulting sign either equals the smallest of the lists of the two combined signs, or it equals the single index 0. The effect of this simplification is apparent from the diagram in Fig. 12 which displays the possible transitions of one node-sign to another when propagation is done using the enhanced $\oplus_e$-operator. From this diagram we have that one of the self-loops for the strong signs from Fig. 11 is replaced by a loop between the strong signs and signs with unknown strength ($i = 0$) in Fig. 12. In addition, upon adding two or more weak signs, the self-loop among the weak signs is invoked only once. Using the enhanced $\oplus_e$-operator may therefore turn out to be more efficient in practice, than using the rich enhanced $\oplus_r$-operator. We illustrate this observation with an example.

Example 5.2 Consider an enhanced qualitative network and one of its variables A. Suppose that sign-propagation has, at some stage, resulted in the node sign $++^3$ for A. In addition, assume that variable A is connected to the observed variable by another nine simple trails and that no messages travelling these trails have yet reached A. We number the trails, and variable A's neighbours on these trails, 1 through 9, and assume that variable A is inspected by its neighbours in that order. Now suppose that the signs of the messages computed by neighbours 1 through 9 for variable A are, respectively, 1)

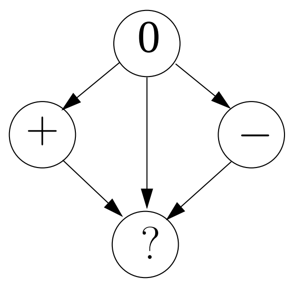


Figure 10: Possible node-sign updates during sign-propagation in a basic qualitative probabilistic network.

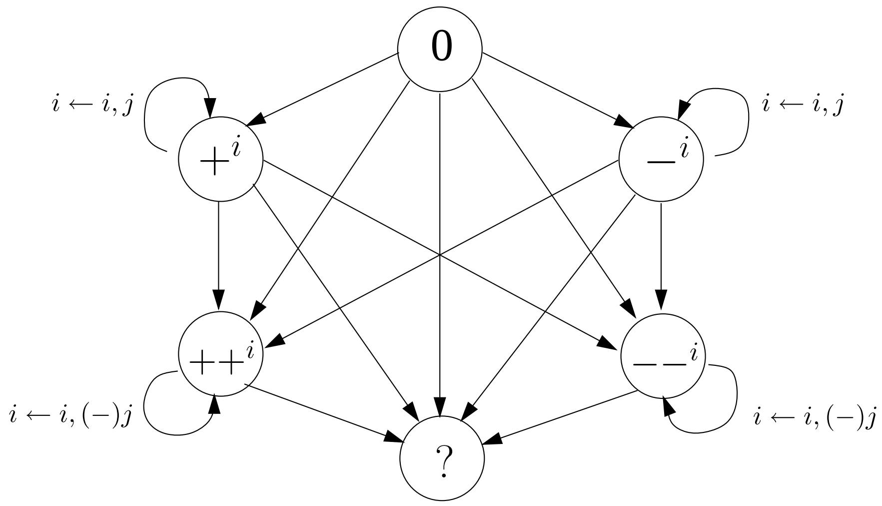


Figure 11: Possible node-sign updates during sign-propagation, using the rich enhanced $\oplus_r$-operator, in an enhanced qualitative network.

$++^2$, 2) $-^4$, 3) $++^4$, 4) $++^2$, 5) $-^3$, 6) $+^3$, 7) $++^3$, 8) $-^2$, and 9) $-^4$.

When the sign-propagation algorithm uses the enhanced $\oplus_e$-operator, the following will now happen at variable A:

1. Variable A has node-sign $++^3$ and neighbour 1) has computed the message $++^2$ for A. Since $++^3 \oplus_e ++^2 = ++^2$, variable A needs to update its node-sign and thus the message from neighbour 1 is sent to A;

2. variable A now has node-sign $++^2$. As $++^2 \oplus_e -^4 = +^0$, the message from neighbour 2 is also sent to variable A, which then updates its node-sign to $+^0$;

3. since $+^0 \oplus_e ++^4 = ++^4$, A also receives a message from neighbour 3 and updates its node-sign to $++^4$;

4. as $++^4 \oplus_e ++^2 = ++^2$, A receives a message from neighbour 4 as well, and updates its node-sign accordingly;

5. since $++^2 \oplus_e -^3 = +^0$, A receives a message from neighbour 5 and updates its node-sign;

6. now, since $+^0 \oplus_e +^3 = +^0$, variable A does not need to update its node-sign and therefore neighbour 6 sends no message to A;

7. as $+^0 \oplus_e ++^3 = ++^3$, variable A does receive a message from neighbour 7 and updates its node-sign;

8. since $++^3 \oplus_e -^2 = ?$, A also receives a message from neighbour 8 and updates its node-sign. The ambiguous node-sign cannot be changed, hence, variable A will be visited no more;

9. as a consequence, neighbour 9 does not send a message to variable A.

When, on the other hand, the rich enhanced $\oplus_r$-operator is used, propagation will result in the following:

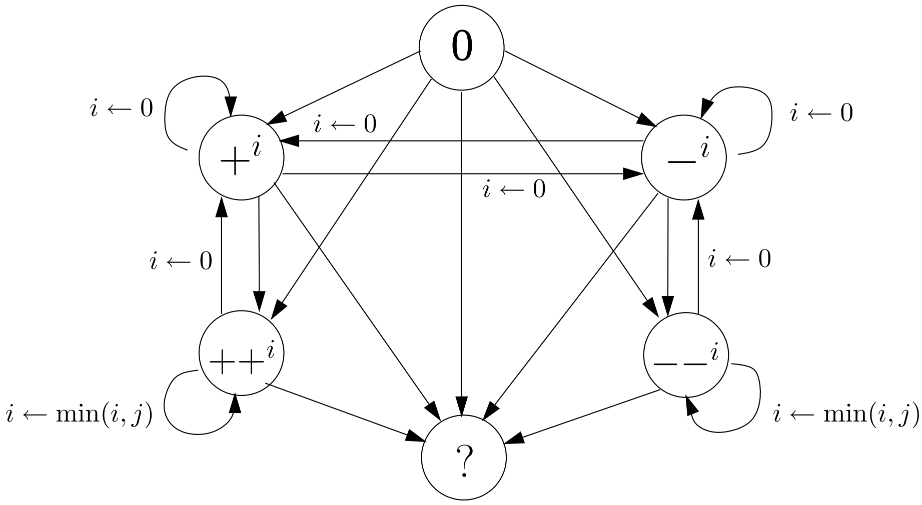


Figure 12: Possible node-sign updates during sign-propagation, using the enhanced $\oplus_e$-operator, in an enhanced qualitative network.

1. Since $++^3 \oplus_r ++^2 = ++^{2,3}$, variable A receives a message from neighbour 1 and updates its node-sign;
2. $++^{2,3} \oplus_r -^4 = ++^{2,3,-4}$, so A receives a message and updates its node-sign;
3. $++^{2,3,-4} \oplus_r ++^4 = ++^{2,3}$, A updates (and simplifies) its node-sign;
4. $++^{2,3} \oplus_r ++^2 = ++^{2,2,3}$, A updates its node-sign;
5. $++^{2,2,3} \oplus_r -^3 = ++^{2,2}$, A updates (and simplifies) its node-sign;
6. since $++^{2,2} \oplus_r +^3 = ++^{2,2}$, variable A does not need to update its node-sign and therefore receives no message from neighbour 6;
7. $++^{2,2} \oplus_r ++^3 = ++^{2,2,3}$, variable A updates its node sign;
8. $++^{2,2,3} \oplus_r -^2 = ++^{2,3}$, variable A updates (and simplifies) its nod-sign;
9. $++^{2,3} \oplus_r -^4 = ++^{2,3,-4}$, A updates its node-sign.

We observe that for this specific pattern of messages, propagation using the enhanced $\oplus_e$-operator halts after fewer steps than propagation using the rich enhanced $\oplus_r$-operator. In addition, the signs required for the use of the $\oplus_r$-operator have to store more information. We also observe, however, that use of the rich enhanced $\oplus_r$-operator has led to a more informative answer for variable A. □

The higher level of detail provided by the lists of multiplication indices and the rich enhanced operator thus may introduce a higher computational cost of inference than when the basic enhanced operator for summing signs is used. In fact, using the enhanced $\oplus_e$-operator, propagation will take exponential time only if, for a certain variable, the signs of the messages sent to that variable in order for it to update its node-sign, obey a certain pattern. This may occur, for example, if the node-sign of the variable is $++^i$, it subsequently is updated with between zero and $i-1$ strongly positive signs (self-loop), followed by a weakly negative sign with a larger multiplication index (resulting in a $+^0$), followed by a strongly positive sign (resulting in a $++^i$), etc. The same process will occur with positive and negative signs switched. Such a pattern of node-sign updates will require at least the same amount of time when the $\oplus_r$-operator is used instead.

We conclude that there exists a trade-off between the amount of information present in inference results after sign-propagation and the complexity of the propagation algorithm. Inference using the basic sign-propagation algorithm has a runtime complexity that is polynomial in the number of nodes of a qualitative network's digraph, but always leads to ambiguous results when the network models a trade-off. Inference using the enhanced $\oplus$-operators may become exponential, but does enable the resolving of trade-offs.

6 Related work

The problem of trade-off resolution within the framework of qualitative networks has been addressed before by different researchers. In this section we briefly review this related work.

S. Parsons introduced the concept of categorical influence [19]. A categorical influence is a qualitative influence that serves either to increase a probability to 1 or to decrease a probability to 0, disregarding all other influences. For example, a positive categorical influence $S^{[++]}(A, B)$ of a variable A on a variable B is defined as $\Pr(c \mid ax) = 1$ for all relevant variables X. A categorical influence thus serves to resolve any trade-off in which it is involved, but can only capture deterministic relationships between nodes; in real-life applications few to none of such relationships will exist. In addition, this extension requires new signs for indicating the increase to 1 and decrease to 0, respectively, and the incorporation of these new signs in the tables for the $\otimes$- and $\oplus$-operators.

Parsons also studied the use of both relative and absolute order-of-magnitude reasoning in the context of qualitative probabilistic networks [19]. Relative orders of magnitude can be used to relate different qualitative influences to each other. Using the relative order-of-magnitude system ROM[K] [6], one qualitative influence can be specified as being, respectively, *negligible* with respect to, *distant* from, *comparable* to, or *close* to another influence. The use of relative orders of magnitude thus serves to relate the strengths of different influences, but it requires the specification of a relation between all pairs of influences, instead of a notion of strength per influence. In addition, the relations used seem to be ill-defined, which makes reasoning with them anything but intuitive. For absolute order-of-magnitude reasoning, Parsons proposes a method that revolves around the propagation of abstract intervals between -1 and 1, that correspond to labels like 'Strongly Positive', 'Weakly Positive', etc. Two different sets of labels are required: one for modelling influences that are associated with the arcs in the network's digraph, and one for modelling changes that occur at the nodes in the graph (comparable to 'node-signs'). The intervals corresponding to a set of labels do not overlap and together span the interval $[-1, 1]$. The boundaries of the intervals, however, are not actually quantified, but set to be α, β, etc.; this approach is therefore comparable to our treatment of the cut-off value. Probabilistic inference is based on propagating and combining the abstract intervals; the interval comparisons required to this end are done using $\geq_{int}$, where $[\alpha, \beta] \geq_{int} [\gamma, \delta]$ iff $\alpha \geq \gamma$ and $\beta \geq \delta$. Note that if one interval is considered larger than another with this operator, then they may in fact overlap. To prevent considerable loss of information, assumptions about the actual values of the interval boundaries have to be made.

κ-calculus [24] can be considered another absolute order-of-magnitude system. Using a probabilistic interpretation of the κ-calculus, probabilities can be abstracted to κ-values, where a κ-value of n indicates that the associated probability has the same order of magnitude as ϵ^n for some infinitesimal number ϵ [7]. Drawbacks of the use of these κ-values are that they are not concerned with *changes* in probabilities, but rather with the probabilities themselves, and that the probabilistic interpretation is suitable for infinitesimal probabilities only. The former problem can be lifted as an absolute change in probability is a number between zero and one and can therefore be abstracted to a κ-value. This principle has more recently been used in another approach to enhance the expressiveness of qualitative probabilistic networks [23]. With this approach, an interval of κ-values is associated with the sign of an influence to capture its possible strengths. These κ-intervals are propagated along with the qualitative network's signs. Propagation results, however, are only guaranteed to be correct for infinitesimal probabilities. Finally, the definition of a κ-value is not very intuitive and such values are therefore hard for domain experts to specify and interpret.

Categorical influences, order-of-magnitude reasoning and κ-calculus are of a purely qualitative nature, yet serve for resolving some trade-offs. C.-L. Liu and M.P. Wellman designed methods for resolving trade-offs based upon the idea of reverting to numerical probabilities whenever necessary [17]. They propose to reason with a probabilistic network in a qualitative way, thereby exploiting the efficiency of sign-propagation, and only reverting to the full quantification whenever a trade-off leads to an ambiguous result. Two methods are described for resolving the trade-off. The first method provides for incrementally applying numeric inference to the point where qualitative reasoning can produce a decisive result. That is, a trade-off between two variables is resolved numerically and then abstracted into a net qualitative influence between the two variables. The second method amounts to estimating bounds on the net influence along the trails that give rise to a trade-off. These bounds are then again used to compute the qualitative sign of the net influence. The methods presented by Liu and Wellman resolve any trade-off present in the network, but require a fully specified, numerical probabilistic network.

We would like to mention that some other approaches to dealing with uncertainty in a qualitative way have been proposed in the literature. As these approaches are not tailored for use within the framework of qualitative probabilistic networks, we do not review them here.

7 Conclusions and further research

Qualitative probabilistic networks have been designed to overcome, to at least some extent, the quantification problem known to probabilistic networks. Qualitative networks in essence are qualitative abstractions of their quantitative counterparts: while in a probabilistic network relationships between variables are quantified by probabilities, these relationships are expressed by qualitative signs in qualitative probabilistic networks. As a result of their coarse level of representation detail, qualitative networks lack the expressive power that allows for resolving trade-offs the way probabilistic networks do. To provide for trade-off resolution we have therefore enhanced the formalism of qualitative probabilistic networks. To this end, we have distinguished between strong and weak influences. We have further enhanced the multiplication and addition operators to guarantee the transitivity and parallel-composition properties of influences. To handle the asymmetry of an influence's strength we have proposed specifying two influences for each arc. With these enhancements we have generalised the basic sign-propagation algorithm to apply to enhanced qualitative networks. We have shown that our formalism provides for resolving at least some trade-offs in a qualitative way, that is, without having to resort to numerical computation.

To distinguish between weak and strong influences, we have introduced additional signs and augmented all signs with multiplication-index lists. As it is hard to interpret the meaning of such lists of indices, it is not our intention to output the augmented signs. The multiplication indices are merely used internally for trade-off resolution. The output of inference, as in a basic qualitative network, is a basic sign for each variable that indicates whether the net influence of an observation on that variable is positive, negative, zero or ambiguous. If desirable, an additional level of strength can be added by introducing, for example, '$+++$' and '$---$' signs using an additional cut-off value. Alternatively, signs with a multiplication index other than 1 could be allowed on the arcs of the enhanced network's digraph. Both options, however, would require domain experts to be able to distinguish between more than two levels of strength and the first option would, in addition, render the necessary $\otimes$- and $\oplus$-operators more complex.

When the sign-propagation algorithm is used with the enhanced $\oplus$-operators, it becomes less efficient than the basic sign propagation algorithm. In fact, inference may then in theory become infeasible. Further research will be necessary to determine the actual complexity of sign-propagation with the enhanced operators in real-life qualitative networks. Two approaches can, however, be used to bound the complexity of inference. The first approach amounts to posing a limit on the number of sign-additions performed for a single variable. If this limit is reached, the node-sign of the variable is changed into a basic sign ('$+$', '$-$', '0', or '$?$') and the basic sign-propagation algorithm is used for further propagation. Note that this approach may lead to weaker, but correct, results. The other approach, especially suitable when using the $\oplus_r$-operator, is to use enhanced signs only in small parts of the network, that is, in those parts where trade-offs reside. In constructing the enhanced network, we then focus on the multiply connected parts of the network's digraph and ask the domain experts whether the possible parallel trails between variables consist of conflicting influences. If so, enhanced signs are elicited for the influences on these trails. During inference, the trade-off can be locally resolved using the enhanced sign-propagation algorithm, and the basic sign for the net influence is then used for further propagation with the basic sign-propagation algorithm. Another advantage of such local computation with enhanced signs is that it requires only local specification of such signs. As a consequence, during the elicitation of signs, domain experts then only have to compare differences in strengths for small sets of influences. As correctly specifying strengths will be harder for experts than correctly specifying the basic sign for an influence, local specification of enhanced signs will make the resulting signs less prone to error. Local specification also allows for different interpretations of strong and weak influences for different parts of the network, that is, it allows for different cut-off values to be (implicitly) used in different parts of the network.

Qualitative probabilistic networks can play an important role in the construction of quantitative probabilistic networks for real-life application domains. As the assessment of the various probabilities required is a hard task, it is performed only when the probabilistic network's graph is considered robust. Now, by assessing signs for the influences modelled in the graph, a qualitative network is obtained that can be exploited for studying the projected probabilistic network's reasoning behaviour prior to the assessment of probabilities. The qualitative signs can in addition be used in several ways as constraints on the quantification. For example, by interpreting the signs as continuous subintervals of the interval $[-1, 1]$, the constraints they impose on the conditional probability distributions involved can be used for stepwise quantification of a probabilistic network: once a conditional probability table for a certain variable is filled, the interval associated with all direct influences upon that variable can be decreased. An interval-propagation algorithm, very similar to the sign-propagation algorithm then serves to study the behaviour of the partially quantified network [22]. Patterns of qualitative influences can also be used to recognise different types of causal interaction, such as the noisy-or, which greatly simplify the quantification effort [18]. At a somewhat higher level, the constraints imposed by qualitative influences can be used to bound the entire space of possible joint probability distributions over the network's variables [11]. Finally, the qualitative signs can be used for explanation of the (qualitative) probabilistic network's reasoning processes [8]. We therefore argue that it is important to derive as much information as possible from a qualitative network. The formalism of enhanced qualitative networks provides for a step into making qualitative networks more applicable.

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