Set Induced Relations and Relational Semantics

Paul Harrenstein

institute of information and computing sciences,
utrecht university

technical report UU-CS-2004-032

www.cs.uu.nl

Set Induced Relations and Relational Semantics

Paul Harrenstein

Contents

Chapter 1

Introduction

The languages of classical propositional logic (CPC) have a neat interpretation by taking as models the truth-value assignments to the propositional variables, or *valuations*, of the language. Each formula φ of a propositional language can be associated with a set of valuations, *viz.*, the formula's extension, denoted by $[\![\varphi]\!]$. Moreover, the extension of a formula allows for a neat compositional definition. We conceive of classical consequence as a relation that intuitively holds between two sets of formulas (or *theories*[1]) whenever at least one formula of the latter set is true, if all formulas of the former are. Assuming this relation independently be given by $\vdash$, the following soundness and completeness result is obtained:

$$\Gamma \vdash \Theta \quad \text{iff} \quad \bigcap_{\gamma \in \Gamma} [\![\gamma]\!] \subseteq \bigcup_{\vartheta \in \Theta} [\![\vartheta]\!].$$

Each theory Γ may be said to impose a set theoretic structure on the set of valuations, *viz.*, the set of extensions of its formulas $\left\{[\![\gamma]\!] : \gamma \in \Gamma\right\}$. Observe that by taking the intersection or the union of this set much of this structure may be lost: although two sets $\left\{[\![\gamma]\!] : \gamma \in \Gamma\right\}$ and $\left\{[\![\gamma]\!] : \gamma \in \Gamma'\right\}$ may be distinct in many ways, their intersections or unions may very well be identical.

In this report, we propose an interpretation of a theory Γ as a relation over valuations. This makes that relation theoretic operations — such as composition, inverse and the like — become available for the semantical analysis of propositional logic. The way a theory induces this relation over the valuations entertains a close relation with the notion of (relative) logical strength, which constitutes

[1]Throughout this report by a *theory* we understand a mere set of formulas that is not necessarily closed under logical consequence. A theory that is closed under a notion of logical consequence Λ we will refer to as a Λ-*closed theory*, or a *closed theory* if Λ is clear from the context.

a natural starting point for semantical analyses of such non-standard phenomena as, *e.g.*, belief revision, paraconsistency, and and non-monotonic and default reasoning. Veltman's update semantics for default reasoning (Veltman (1996)) is a particularly apt example. Relations in a similar way induced by formulas and theories played an important role in the definition of the distributed evaluation games that formed the basis of the game-theoretical concept of consequence advanced in Harrenstein (2004, to appear).[2]

The relative logical strength of two formulas can be captured in terms of the extensions of the formulas involved. A formula φ is said to be at least as strong as another formula ψ whenever any formula that follows from ψ is also a consequence of φ. In terms of sets of valuations, φ is then at least as strong as ψ if and only if the extension of φ is included in that of ψ. This definition of relative logical strength can straightforwardly be extrapolated as to hold between theories. A theory Θ is said to be logically at least as strong as another theory Γ if the consequences of Γ are contained in those of Θ. In this manner, relative logical strength induces a reflexive and transitive relation, *i.e.*, a preorder, on the subtheories of a theory.

This ordering on the formulas in a theory Γ in turn engenders an ordering over the valuations as follows. Let Γ_s be the subtheory of Γ containing exactly those formulas from Γ satisfied by s. Then, Γ_s is easily recognized as the (unique) logically strongest subtheory of Γ satisfied by s. On this basis, we might define a valuation s to be at least as strong as a valuation s' with respect to Γ if and only if Γ_s is logically at least as strong as $\Gamma_{s'}$. This ordering on the valuations is reflexive and transitive.

For an example, let Γ be the theory $\{a \vee b, \neg a, \neg a \wedge \neg b\}$ and consider the valuations $\{a\}$ and $\{b\}$. Then $\{a \vee b, \neg a\}$ is the strongest subtheory of Γ the valuation $\{b\}$ satisfies. The valuation $\{a\}$, on the other hand, satisfies no subtheory stronger than $\{a \vee b\}$. Because $\{a \vee b, \neg a\}$ is logically stronger than $\{a \vee b\}$, the valuation $\{b\}$ is ranked higher with respect to Γ than the valuation $\{a\}$. For a similar reason, the valuations $\{a\}$ and $\emptyset$ are incomparable with respect to Γ. Also consider Figure 1.1 for a pictorial illustration of these considerations.

In the second section of Chapter 2 we show how each subsets of a set S define relations over S and similarly how each set of subsets of S does the same. The relations induced by formulas over the valuations are then defined as the relations induced thus by the extensions of the formulas, the latter being subsets of valuations. For each subset X of a set S, a relation $\rho_0(X)$ is defined such that for all

[2]The material of this report largely derives from Harrenstein (2004). An exception is Chapter 6, which was not included in my thesis.

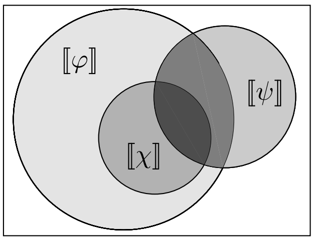

Figure 1.1. The extensions of three formulas φ, ψ and χ in logical space. The ordering on the valuations determined by the theory $\{\varphi, \psi, \chi\}$ on the basis of relative logical strength is indicated by the different shades of grey. The darker the area a valuation is in the higher that valuation is in the ordering on valuations determined by the three formulas, on the understanding that the valuations in $[\![\varphi]\!] - [\![\psi]\!]$ and those in $[\![\psi]\!] - [\![\varphi]\!]$ are incomparable. The valuations in the darkest area satisfy all of φ, ψ and χ and are ranked highest; those in the lighter areas satisfy no so strong a subtheory of $\{\varphi, \psi, \chi\}$ and are consequently ranked lower.

elements x and x' in S:

$$\big(x, x'\big) \in \rho_0(X) \quad \text{iff} \quad x \in X \text{ implies } x' \in X.$$

Let further for each set $\boldsymbol{X}$ of subsets of S, $\boldsymbol{\rho}_0\,(\boldsymbol{X})$ the relation $\boldsymbol{\rho}_0\,(\boldsymbol{X})$ be defined as $\bigcap_{X \in \boldsymbol{X}} \rho_0(X)$. As a special case, for each extension $[\![\varphi]\!]$ of a propositional formula φ, $\rho_0([\![\varphi]\!])$ defines a relation over the valuations of the respective language. In a similar fashion, each theory Γ gives rise to the relation $\bigcap_{\gamma \in \Gamma} \rho_0([\![\gamma]\!])$, which we also denote by $\boldsymbol{\rho}_0\,(\Gamma)$. We find that the relation $\boldsymbol{\rho}_0\,(\Gamma)$ coincides with the relation over the valuations based on the notion of relative logical strength of the formulas in Γ.

Defined thus, however, this relation has a considerable drawback: the relation $\rho_0(\varphi)$ does not allow for a neat compositional definition in φ. The malefactor is here the fact both theories containing merely contradictions and theories solely made up of tautologies induce the universal relation over the valuations. In order to circumvent this problem we define for each subset X of a set S the relation $\rho\,(X)$ such that $\rho\,(\emptyset)$ is the empty relation and $\rho\,(X)$ coincides with $\rho_0(X)$, whenever X is nonempty. For sets $\boldsymbol{X}$ of subsets of S, the relation $\boldsymbol{\rho}\,(\boldsymbol{X})$ is again defined as the intersection of the relations $\rho\,(X)$ for all $X \in \boldsymbol{X}$, *i.e.*, $\boldsymbol{\rho}\,(\boldsymbol{X})$ is defined as $\bigcap_{X \in \boldsymbol{X}} \rho\,(X)$. Obviously, for each formula φ and each theory Γ, both $\rho\,([\![\varphi]\!])$ and $\bigcap_{\gamma \in \Gamma} \rho\,([\![\gamma]\!])$ induces a relation over the valuations of the language in question. In Chapter 6 we prove that the relation $\rho\,([\![\varphi]\!])$ can be provided with a definition that is compositional in φ.

The remainder of this report concerns the formal properties of these relations induced by formulas, theories and sets. Apart from Chapter 3, which concerns Veltman's semantics for defaults, their application to the formal analysis of non-standard reasoning mechanisms decidedly falls beyond its scope. Chapter 4 concerns the characterization of the relations induced by sets, sets of subsets, formulas and theories. We find that the class of relations over a set S as induced by a subset X of S coincides with the class of so-called *bisective relations* over X (Proposition 4.3.2). Here, a relation is understood to be bisective whenever it is transitive and in addition satisfies the following condition:

$$\text{for all } x, x', x'' \in S\colon \quad x \leqslant x' \ \text{ implies } \ x'' \leqslant x \ \text{ or } \ x' \leqslant x''.$$

A relation that is either empty or both reflexive and transitive we will hereafter refer to as a *proto-order*. Corollary 6.1.9 establishes that the relations over a set S defined by a set of subsets is precisely the set of proto-orders over S.

Since for propositional languages with a *finite* number of propositional variables each subset of valuations is the extension of some formula, these results immediately settle the case for the relations defined by formulas and theories of such finite languages. For propositional languages with an *infinite* number of propositional variables, things are slightly different. It is then no longer the case that each subset of valuations is the extension of some formula or even of some theory. Similarly, the bisective relations over the valuations are not exhausted by the relations induced by formulas and neither are the proto-orders over the valuations by the relations induced by theories. A characterization of the relations induced by the formulas and theories of a propositional language with an infinite number of propositional variables, however, can be given relying on the apparatus provided by *rough set theory* (Corollary 4.3.5 and Theorem 4.3.10).

In classical logic a theory may be closed under its consequences without affecting its deductive properties. At a semantical level, this fact is reflected in that the extension of a theory Γ is identical to the extension of its closure under logical consequence, *i.e.*, in general $[\![\Gamma]\!] = [\![Cn(\Gamma)]\!]$. The relations over the valuations induced by theories, however, are more sensitive in this respect. In particular, it is not in general the case that the relations $\boldsymbol{\rho}(\Gamma)$ and $\boldsymbol{\rho}(Cn(\Gamma))$ are identical, where $Cn(\Gamma)$ stands for the set of classical consequences of the theory Γ.

At a set-theoretic level, a set of sets $\boldsymbol{X}$ cannot in general be closed under supersets without affecting the relation $\boldsymbol{\rho}(\boldsymbol{X})$. On the other hand, different sets of sets may very well induce the same relation on a universe, *i.e.*, $\boldsymbol{\rho}(\boldsymbol{X})$ and $\boldsymbol{\rho}(\boldsymbol{Y})$ may be identical even if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are distinct. Chapter 5 aims at making precise the conditions on sets of sets $\boldsymbol{X}$ and $\boldsymbol{Y}$ that have to be satisfied for the relations $\boldsymbol{\rho}(\boldsymbol{X})$

and $\boldsymbol{\rho}(\boldsymbol{Y})$ to be identical. Closure conditions on theories that preserve relations induced by theories can then be obtained as corollaries.

Chapter 2

Set Induced Relations and Relational Semantics

2.1 Set Induced Relations

In this chapter we advance a relational semantics for classical propositional logic. The set-theoretic basis for the semantics is provided by relations on a universe S that sets and sets of sets give rise to. With each subset X of a set S we associate a relation $\rho_0(X)$, which relates all elements outside X to any other element of S as well as all elements in X to one another. Intuitively, the objects in X are considered 'higher' than those outside X. Formally we define for each subset X of S and all elements x and x' of S:

$$\left(x, x'\right) \in \rho_0(X) \quad \text{iff} \quad x \in X \ \text{ implies } \ x' \in X$$

On this basis we also define for each set $\boldsymbol{X}$ of subsets of S a relation $\boldsymbol{\rho}_0\left(\boldsymbol{X}\right)$ on S:

$$\boldsymbol{\rho}_0\left(\boldsymbol{X}\right) \quad =_{df.} \quad \bigcap_{X \in \boldsymbol{X}} \rho_0(X)\,.$$

As can easily be checked, for each $X \subseteq S$, the relation $\rho_0(X)$ is a total pre-order, *i.e.*, it is reflexive, transitive and connected. For each set $\boldsymbol{X}$ of subsets of S, the relation $\boldsymbol{\rho}_0\left(\boldsymbol{X}\right)$ is a *partial* pre-order over S, *i.e.*, it is both reflexive and transitive but not necessarily connected. As auxiliary notions we have $\bar{\rho}_0\left(X\right)$ and $\bar{\boldsymbol{\rho}}_0\left(\boldsymbol{X}\right)$, defined as, respectively, $\rho_0\left(\overline{X}\right)$ and $\bigcap_{X \in \boldsymbol{X}} \bar{\rho}_0\left(X\right)$.

Of particular interest to us are the relations on the valuations induced by the extensions of formulas of a propositional language. Let φ be a formula and Γ a theory. We denote $\rho_0(\llbracket\varphi\rrbracket)$ by $\rho_0(\varphi)$ and $\bigcap_{\gamma\in\Gamma}\rho_0(\gamma)$ by $\boldsymbol{\rho}_0(\Gamma)$. Observe that, defined thus, $\boldsymbol{\rho}_0(\Gamma)$ *does not* in general coincide with $\rho_0(\llbracket\Gamma\rrbracket)$.

For Γ a theory, $\boldsymbol{\rho}_0(\Gamma)$ is exactly the relation the subtheories of Γ defines over the valuations relative to their respective logical strength. Recall that we understood a theory Γ to be logically stronger than another theory Θ in case the consequences of Θ were included in those of Γ, *i.e.*, if $Cn(\Theta) \subseteq Cn(\Gamma)$. To appreciate this, define for each theory Γ the relation $\boldsymbol{\rho}_1(\Gamma)$ over the valuations, such that for all valuations s and s':

$$(s, s') \in \boldsymbol{\rho}_1(\Gamma) \quad \text{iff} \quad Cn\big(\{\gamma\in\Gamma :\ s\Vdash\gamma\}\big) \subseteq Cn\big(\{\gamma\in\Gamma :\ s'\Vdash\gamma\}\big).$$

Recall $Cn(\Gamma)$ stands for the set of classical consequences of the theory Γ. Intuitively, $\boldsymbol{\rho}_1(\Gamma)$ relates a valuation s with another s' if the latter satisfies at least all those subtheories of Γ that s satisfies as well. As such, it is in effect the relation on the valuations based on the classical notion of logical strength that was proposed in the introduction to this report. We now have the following easy proposition.

Proposition 2.1.1 *Let Γ be a theory in a propositional language $L(A)$. Then the relations $\boldsymbol{\rho}_0(\Gamma)$ and $\boldsymbol{\rho}_1(\Gamma)$ coincide.*

Proof: First assume $(s, s') \in \boldsymbol{\rho}_0(\Gamma)$. Then, for all $\gamma\in\Gamma$, if $s\Vdash\gamma$ then $s'\Vdash\gamma$. Hence, $\{\gamma\in\Gamma :\ s\Vdash\gamma\} \subseteq \{\gamma\in\Gamma :\ s'\Vdash\gamma\}$. By monotonicity of *Cn*, immediately $Cn(\{\gamma\in\Gamma :\ s\Vdash\gamma\}) \subseteq Cn(\{\gamma\in\Gamma :\ s'\Vdash\gamma\})$, *i.e.*, $(s, s') \in \boldsymbol{\rho}_1(\Gamma)$. For the opposite direction, assume $Cn(\{\gamma\in\Gamma :\ s\Vdash\gamma\}) \subseteq Cn(\{\gamma\in\Gamma :\ s'\Vdash\gamma\})$ as well as for an arbitrary $\gamma\in\Gamma$ that $s\Vdash\gamma$. Then, $\gamma\in\{\gamma\in\Gamma :\ s\Vdash\gamma\}$ and by monotonicity of *Cn* also $\gamma\in Cn(\{\gamma\in\Gamma :\ s\Vdash\gamma\})$. By the assumption, $\gamma\in Cn(\{\gamma\in\Gamma :\ s'\Vdash\gamma\})$. Then $\{\gamma\in\Gamma :\ s'\Vdash\gamma\}\vdash_{\mathrm{CPC}}\gamma$, *i.e.*, for all valuations s'', if $s''\Vdash\varphi$ for all $\varphi\in\{\gamma\in\Gamma :\ s'\Vdash\gamma\}$, then $s''\Vdash\gamma$. Since trivially, $s'\Vdash\varphi$ for all $\varphi\in\{\gamma\in\Gamma :\ s'\Vdash\gamma\}$, in particular $s'\Vdash\gamma$. Therefore $(s, s') \in \boldsymbol{\rho}_0(\Gamma)$ and we are done. $\dashv$

For formulas φ, however, the relation $\rho_0(\varphi)$ does not have in general a neat compositional definition in the complexity of φ. To appreciate this, observe that both $\rho_0(\top)$ and $\rho_0(\bot)$ are the universal relation over the valuations. In contrast, for each propositional variable a, the relation $\rho_0(a)$ is not universal. Whenever a valuation s forces a but another valuation s' does not, the pair (s, s') will not be in $\rho_0(a)$. This is in particular the case for the valuations $\{a\}$ and $\varnothing$, which are guaranteed to exist for any language with a as a propositional variable. Now consider the formulas $\bot\wedge a$ and $\top\wedge a$. Since $\llbracket\top\wedge a\rrbracket = \llbracket a\rrbracket$, it should also be the case that the relations $\rho_0(\top\wedge a)$ and $\rho_0(a)$ coincide. So, with $\rho_0(a)$ not being the universal relation, neither is $\rho_0(\top \wedge a)$. However, $\rho_0(\bot \wedge a)$ is the universal relation on the valuations, in virtue of $\bot \wedge a$ and $\bot$ being logically equivalent, and as such having the same extension. Hence $\rho_0(\top \wedge a)$ is distinct from $\rho_0(\bot \wedge a)$. However, with $\rho_0(\top)$ and $\rho_0(\bot)$ being identical, this distinction cannot be made on the basis of the relations $\rho_0(\top)$, $\rho_0(\bot)$ and $\rho_0(a)$ alone.

The problem here of course is that $\rho_0(\varnothing)$ and $\rho_0(S)$ are the same relation. For any two non-empty subsets X and Y of S the reader can easily verify that $\rho_0(X)$ and $\rho_0(Y)$ coincide if and only if X and Y are identical (also compare Fact 2.1.2, below). By treating the empty set as a special case, many of the problems dissolve. So, define, for each subset X of a set S, the relation $\rho(X)$ on S as follows:[1]

$$\rho(X) \quad =_{df.} \quad \begin{cases} \left\{(x, x') : \; x \in X \text{ implies } x' \in X\right\} & \text{if } X \neq \varnothing, \\ \varnothing & \text{otherwise.} \end{cases}$$

Defined thus, $\rho(X)$ coincides with $\rho_0(X)$ for non-empty subsets X, and is empty otherwise (*cf.*, Fact 2.1.3 below). For A a set of propositional variables, let, $\mathscr{R}(A)$ denote the set $\left\{\rho(\varphi) : \; \varphi \text{ a formula in } L(A)\right\}$. Similarly, for each set $\boldsymbol{X}$ of subsets of S, we define the relation $\boldsymbol{\rho}(\boldsymbol{X})$ over S as:

$$\boldsymbol{\rho}(\boldsymbol{X}) \quad =_{df.} \quad \bigcap_{X \in \boldsymbol{X}} \rho(X).$$

Let further $\rho(\varphi)$ and $\boldsymbol{\rho}(\Gamma)$ denote $\rho(\llbracket\varphi\rrbracket)$ and $\bigcap_{\gamma\in\Gamma} \rho(\llbracket\gamma\rrbracket)$, respectively. Observe that $\boldsymbol{\rho}(\boldsymbol{X}) = \boldsymbol{\rho}_0(\boldsymbol{X})$ if and only if $\varnothing \notin \boldsymbol{X}$ (*cf.*, Fact 2.1.3). For $\varnothing \in \boldsymbol{X}$, the relation $\boldsymbol{\rho}(\boldsymbol{X})$ is empty. As a dual notion we also introduce for each subset X of S the relation $\bar{\rho}(X)$ on S defined as $\rho\left(\overline{X}\right)$. Also, for each set $\boldsymbol{X}$ of subsets of S, let $\bar{\rho}(\boldsymbol{X})$ denote the relation on S given by $\bigcap_{X\in\boldsymbol{X}} \bar{\rho}(X)$. We have $\bar{\rho}(\varphi)$ and $\bar{\boldsymbol{\rho}}(\Gamma)$ abbreviate $\bar{\rho}(\llbracket\varphi\rrbracket)$ and $\bigcap_{\gamma\in\Gamma} \bar{\rho}(\llbracket\gamma\rrbracket)$. In contradistinction to $\rho_0(\varphi)$, the relation $\rho(\varphi)$ does allow for a compositional definition in φ. This we show in Chapter 6 below.

We first review some of the more elementary properties of the relations $\rho(X)$ and $\boldsymbol{\rho}(\boldsymbol{X})$. As a first fact we find that no two different subsets X and Y of a set S such that $\rho(X)$ and $\rho(Y)$ are identical relations on S.

[1]It might seem that the identity relation *Id* would have been an equally suitable choice for $\rho(\varnothing)$, as there is no subset X of S such that $\rho_0(X) = Id$. Had the definition been chosen thus, however, an exception should be made in Proposition 2.2.2 below for propositional languages with no propositional variables. For such languages there is only one valuation, *viz.*, $\varnothing$, and again $\rho(\varnothing)$ would coincide with the universal relation over all valuations. Then it would have been the case that $max(\rho(\bot)) = \{\varnothing\}$ and $\overline{max(\bar{\rho}(\top))} = \varnothing$. Hence, $max(\rho(\bot)) \nsubseteq \overline{max(\bar{\rho}(\top))}$. In classical logic, however, $\bot \vdash \top$, even for languages lacking in propositional variables.

Fact 2.1.2 *Let X and Y be subsets of some set S. Then:*

$$\rho(X) = \rho(Y) \quad \textit{iff} \quad X = Y.$$

Proof: The right-to-left direction is trivial. For the opposite direction suppose $X \neq Y$. Without loss of generality we may assume there be some $x \in X$ for which $x \notin Y$. In case Y is empty, $(x, x) \notin \rho(Y)$ but $(x, x) \in \rho(X)$ and *a fortiori* $\rho(X) \neq \rho(Y)$. If on the other hand Y is not empty there is some $y \in Y$. Then, $(y, x) \notin \rho(Y)$ and $(y, x) \in \rho(X)$. Again we may conclude that $\rho(X) \neq \rho(Y)$. ⊣

By contrast, $\rho_0(X)$ and $\rho_0(Y)$ may be identical even for distinct X and Y, be it only if either X or Y is the universe and the other the empty set. Both $\rho_0(X)$ and $\rho_0(Y)$ are then the universal relation. For X any subset other than the empty set, the relations $\rho_0(X)$ and $\rho(X)$ coincide. This observation also sustains a corresponding result for relations $\boldsymbol{\rho}(\boldsymbol{X})$ induced by sets of subsets $\boldsymbol{X}$.

Fact 2.1.3 *Let X be a subset of some non-empty set S. Then:*

$$\rho_0(X) = \rho(X) \quad \textit{iff} \quad X \neq \varnothing \qquad \textit{and} \qquad \boldsymbol{\rho}_0(\boldsymbol{X}) = \boldsymbol{\rho}(\boldsymbol{X}) \quad \textit{iff} \quad \varnothing \notin \boldsymbol{X}.$$

Proof: For the first claim, the proof from right to left is trivial. So assume $X = \varnothing$. Then, $\rho(X) = \varnothing$ and $\rho_0(X) = S \times S$. Since S had been assumed to be non-empty, also $\rho_0(X) \neq \rho(X)$. For the second claim merely observe the following equalities:

$$\boldsymbol{\rho}_0(\boldsymbol{X}) \;=\; \bigcap_{X \in \boldsymbol{X}} \rho_0(X) \;=_{\varnothing \notin \boldsymbol{X}}\; \bigcap_{X \in \boldsymbol{X}} \rho(X) \;=\; \boldsymbol{\rho}(\boldsymbol{X}).$$ ⊣

We also have the following equally easy fact.

Fact 2.1.4 *Let* $\boldsymbol{X}$ *be a set of subsets of a non-empty set S. Then:*

$$\boldsymbol{\rho}(\boldsymbol{X}) = \varnothing \quad \textit{iff} \quad \varnothing \in \boldsymbol{X}.$$

Proof: Straightforward. From right to left the proof is almost trivial. Merely observe that then $\rho(\varnothing) \in \{\rho(X) : X \in \boldsymbol{X}\}$ and, since $\rho(\varnothing) = \varnothing$, $\boldsymbol{\rho}(\boldsymbol{X}) = \bigcap_{X \in \boldsymbol{X}} \rho(X) = \varnothing$. For the opposite direction assume that $\varnothing \notin \boldsymbol{X}$. In virtue of Fact 2.1.3, then $\boldsymbol{\rho}(\boldsymbol{X}) = \boldsymbol{\rho}_0(\boldsymbol{X})$. With the latter begin reflexive, it follows that $\boldsymbol{\rho}(\boldsymbol{X})$ is reflexive as well. Having assumed S to be non-empty, we may conclude that $\boldsymbol{\rho}(\boldsymbol{X})$ is non-empty. ⊣

For any subset X, the relation $\rho(X)$ is *not* in general monotone in X. To appreciate this, let X and Y be two non-empty proper subsets of a set S such that Y is also a proper subset of X. Assume that $y \in Y$, $x \in X - Y$ and $z \notin X$. Then, $(y, x) \in \rho(X)$ but $(y, x) \notin \rho(Y)$. Moreover, $(x, z) \in \rho(Y)$ but $(x, z) \notin \rho(X)$. More in general we have the following fact.

Fact 2.1.5 *Let X and Y be* distinct *subsets of some set S. Then:*

$$\rho(X) \subseteq \rho(Y) \quad \textit{iff} \quad X = \emptyset \;\; \textit{or} \;\; Y = S.$$

Proof: From right-to-left the claim is trivial. The opposite direction is by contraposition. So, assume $X \neq \emptyset$ and $Y \neq S$. Hence, $x \in X$ and $z \notin Y$, for some $x, z \in S$. In case Y is empty, we are done immediately, for then $(x, x) \in \rho(X)$ and $(x, x) \notin \rho(Y)$. So for the remainder of the proof we may assume there to be some $y \in S$ such that $y \in Y$. By Fact 2.1.3, moreover, both $\rho(X) = \rho_0(X)$ and $\rho(Y) = \rho_0(Y)$, which simplifies the reasoning.

With the assumption that X and Y be distinct, either $Y \nsubseteq X$, or $Y \subsetneq X$. In the former case, $y' \in Y$ and $y' \notin X$, for some $y' \in S$. Hence, $(y', z) \in \rho(X)$ and $(y', z) \notin \rho(Y)$. In the latter case, $x' \in X$ and $x' \notin Y$, for some $x \in S$. Because $Y \subseteq X$, also $y \in X$. Therefore, $(y, x') \in \rho(X)$ whereas $(y, x') \notin \rho(Y)$. ⊣

By contrast, both $\boldsymbol{\rho}_0(\boldsymbol{X})$ and $\boldsymbol{\rho}(\boldsymbol{X})$ are tidily downward monotone in $\boldsymbol{X}$.

Fact 2.1.6 *(Monotonicity) Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be sets of subsets of a set S. Then:*

$$\boldsymbol{X} \subseteq \boldsymbol{Y} \quad \textit{implies} \quad \boldsymbol{\rho}_0(\boldsymbol{Y}) \subseteq \boldsymbol{\rho}_0(\boldsymbol{X}) \;\; \textit{and} \;\; \boldsymbol{\rho}(\boldsymbol{Y}) \subseteq \boldsymbol{\rho}(\boldsymbol{X}).$$

Proof: Straightforward. Assume $\boldsymbol{X} \subseteq \boldsymbol{Y}$. Then also $\{\rho_0(X) : X \in \boldsymbol{X}\} \subseteq \{\rho_0(Y) : Y \in \boldsymbol{Y}\}$. Hence:

$$\boldsymbol{\rho}_0(\boldsymbol{Y}) = \bigcap \{\rho_0(Y) : Y \in \boldsymbol{Y}\} \subseteq_{\boldsymbol{X} \subseteq \boldsymbol{Y}} \bigcap \{\rho_0(X) : X \in \boldsymbol{X}\} = \boldsymbol{\rho}_0(\boldsymbol{X}).$$

The reasoning for $\boldsymbol{\rho}(\boldsymbol{Y}) \subseteq \boldsymbol{\rho}(\boldsymbol{X})$ runs along analogous lines. ⊣

2.2 Relational Semantics

We are now in a position to furnish classical propositional logic with a relational semantics. With each formula φ we associate the relations $\rho(\varphi)$ and $\bar{\rho}(\varphi)$ over the valuations and, similarly, with each theory Γ the relations $\boldsymbol{\rho}(\Gamma)$ and $\bar{\boldsymbol{\rho}}(\Gamma)$. By a *maximum element* of a relation ρ on a set S we understand an element x of S such that for all elements x' of S it is the case that $(x', x) \in \rho$ Denoting the set of maximum elements of a relation ρ by $max(\rho)$, we now have the following proposition.

Proposition 2.2.1 *Let Γ be a theory in a propositional language $L(A)$. Then:*

$$[\![\Gamma]\!] = max(\boldsymbol{\rho}(\Gamma)) \qquad \textit{and} \qquad \langle\!\langle \Gamma \rangle\!\rangle = \overline{max(\bar{\boldsymbol{\rho}}(\Gamma))}.$$

Proof: First assume $[\![\Gamma]\!]$ to be empty. Assume further for a *reductio ad absurdum* that s is a maximum element of $\boldsymbol{\rho}(\Gamma)$ and consider an arbitrary $\gamma \in \Gamma$. Then, $(s', s) \in \rho(\gamma)$, for all valuations s'. So, in particular, $(s, s) \in \rho(\gamma)$ and from the definition of $\rho(\gamma)$ then follows that $[\![\gamma]\!] \neq \emptyset$. Hence, $s^* \in [\![\gamma]\!]$, for some s^*. Then also $(s^*, s) \in \rho(\gamma)$ and consequently $s \in [\![\gamma]\!]$ as well. With γ having been chosen as an arbitrary element of Γ, we have that $s \in [\![\Gamma]\!]$, which is at variance with the assumption that $[\![\Gamma]\!]$ be empty.

So, for the remainder of the proof we will assume $[\![\Gamma]\!]$ to be not empty. Consider an arbitrary valuation s. First assume that $s \notin [\![\Gamma]\!]$. Then $s \notin [\![\gamma]\!]$, for some $\gamma \in \Gamma$. With $[\![\Gamma]\!]$ not empty, we may assume there is some $s' \in [\![\gamma]\!]$. Then, however, $(s', s) \notin \rho(\gamma)$ and $(s', s) \notin \boldsymbol{\rho}(\Gamma)$. Hence, s is no maximum element of $\boldsymbol{\rho}(\Gamma)$. Finally, assume $s \in [\![\Gamma]\!]$. Now consider an arbitrary valuation s' along with an arbitrary $\gamma \in \Gamma$. Then, $s \in [\![\gamma]\!]$ and so $(s', s) \in \rho(\gamma)$. With γ having been chosen arbitrarily, also $(s', s) \in \boldsymbol{\rho}(\Gamma)$ and we may conclude that s is a maximum element of $\boldsymbol{\rho}(\Gamma)$. This concludes the first part of the proof

The second part of the proof can be obtained using the first one (duality). Merely consider the following equalities:

$$\begin{aligned} \langle\!\langle \Theta \rangle\!\rangle &= \textstyle\bigcup_{\vartheta\in\Theta} [\![\vartheta]\!] = \overline{\textstyle\bigcap_{\vartheta\in\Theta} \overline{[\![\vartheta]\!]}} = \overline{[\![\, \{\neg\vartheta : \vartheta \in \Theta\} \,]\!]} \\ &= \overline{max\left(\boldsymbol{\rho}\left(\{\neg\vartheta : \vartheta \in \Theta\}\right)\right)} = \overline{max\left(\textstyle\bigcap_{\vartheta\in\Theta} \rho\left(\overline{[\![\vartheta]\!]}\right)\right)} = \overline{max\left(\bar{\boldsymbol{\rho}}(\Theta)\right)}. \end{aligned}$$

This concludes the proof. ⊣

As an immediate consequence of this result, we have the following corollary, which characterizes classical logical consequence in terms of the relations theories define. A theory Θ follows classically from another theory Γ if and only if the maximum elements of the relation $\boldsymbol{\rho}(\Gamma)$ are *no* maximum elements of the relation $\bar{\boldsymbol{\rho}}(\Theta)$.

Corollary 2.2.2 *Let Γ be a theory and φ a formula. Then:*

$$\Gamma \vdash_{\mathrm{CPC}} \Theta \quad \textit{iff} \quad max\left(\boldsymbol{\rho}(\Gamma)\right) \subseteq \overline{max\left(\bar{\boldsymbol{\rho}}(\Theta)\right)}.$$

Proof: Immediate by Proposition 2.2.1. ⊣

As an alternative to classical consequence, one could define a consequence relation $\vdash^*$ as follows in terms of the maximum elements of the relations $\boldsymbol{\rho}_0(\Gamma)$ and $\bar{\boldsymbol{\rho}}_0(\Gamma)$:

$$\Gamma \vdash^* \Theta \quad \text{iff} \quad max\left(\boldsymbol{\rho}_0(\Gamma)\right) \subseteq \overline{max\left(\bar{\boldsymbol{\rho}}_0(\Theta)\right)}.$$

This consequence relation is as the classical one, except for its behavior with respect to classical contradictions and tautologies. In virtue of Fact 2.1.3 below —

which states that $\boldsymbol{\rho}(\Gamma) = \boldsymbol{\rho}_0(\Gamma)$ if and only if $\varnothing \notin \boldsymbol{X}$ — it can easily be appreciated that $\Gamma \vdash^* \Theta$ if and only if $\Gamma \vdash_{\mathrm{CPC}} \Theta$, provided that Γ contains no contradictions and Θ no tautologies. We have already seen, however, that $\rho_0(\bot)$ and $\rho_0(\top)$ are both interpreted as the universal relation over the valuations. Consequently, $\vdash^*$ treats classical contradictions on a par with classical tautologies. This makes that, *e.g.*, the classical rule *ex falso quod libet* fails for $\vdash^*$. For a counterexample, let a be a propositional variable of a language $L(A)$. Then observe that $\bot \nvdash^* a$, as $\overline{max(\boldsymbol{\rho}_0(\{\bot\})) = 2^A}$ and $\overline{max(\bar{\boldsymbol{\rho}}_0(\{a\}))} = \overline{max(\boldsymbol{\rho}_0(\{\neg a\}))} = [\![a]\!]$. A similar remark would have applied, had $\vdash^*$ been defined in terms of the *maximal* elements of the relations $\boldsymbol{\rho}_0(\Gamma)$ and $\bar{\boldsymbol{\rho}}_0(\Theta)$. Observe in this respect that the maximal elements of $\boldsymbol{\rho}_0(\{\bot\})$ exhaust logical space just as well as the maximum elements of $\boldsymbol{\rho}_0(\{\bot\})$ do. The consequence relations defined in terms of maximal elements also exhibit non-monotonic features, but we will not pursue this issue here.

The advantage of the relational semantics is that it preserves more of the structure that formulas and theories impose on logical space. From the extension $[\![\Gamma]\!]$ of a theory Γ the extensions $[\![\gamma]\!]$ of the formulas γ in Γ cannot in general be recovered; this structure may have been lost beyond repair. In a strict sense a similar thing can be said of the relation $\boldsymbol{\rho}(\Gamma)$ and the relations $\rho(\gamma)$: it is not in general the case that from the relation $\boldsymbol{\rho}(\Gamma)$ the theory Γ can be reconstructed. Nevertheless, the relation $\boldsymbol{\rho}(\Gamma)$ can distinguish valuations s and s' even if neither of them is maximum in $\boldsymbol{\rho}(\Gamma)$, indicating that Γ contain a formula that is validated in the one but not in the other, or if Γ is inconsistent. *E.g.*, the valuation $\varnothing$ is strictly less than the valuation $\{a\}$ in the relation $\boldsymbol{\rho}(\{a, b\})$, yet neither of them is a maximum element in this respect. This feature of the relational semantics is especially serviceable when one is interested in the maximal or maximum elements of the relation determined by a theory as restricted to a subset of valuations, even if the maximal or maximum elements of the unrestricted relation are disjoint from that subset.

Chapter 3

Intermezzo: Veltman's Updates for Defaults

The additional ordinal structure the relational semantics for propositional logic engenders over the set of valuations, is quite superfluous if one's concerns are with classical consequence only. However, the semantics of a considerable number of non-standard variants or extensions of classical propositional logic appeal to a relational structure over the valuations or possible worlds. We have already mentioned qualitative decision theory (*e.g.*, Boutilier (1994)), belief revision (*e.g.*, Gärdenfors (1988)), and non-monotonic consequence relations (*e.g.*, Shoham (1988), Kraus, Lehmann, and Magidor (1990) and Makinson (1994)). The prime example in this respect is, of course, Kripke semantics for modal languages. As in Kripke semantics, this relational structure is often assumed to be given independently by the semantics, rather than induced by syntactic objects, such as formulas.

Veltman's analysis of defaults (Veltman (1996)), however, is different in this respect. There it is suggested that in a proper treatment of defeasible reasoning, some formulas are interpreted as imposing a relational structure on logical space. This enables one to distinguish among any subset of valuations those that are optimal with respect to this structure. In the semantics of formulas of another logical form these optimal valuations play a crucial role. There being a clear parallel between the concluding remarks of the previous section and Veltman's semantical ideas, we will here give a synopsis of the third section of 'Defaults in Update Semantics'.

Classical logic is monotonic in the sense that if a conclusion follows from a collection of premisses Γ, then the same conclusion also follows from any collection of premisses that includes Γ. If premisses are taken to represent the information available to an agent and conclusions the inferences that agent may reasonably draw from the premisses, it has been argued that much of human reasoning exhibits non-monotonic features. In the face of new evidence one may be happy to withdraw conclusions arrived at on the basis of information obtained previously. The new evidence is then said to defeat the conclusion and the conclusion itself is said to be defeasible. *E.g.*, if the only piece of information available is that it normally rains, one could arguably infer that it presumably rains. However, if one obtains as an additional piece of information that it as a matter of fact does not rain, one might be quite willing to retract the conclusion that it rains, as it does not.

Veltman gives a formal account of these and similar phenomena having to do with defeasible reasoning and the order in which information is received. Using a dynamic framework Veltman can account for the contrast between the *acceptability* of texts as (1) and (2), and the *unacceptability* of the sequence (3):

(1) "Normally, it rains. ... Presumably, it rains."

(2) "Normally, it rains. ... Presumably, it rains. ... It does not rain."

(3) "Normally, it rains. ... It does not rain. ... Presumably, it rains."

Although many of the merits of Veltman's approach lie in its ability to deal with such examples using a dynamic framework, we concentrate on some of its static aspects. The intuition behind Veltman's approach is that a sentence like *"Presumably, it rains"* signifies that it rains in all of the most normal states of affairs that are consistent with the information available. This presupposes that the possible states of affairs can somehow be ordered with respect to normality. A distinguishing mark of Veltman's proposal is that this normality order over the possible states of affairs is determined by sentences like *"Normally, it rains"* that occurred earlier in the text, rather than merely fixed exogenously.

Veltman proposes a propositional modal language $L(A, \{\mathit{normally}, \mathit{presumably}\})$, where *normally* and *presumably* are modalities operating on formulas of the propositional language $L(A)$ only. *I.e.*, the formulas of $L(A, \{\mathit{normally}, \mathit{presumably}\})$ are given by the set $\{\varphi, \mathit{normally}\,\varphi, \mathit{presumably}\,\varphi \;:\; \varphi \text{ a formula of } L(A)\}$ and there is no nesting of the modalities. The intended readings of $\mathit{normally}\,\varphi$ and $\mathit{presumably}\,\varphi$ suggest themselves.

The formulas of $L(A, \{\mathit{normally}, \mathit{presumably}\})$ are interpreted in terms of states consisting of a so-called *expectation pattern* ρ and an *information set* X. An expectation pattern is a reflexive and transitive relation over the valuations for $L(A)$ and the information set X is a subset of valuations, intuitively, containing the possible states of affairs that are compatible with one's factual information about the world.Veltman distinguishes the minimal state $\mathbf{0}$ and the absurd state $\mathbf{1}$, defined by $(S \times S, S)$ and $(Id, \varnothing)$, respectively. The valuations s that are minimal with respect to ρ — *i.e.*, such that $s' < s$ in ρ, for no valuation s' — are called *normal*. If the set of normal worlds in an expectation pattern ρ is not empty, ρ is said to be *coherent*. A state (ρ, X) is an *information state* if ρ is coherent and X non-empty, or if (ρ, X) is the absurd state **1**.

Semantically, formula φ in $L(A, \{\textit{normally}, \textit{presumably}\})$ is interpreted as a postfixed *operation* $[\varphi]$ on information states. For each formula φ in $L(A)$, the operation $[\varphi]$ performs an update on the information set of an information state, accommodating the information conveyed by φ without changing the expectation pattern. That is, provided that the update does not render the information set void, for then the absurd state results.

By contrast, if φ is of the form *normally* ψ and the extension $[\![\psi]\!]$ contains a normal world with respect to ρ, we have $[\varphi]$ operate on the expectation pattern ρ of an information state (ρ, X). It leaves the information set X as it was but ρ by removing from it all edges (s, s') with ψ holding in s but not in s'. This renders any valuation s that forces ψ strictly more normal than any valuation in which ψ does not hold but that was as normal as s in the original expectation pattern. As such $[\textit{normally}\,\psi]$ imposes additional structure on logical space rendering ψ worlds more normal that non-φ worlds without affecting the agents factual information about the world. Rather, $[\textit{normally}\,\psi]$ refines the expectation pattern by intersecting it with the inverse of the relation $\rho_0(\psi)$, as defined in the previous section. If, however, $[\![\psi]\!]$ fails to contain a normal world with respect to ρ, then updating (ρ, X) with $[\textit{normally}\,\psi]$ will result in the absurd state **1**.

Finally, $[\textit{presumably}\,\varphi]$ performs a test on information states. In case φ holds in all valuations that are minimal with respect to the expectation pattern of the information state, $[\textit{presumably}\,\varphi]$ returns the original information state. Otherwise, it returns the absurd state. Let φ be a formula in $L(A)$. Then — employing notations used throughout this thesis — Veltman's formally definitions are given by:

$$(\rho, X)[\varphi] \;=_{df.}\; \begin{cases} (\rho, X \cap [\![\varphi]\!]) & \text{if } X \cap [\![\varphi]\!] \neq \emptyset, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

$$(\rho, X)[\textit{normally}\,\varphi] \;=_{df.}\; \begin{cases} (\rho \cap \rho_0(\varphi)^{\smile}, X) & \text{if } [\![\varphi]\!] \text{ contains a normal world,} \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

$$(\rho, X)[\textit{presumably}\,\varphi] \;=_{df.}\; \begin{cases} (\rho, X) & \text{if } s \in [\![\varphi]\!], \text{ for all } s \text{ minimal in } X \text{ w.r.t. } \rho, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

(Here, $\rho_0(\varphi)^{\smile}$ denotes the *inverse* of the relation $\rho_0(\varphi)$.)

For formulas φ of $L(A, \{\textit{normally}, \textit{presumably}\})$ and information states σ define $\sigma \Vdash \varphi$ if and only if $\sigma[\varphi] = \sigma$. Moreover, consequence $\vdash_{\mathrm{V}}$ for this particular

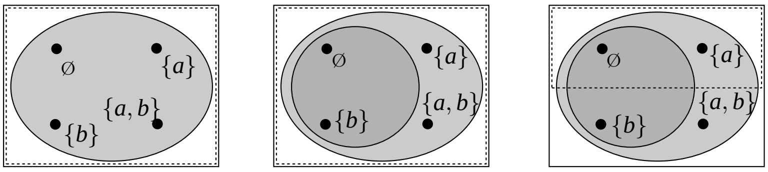

Figure 3.1. The figure on the left depicts $\mathbf{0}$ for a language with a and b as the only propositional variables. The dashed box and the grey balloons indicate the subset of valuations and the expectation patter, respectively. From left to right the figures depict the minimal state $\mathbf{0}$, $\mathbf{0}[\textit{normally}\, a]$ and $\mathbf{0}[\textit{normally}\, a][\neg b]$. The valuation $\{a\}$ is now minimal in $\mathbf{0}[\textit{normally}\, a][\neg b]$, but for instance $\varnothing$ is not. Hence, $\mathbf{0}[\textit{normally}\, a][\neg b] \Vdash \textit{presumably}(a \wedge \neg b)$ but $\mathbf{0}[\textit{normally}\, a][\neg b] \nVdash \textit{presumably}(\neg\,(a \vee b))$.

system is defined as a relation between the *sequences* of formulas $\varphi_0, \ldots, \varphi_n$ and a formula ψ as follows:

$$\varphi_0, \ldots, \varphi_n \vdash_{\mathrm{V}} \psi \quad \text{iff} \quad \mathbf{0}[\varphi_0] \ldots [\varphi_n] \Vdash \psi.$$

The sequential order of the formulas $\varphi_0, \ldots, \varphi_n$ here makes a difference. *E.g.*, as the formal counterparts of (1), (2) and (3), above, we find:

(1′) $\mathbf{0}[\textit{normally}\, a][\textit{presumably}\, a] \neq \mathbf{1}$,

(2′) $\mathbf{0}[\textit{normally}\, a][\textit{presumably}\, a][\neg a] \neq \mathbf{1}$,

(3′) $\mathbf{0}[\textit{normally}\, a][\neg a][\textit{presumably}\, a] = \mathbf{1}$.

The reader be also referred to Figure 3.1 for further illustration.

The guiding principle behind Veltman's update semantics for defaults is that Boolean formulas φ and those of the form *normally* φ build up an information state. Suppose that $\mathbf{0}[\varphi_0] \ldots [\varphi_n]$ is an information state that is being constructed in the course of an update process and distinct from the absurd state. The constituent expectation pattern is then precisely the inverse of the relation $\boldsymbol{\rho}_0\,(\Theta)$, where Θ is given by exactly those formulas ψ such that the formula *normally* ψ is among $\varphi_0, \ldots, \varphi_n$. Formulas of the form *presumably* φ are then evaluated with respect to the information state constructed. A formula *presumably* φ holds in a non-absurd information state (ρ, X), *i.e.*, $\sigma \Vdash \textit{presumably}\, \varphi$, if φ holds in all *optimal* states in ρ that are compatible with the factual information represented by X. Optimality is here taken as minimality with respect to the information pattern, but could equally well be defined as maximality with respect to its inverse. Moreover, the expectation pattern of an information state is built up by formulas, *viz.*, by those formulas that are of the form *normally* φ.

Chapter 4

Characterizing Set Induced Relations

It is a well-known fact of classical propositional logic that, in case the language contains an countably infinite number of propositional variables, it is not the case that each subset of valuations is the extension of some formula. To appreciate this observe that the number of formulas of a language of classical propositional logic is denumerable whereas the number of valuations for an infinite number of propositional variables is uncountable. A similar remark applies to the extensions of the theories of a countably infinite propositional language.

Classical propositional logic is a particular logic that can be given a semantics by associating sets of valuations to formulas. Other and distinct logics with a similar semantics are *prima facie* conceivable and we may come to wonder whether such logics fare better or worse with respect to characterizing subsets of valuations as the extension of formulas. In the first section of this chapter we prove that classical propositional logic is in an important sense the most expressive of such languages. Let a set X of valuations be called *finitary*, or *of finite character* in case it depends on the values assigned to a finite set of propositional variables whether a valuation belongs to X or not. Let further a valuation based semantics for a propositional language be called *finite* in case the extensions it associates with the formulas are all finitary. We find that classical propositional logic is the most "expressive" in this respect: the extensions associated with the formulas in the Tarskian semantics for classical propositional logic exhaust the set of finitary sets of valuations.

In the previous chapter, formulas and theories were associated with relations over valuations rather than with sets of valuations, thus giving rise to a relational semantics. We find that the relation $\rho(\varphi)$ thus associated with a formula φ in general satisfies a particular property, *viz.*, that of *bisectivity* (*cf.*, page 4). Similarly, the relation $\boldsymbol{\rho}(\Gamma)$ associated with a theory Γ will in general be a *proto-order* (*cf.*, page 4). However, if the set of propositional variables is infinite, the relations thus defined with formulas and those with theories fail to exhaust, respectively, the set of bisective relations and that of the proto-orders over the valuations. However, invoking a suitable notion of finite character for relations, analogous as we did for sets, we find that the set of bisective relations defined by formulas can suitably be characterized. Eventually we also obtain a kindred result for the proto-orders defined by theories.

The notions of finitary sets and relations are most conveniently defined using the machinery of rough set theory. We therefore first give a brief summary of some of the latter's elementary facts.

4.1 Some Elementary Facts about Rough Sets

In this section the elementary concepts of rough set theory, *viz.*, the upper and lower approximations of a set, are introduced. In the next sections they are employed for some results for propositional logics. As such our employment of rough sets is divergent from normal use. First we will give some basic facts concerning upper and lower approximations.[1]

For S we have $Part(S)$ denote the set of partitions over S. Moreover, for π a partition in S and for x an element of S, $[x]_\pi$ is the unique block of π containing x.

Let S be a set and let π be a partition of S. Obviously, it is not in general the case that a subset X of S is identical to a union of a number of blocks in π. Still each subset can be characterized by two sets which do have this property. Define the *lower approximation* $\underline{apr}_\pi(X)$ and the *upper approximation* $\overline{apr}_\pi(X)$ by:

$$\begin{aligned} \underline{apr}_\pi(X) &=_{df.} \bigcup\{Y \in \pi : \ Y \subseteq X\}, \\ \overline{apr}_\pi(X) &=_{df.} \bigcup\{Y \in \pi : \ Y \cap X \neq \emptyset\}. \end{aligned}$$

Let ε_π be the equivalence relation over S associated with the partition π. Then we also have the following equivalent characterizations of the lower and upper approximation of a subset X of S. For each $x \in S$:

$$\begin{aligned} x \in \underline{apr}_\pi(X) \quad &\text{iff} \quad \text{for all } s' \in S \text{ such that } s \sim_\pi s' : s' \in X, \\ x \in \overline{apr}_\pi(X) \quad &\text{iff} \quad \text{for some } s' \in S : s \sim_\pi s' \ \text{ and } \ s' \in X. \end{aligned}$$

[1] For a more extensive account the reader be referred to Pawlak (1991)

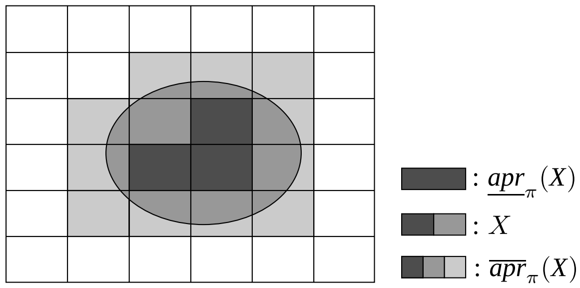

Figure 4.1. Rough sets in a set S partitioned by π. The oval represents the set X. The colored areas indicate the upper approximation and the darkly colored area the lower approximation.

Clearly, $\overline{apr}_\pi$ is a cylindrification operator on 2^S and $\underline{apr}_\pi$ its dual. As such they exemplify a more general mathematical concept that is also instanced by quantification and modality in logic. This observation has by no means escaped attention in the literature (*cf. e.g.*, Yao, Wong, and Lin (1997), Düntsch (1999) and Düntsch (no date)). Figure 4.1 illustrates the lower and upper approximations of a set X. Suppressing the subscript π, the approximation operations $\underline{apr}$ and $\overline{apr}$ satisfy the following elementary properties for subsets X and Y of a set S:[2]

$$\begin{array}{rcl@{\qquad}rcl}
\underline{apr}(\emptyset) & = & \emptyset & \overline{apr}(\emptyset) & = & \emptyset \\
\underline{apr}(S) & = & S & \overline{apr}(S) & = & S \\
\underline{apr}(X) & \subseteq & X & \overline{apr}(X) & \supseteq & X \\
\underline{apr}(X) & \subseteq & \underline{apr}\left(\underline{apr}(X)\right) & \overline{apr}(X) & \supseteq & \overline{apr}(\overline{apr}(X)) \\
X & \subseteq & \underline{apr}(\overline{apr}(X)) & X & \supseteq & \overline{apr}\left(\underline{apr}(X)\right) \\
\overline{apr}(X) & \subseteq & \underline{apr}(\overline{apr}(X)) & \underline{apr}(X) & \supseteq & \overline{apr}\left(\underline{apr}(X)\right) \\
\underline{apr}(X) & = & \overline{\overline{apr}\left(\overline{X}\right)} & \overline{apr}(X) & = & \overline{\underline{apr}\left(\overline{X}\right)} \\
\underline{apr}(X \cap Y) & = & \underline{apr}(X) \cap \underline{apr}(Y) & \overline{apr}(X \cap Y) & \subseteq & \overline{apr}(X) \cap \overline{apr}(Y) \\
\underline{apr}(X \cup Y) & \supseteq & \underline{apr}(X) \cup \underline{apr}(Y) & \overline{apr}(X \cup Y) & = & \overline{apr}(X) \cup \overline{apr}(Y)\,.
\end{array}$$

As two obvious consequences of these properties also both $\underline{apr}\left(\underline{apr}(X)\right) = \underline{apr}(X)$ and $\overline{apr}(\overline{apr}(X)) = \overline{apr}(X)$. Moreover, the latter four inequalities can be general-

[2]These inequalities are taken from Yao, Wong, and Lin (1997).

ized to infinite sets of sets. Let $\boldsymbol{X} \subseteq 2^S$, then:

$$\begin{array}{rclcrcl} \underline{apr}\left(\bigcap \boldsymbol{X}\right) & = & \bigcap_{X \in \boldsymbol{X}} \underline{apr}\,(X) & \qquad & \overline{apr}\left(\bigcap \boldsymbol{X}\right) & \subseteq & \bigcap_{X \in \boldsymbol{X}} \overline{apr}\,(X) \\ \underline{apr}\left(\bigcup \boldsymbol{X}\right) & \supseteq & \bigcup_{X \in \boldsymbol{X}} \underline{apr}\,(X) & \qquad & \overline{apr}\left(\bigcup \boldsymbol{X}\right) & = & \bigcup_{X \in \boldsymbol{X}} \overline{apr}\,(X)\,. \end{array}$$

Both $\underline{apr}$ and $\overline{apr}$ satisfy upward monotonicity:

$$\begin{array}{lll} X \subseteq Y & \text{implies} & \underline{apr}\,(X) \subseteq \underline{apr}\,(Y)\,, \\ X \subseteq Y & \text{implies} & \overline{apr}\,(X) \subseteq \overline{apr}\,(Y)\,. \end{array}$$

We also have the following fact, which says that, given a partition, the fixed points of the upper and lower approximations coincide.

Fact 4.1.1 *Let S be a set, $X \subseteq S$ and $\pi \in Part\,(S)$. Then:*

$$X = \overline{apr}_\pi(X) \quad \textit{iff} \quad X = \underline{apr}_\pi(X)\,.$$

Proof: First assume $X = \overline{apr}\,(X)$. Observe that both $\overline{apr}\,(X) \subseteq \underline{apr}\,(\overline{apr}\,(X))$ and $\underline{apr}\,(\overline{apr}\,(X)) \subseteq \overline{apr}\,(X)$ are instances of rough set laws. Hence $\overline{apr}\,(X) = \overline{apr}\left(\underline{apr}\,(X)\right)$ and we may reason as follows:

$$X \;=_{\text{ass.}}\; \overline{apr}\,(X) \;=\; \underline{apr}\,(\overline{apr}\,(X)) \;=_{X\,=\,\overline{apr}\,(X)}\; \underline{apr}\,(X)\,.$$

The reasoning in the opposite direction is analogous.[3] ⊣

As a generalization of this fact, we also have the following.

Fact 4.1.2 *Let π be a partition of a set S. Let further $\boldsymbol{X}$ a set of subsets of S such that $\boldsymbol{X} \subseteq \pi$. Then:*

$$\overline{apr}_\pi(\bigcup \boldsymbol{X}) \;=\; \underline{apr}_\pi(\bigcup \boldsymbol{X}) \;=\; \bigcup \boldsymbol{X}.$$

Proof: Straightforward. ⊣

The partitions by means of which sets are approximated may be finer or coarser. The facts that follow concern the behavior of the approximation operations with respect to partitions of various degrees of coarseness. For π and π' partitions of a set S, *i.e.*, $\pi, \pi' \in Part\,(S)$, let $\pi \leqslant \pi'$ be formally defined as:

$$\pi \leqslant \pi' \quad \text{iff} \quad \text{for all } x \in \pi \text{ there is a } y \in \pi' \text{ such that } x \subseteq y.$$

[3]For this elegant proof I am indebted to Boudewijn de Bruin.

Intuitively, $\pi \leqslant \pi'$ denotes that π at least as fine as π'. As such $\leqslant$ defines a partial order on $Part\,(S)$. Then $\leqslant$ defines a partial order on $Part\,(S)$. Rather, $Part\,(S)$ constitutes a complete lattice if ordered thus. We now also have the following monotonicity properties for the lower and upper approximation operations:

Fact 4.1.3 *Let π and π' be partitions of some set S. Then for all $X \subseteq S$:*

$$\pi \leqslant \pi' \quad \textit{implies} \quad \underline{apr}_{\pi}(X) \;\supseteq\; \underline{apr}_{\pi'}(X)\,,$$
$$\pi \leqslant \pi' \quad \textit{implies} \quad \overline{apr}_{\pi}(X) \;\subseteq\; \overline{apr}_{\pi'}(X)\,.$$

Proof: Both cases are analogous; here we prove only the first. Assume $\pi \leqslant \pi'$ and consider an arbitrary $x \in \underline{apr}_{\pi'}(X)$. Consider the block $[x]_{\pi}$ of π; then, $[x]_{\pi'} \subseteq X$. By the assumption there is a block Y of π' such that $[x]_{\pi} \subseteq Y$. Then, $x \in Y$ and, therefore, $Y = [x]_{\pi'}$. Hence, $[x]_{\pi} \subseteq [x]_{\pi'} \subseteq X$. Since $x \in [x]_{\pi}$, we may conclude that $x \in \underline{apr}_{\pi}(X)$. ⊣

In words, the coarser the partition, the larger the upper approximation of a set and the smaller its lower approximation. The following fact conveys a stronger and closely related result.

Fact 4.1.4 *Let π and π' be partitions of some set S. Then:*

$$\pi \leqslant \pi' \quad \textit{iff} \quad \textit{for all } X \subseteq S\colon\ \underline{apr}_{\pi}(X) \;\supseteq\; \underline{apr}_{\pi'}(X)\,,$$
$$\pi \leqslant \pi' \quad \textit{iff} \quad \textit{for all } X \subseteq S\colon\ \overline{apr}_{\pi}(X) \;\subseteq\; \overline{apr}_{\pi'}(X)\,.$$

Proof: The proofs of both cases run along analogous lines; we will here give that of the first. The left-to-right direction is immediate by Fact 4.1.3. For the opposite direction, assume that for all $X \subseteq S$ we have $\underline{apr}_{\pi}(X) \supseteq \underline{apr}_{\pi'}(X)$. Consider an arbitrary $X \in \pi$. By definition, X is a non-empty subset of S. As $S = \bigcup \pi'$, there is a $Y \in \pi'$ such that $X \cap Y \neq \varnothing$. By the assumption, then $\underline{apr}_{\pi}(Y) \supseteq \underline{apr}_{\pi'}(Y) =_{\text{Fact 4.1.2}} Y$. Because also $\underline{apr}_{\pi}(Y) \subseteq Y$, it follows that $Y = \underline{apr}_{\pi}(Y)$. By definition, $Y = \bigcup \left\{X' \in \pi :\ X' \subseteq Y\right\}$. Since, $X \cap Y \neq \varnothing$, we have $X \cap X'' \neq \varnothing$, for some $X'' \in \left\{X' \in \pi :\ X' \subseteq Y\right\}$. With X and X'' blocks in the partition π, it follows that $X = X''$, and hence, $X \subseteq Y$. Having chosen X arbitrarily from π, we may conclude that $\pi \leqslant \pi'$. ⊣

In the sequel, we will mostly be interested in a particular class of partitions of a universe with respect to which the approximations are defined. If the universe set is the powerset of a set A, we define an equivalence relation holding between any two subsets of A if each element of a third subset of A is in the one subset if and only if it is an element of the other. Let A be a set and define for each $Z \subseteq A$ the equivalence relation ε_Z on 2^A such that for all $X, Y \subseteq A$:

$$(X,Y) \in \varepsilon_Z \quad \text{iff} \quad Z \cap X = Z \cap Y.$$

Sometimes we use the infix notation $X \sim_Z Y$ to convey that $(X,Y) \in \varepsilon_Z$. Observe that it is both a necessary and a sufficient condition for $X \sim_Z Y$ to hold that for all $z \in Z$, it is the case that $z \in X$ if and only if $z \in Y$. Note that $\varepsilon_\varnothing$ and ε_A are the universal relation and the identity relation on 2^A, respectively. More in general, we have $X \subseteq Y$ *if* and only if $\varepsilon_Y \subseteq \varepsilon_X$. The only-if direction is trivial. For the other direction assume for the contrapositive that $x \in X$ but $x \notin Y$. Then, $\varnothing \nsim_X \{x\}$ but $\varnothing \sim_Y \{x\}$, *i.e.*, the relations ε_X and ε_Y are distinct. Hence, the set $\{\varepsilon_X : X \subseteq A\}$ constitute a complete lattice with relation composition and intersection as join and meet, respectively. To appreciate this, consider the following fact.

Fact 4.1.5 *Let X and Y be subsets of a set A. Then:*

$$\varepsilon_{X\cap Y} = \varepsilon_X \circ \varepsilon_Y \qquad \textit{and} \qquad \varepsilon_{X\cup Y} = \varepsilon_X \cap \varepsilon_Y.$$

Proof: For the $\subseteq$-direction of the first claim, assume for arbitrary $s, s' \in 2^A$ that $(s,s') \in \varepsilon_{X\cap Y}$. Hence, $s \cap X \cap Y = s' \cap X \cap Y$. Define $s^* =_{df.} (s \cap X) \cup \big(s' \cap \overline{X}\big)$. Then:

$$s \cap X = (s \cap X \cap X) \cup \big(s' \cap \overline{X} \cap X\big) = \big((s \cap X) \cup (s' \cap \overline{X})\big) \cap X.$$

Hence, $(s, s^*) \in \varepsilon_X$. Also consider the following equalities:

$$\begin{aligned} s' \cap Y &= \big((s' \cap X) \cup (s' \cap \overline{X})\big) \cap Y = \big((s' \cap X \cap Y) \cup (s' \cap \overline{X} \cap Y)\big) \\ &= \big((s \cap X \cap Y) \cup (s' \cap \overline{X} \cap Y)\big) = \big((s \cap X) \cup (s' \cap \overline{X})\big) \cap Y. \end{aligned}$$

Accordingly also $(s^*, s') \in \varepsilon_Y$ and finally also $(s,s') \in \varepsilon_X \circ \varepsilon_Y$.

For the $\supseteq$-direction, assume $(s,s') \in \varepsilon_X \circ \varepsilon_Y$. So, for some $s'' \in 2^A$ both $(s,s'') \in \varepsilon_X$ and $(s'',s') \in \varepsilon_Y$. *I.e.*, both $s \cap X = s'' \cap X$ and $s'' \cap Y = s' \cap Y$. Consider the following equalities:

$$s \cap X \cap Y = s'' \cap X \cap Y = s'' \cap Y \cap X = s' \cap Y \cap X = s' \cap X \cap Y.$$

Hence, $(s,s') \in \varepsilon_{X\cap Y}$.

For the second claim, first assume for arbitrary $s, s' \in 2^A$ that $(s,s') \in \varepsilon_{X\cup Y}$. Then, $s \cap (X \cup Y) = s' \cap (X \cup Y)$. Since, $X, Y \subseteq X \cup Y$ then also both $s \cap X = s' \cap X$ and $s \cap Y = s' \cap Y$. We may conclude that $(s,s') \in \varepsilon_X \cap \varepsilon_Y$. For the opposite direction, assume $(s, s') \in \varepsilon_X \cap \varepsilon_Y$, *i.e.*, both $s \cap X = s' \cap X$ and $s \cap Y = s' \cap Y$. Now reason as follows:

$$s \cap (X \cup Y) = (s \cap X) \cup (s \cap Y) = \big(s' \cap X\big) \cup \big(s' \cap Y\big) = s' \cap (X \cup Y)\,.$$

We may conclude that $(s, s') \in \varepsilon_{X \cup Y}$. ⊣

The partition of 2^A as determined by ε_X, we denote by π_X. The notation $[x]_{\varepsilon_X}$ for the equivalence class under ε_X containing x we usually abbreviate to $[x]_X$. Obviously, π_A is the finest and $\pi_\varnothing$ the coarsest partition of 2^A. More in general we have that the larger the set X, the finer the partition π_X.

Fact 4.1.6 *Let X and Y be subsets of some set A. Then:*

$$X \subseteq Y \quad \textit{iff} \quad \pi_Y \leqslant \pi_X.$$

Proof: First assume $X \subseteq Y$ and consider an arbitrary $\boldsymbol{X} \in \pi_X$. We without loss of generality we may assume that $\boldsymbol{X} = [Z]_X$, for some $Z \subseteq A$. Now consider $[Z]_Y$ as well as an arbitrary $Z' \in [Z]_Y$. Then, $Z' \cap Y = Z \cap Y$. With the assumption that $X \subseteq Y$, then also $Z' \cap X = Z \cap X$. Hence, $Z' \in [Z]_X$. We may conclude that $\pi_Y \leqslant \pi_X$.

For the opposite direction, assume that $X \nsubseteq Y$, *i.e.*, that there be an $x \in X$ with $x \notin Y$. Consider this x along with the block $[\{x\}]_Y$ of π_Y. Observe that both $\{y\} \in [\{x\}]_Y$ and $\varnothing \in [\{x\}]_Y$. It suffices to show that for all $\boldsymbol{X} \in \pi_X$ if $\{x\} \in \boldsymbol{X}$ then $\varnothing \notin \boldsymbol{X}$. So consider an arbitrary $\boldsymbol{X} \in \pi_X$ with $\{x\} \in \boldsymbol{X}$ as well as an arbitrary $X' \in \boldsymbol{X}$. Then $\{x\} \sim_X X'$, and with $x \in \{x\}$ and $x \in X$ we may conclude that $x \in X'$. Hence, $X' \neq \varnothing$. ⊣

For X a subset of a set A we denote the approximation operators $\underline{apr}_{\pi_X}$ and $\overline{apr}_{\pi_X}$, as defined for subsets of 2^A, by $\underline{apr}_X$ and $\overline{apr}_X$, respectively. As an immediate result of the facts 4.1.4 and 4.1.6 we have the following corollary.

Corollary 4.1.7 *Let X and Y be subsets of some set A. Then:*

$$X \subseteq Y \quad \textit{iff} \quad \textit{for all } \boldsymbol{X} \subseteq 2^A\colon\ \underline{apr}_X(\boldsymbol{X}) \subseteq \underline{apr}_Y(\boldsymbol{X})\,,$$
$$X \subseteq Y \quad \textit{iff} \quad \textit{for all } \boldsymbol{X} \subseteq 2^A\colon\ \overline{apr}_Y(\boldsymbol{X}) \subseteq \overline{apr}_X(\boldsymbol{X})\,.$$

Proof: Immediate from Fact 4.1.4 and Fact 4.1.6. ⊣

With respect to the behavior of lower and upper approximations of a set given partitions π_X and π_Y, we have the following two facts.

Fact 4.1.8 *Let A be some set, of which X and Y are subsets. Let, moreover, $\boldsymbol{X}$ be a subset of 2^A. Then:*

$$
\begin{aligned}
\underline{apr}_{X\cap Y}(\boldsymbol{X}) &= \underline{apr}_X\left(\underline{apr}_Y(\boldsymbol{X})\right),\\
\overline{apr}_{X\cap Y}(\boldsymbol{X}) &= \overline{apr}_X\left(\overline{apr}_Y(\boldsymbol{X})\right),\\
\underline{apr}_{X\cup Y}(\boldsymbol{X}) &\supseteq \underline{apr}_X(\boldsymbol{X}) \cap \underline{apr}_Y(\boldsymbol{X}),\\
\overline{apr}_{X\cup Y}(\boldsymbol{X}) &\subseteq \overline{apr}_X(\boldsymbol{X}) \cap \overline{apr}_Y(\boldsymbol{X}).
\end{aligned}
$$

Proof: The proofs of the first two claims are analogous; here we only give that of the latter.

$$
\begin{aligned}
s \in \overline{apr}_{X\cap Y}(\boldsymbol{X}) \quad &\text{iff} \quad s \sim_{X\cap Y} s' \text{ and } s' \in \boldsymbol{X}, \text{ for some } s' \in 2^A\\
&\text{iff}_{\text{Fact 4.1.5}} \quad s \sim_X s'' \sim_Y s' \text{ and } s' \in \boldsymbol{X}, \text{ for some } s', s'' \in 2^A\\
&\text{iff} \quad s \sim_X s'' \text{ and } s \in \overline{apr}_Y(\boldsymbol{X}), \text{ for some } s'' \in 2^A\\
&\text{iff} \quad s \in \overline{apr}_X\left(\overline{apr}_Y(\boldsymbol{X})\right).
\end{aligned}
$$

For the latter two claims merely observe that, in virtue of Coroll 4.1.7, both $\underline{apr}_X(\boldsymbol{X}) \subseteq \underline{apr}_{X\cup Y}(\boldsymbol{X})$ and $\underline{apr}_Y(\boldsymbol{X}) \subseteq \underline{apr}_{X\cup Y}(\boldsymbol{X})$, as well as both $\overline{apr}_{X\cup Y}(\boldsymbol{X}) \subseteq \overline{apr}_X(\boldsymbol{X})$ and $\overline{apr}_{X\cup Y}(\boldsymbol{X}) \subseteq \overline{apr}_Y(\boldsymbol{X})$. ⊣

Fact 4.1.9 *Let A a set and let I be a set of indices. Let further $\{X_i\}_{i\in I}$ and $\{\boldsymbol{X}_i\}_{i\in I}$ be indexed families of subsets of A and of subsets of 2^A, respectively. Then:*

$$
\begin{aligned}
\bigcap_{i\in I} \underline{apr}_{X_i}(\boldsymbol{X}_i) &\subseteq \underline{apr}_{\bigcup_{i\in I} X_i}\left(\textstyle\bigcap_{i\in I} \boldsymbol{X}_i\right),\\
\bigcap_{i\in I} \overline{apr}_{X_i}(\boldsymbol{X}_i) &\supseteq \overline{apr}_{\bigcup_{i\in I} X_i}\left(\textstyle\bigcap_{i\in I} \boldsymbol{X}_i\right).
\end{aligned}
$$

Proof: First consider an arbitrary $Y \subseteq A$ and assume $Y \in \bigcap_{i\in I} \underline{apr}_{X_i}(\boldsymbol{X}_i)$, *i.e.*, for all $i \in I$, it is the case that $Y \in \underline{apr}_{X_i}(\boldsymbol{X}_i)$. Since, obviously, $X_i \subseteq \bigcup_{i\in I} X_i$, by Corollary 4.1.7 for all $i \in I$, also $Y \in \underline{apr}_{\bigcup_{i\in I} X_i}(\boldsymbol{X}_i)$. Therefore, $Y \in \bigcap_{i\in I} \underline{apr}_{\bigcup_{i\in I} X_i}(\boldsymbol{X}_i)$. Finally, by distribution of $\bigcap$ over $\underline{apr}$, we may conclude that $Y \in \underline{apr}_{\bigcup_{i\in I} X_i}\left(\bigcap_{i\in I} \boldsymbol{X}_i\right)$.

For the second claim, assume for an arbitrary $Y \subseteq A$, that $Y \in \overline{apr}_{\bigcup_{i\in I} X_i}\left(\bigcap_{i\in I} \boldsymbol{X}_i\right)$. Then there is some $Z \subseteq A$ such that $Y \sim_{\bigcup_{i\in I} X_i} Z$ and

$Z \in \bigcap_{i \in I} \boldsymbol{X}_i$. It follows that for each $i \in I$, both $Y \sim_{X_i} Z$ and $Z \in \boldsymbol{X}_i$, *i.e.*, $Y \in \overline{apr}_{Y_i}(\boldsymbol{X}_i)$. We may conclude that $Y \in \bigcap_{i \in I} \overline{apr}_{X_i}(\boldsymbol{X}_i)$. ⊣

For the special partitions π_A and $\pi_\emptyset$, moreover the following equalities hold:

Fact 4.1.10 *Let A be a set and let $\boldsymbol{X} \subseteq 2^A$. Then:*

$$\underline{apr}_A(\boldsymbol{X}) = \overline{apr}_A(\boldsymbol{X}) = \boldsymbol{X}$$

$$\underline{apr}_\emptyset(\boldsymbol{X}) = \begin{cases} \boldsymbol{X} & \textit{if } \boldsymbol{X} = 2^A \\ \emptyset & \textit{otherwise} \end{cases}$$

$$\overline{apr}_\emptyset(\boldsymbol{X}) = \begin{cases} \boldsymbol{X} & \textit{if } \boldsymbol{X} = \emptyset \\ 2^A & \textit{otherwise.} \end{cases}$$

Proof: Observe that $\pi_A = \big\{ \{a\} : a \in A \big\}$ and that $\pi_\emptyset = \big\{ 2^A \big\}$. Then the claims follow almost immediately from the definitions of upper and lower approximation. ⊣

4.2 Extensions in Propositional Logic

A *propositional language* $L(A)$ consists of a set of formulas $\Phi(A)$ over some set of propositional variables A. Unless stated otherwise, we assume A to be countable. A *proper logic* for a propositional language $L(A)$ is a set of pairs of theories in $L(A)$, *i.e.*, a subset of $\Phi(A) \times \Phi(A)$.[4] For any logic Λ for $L(A)$ and any pair of theories Γ and Θ we say that Θ *logically follows from* Γ *in* Λ if $(\Gamma, \Theta) \in \Lambda$. In the sequel we will usually denote $(\Gamma, \Theta) \in \Lambda$ by $\Gamma \vdash_\Lambda \Theta$ and $(\Gamma, \Theta) \notin \Lambda$ by $\Gamma \nvdash^\Lambda \Theta$, also omitting the subscript whenever possible. At an intuitive level, it can be a help to read $\Gamma \vdash \Theta$ as signifying that some formulas in Θ hold whenever all formulas in Γ hold. We take a very liberal attitude towards what to consider a logic and impose no further restrictions.

A *valuation based semantics* (or just ‘semantics’) for a language $L(A)$ associates with each formula φ of the language a subset of 2^A which we call the *extension of* φ and denote by $[\![\varphi]\!]$. Here, 2^A is taken as the set of *valuations*, which will in the sequel frequently be referred to by S. Let $s \Vdash \varphi$ if $s \in [\![\varphi]\!]$ and $s \nVdash \varphi$ if $s \notin [\![\varphi]\!]$. The set of extensions of the formulas in a Γ we denote by $\mathscr{E}(\Gamma)$, *i.e.*, $\mathscr{E}(\Gamma) =_{df.} \big\{ [\![\gamma]\!] : \gamma \in \Gamma \big\}$. The set of all formula extensions of a language $L(A)$,

[4] We follow Segerberg (1982) in this definition and the following remarks on logics.

$\mathscr{E}(\Phi(A))$ we usually denote by simply $\mathscr{E}(A)$ or even just $\mathscr{E}$, if A is clear from the context. Let, furthermore, $[\![\Gamma]\!] =_{df.} \bigcap_{\gamma\in\Gamma} [\![\gamma]\!]$ and $\langle\!\langle\Gamma\rangle\!\rangle =_{df.} \bigcup_{\gamma\in\Gamma} [\![\gamma]\!]$. *Semantical consequence* is then defined as:

$$\Gamma \vDash \Theta \quad \text{iff} \quad [\![\Gamma]\!] \subseteq \langle\!\langle\Theta\rangle\!\rangle.$$

In a similar vein, a theory is said to be *satisfiable* if $[\![\Gamma]\!] \neq \varnothing$ and *valid* if $[\![\Gamma]\!] = 2^A$. A formula φ is *satisfiable* or *valid* if $\{\varphi\}$ is, respectively, satisfiable or valid. Any binary relation on the theories of a language $L(A)$ is said to be *sound* with respect to a logic Λ if it is a subset of Λ and *complete* whenever it is a superset of Λ.

We call a valuation-based semantics for a language $L(A)$ *finite* if for each formula φ of $L(A)$ there is a finite subset $X \subseteq_\omega A$ such that:

$$s \in [\![\varphi]\!] \text{ and } s \sim_X s' \quad \text{implies} \quad s' \in [\![\varphi]\!]$$

Intuitively, this says that whether a valuation belongs to the extension of a formula depends only on the values it assigns to a finite number of propositional variables. In terms of rough sets this means that for all formulas φ of $L(A)$ there is a finite set $X \subseteq_\omega A$ such that:

$$[\![\varphi]\!] \;=\; \overline{apr}_X([\![\varphi]\!])\,.$$

It now happens that if a semantics is finite, then for each formula φ in $L(A)$ there is a *smallest* finite set such that $[\![\varphi]\!] = \overline{apr}_X([\![\varphi]\!])$. This proposition is a corollary of the following lemma in rough set theory:

Lemma 4.2.1 *Let A be a countable set and $\boldsymbol{Z} \subseteq 2^A$. Assume further that there exists some finite $Z_0 \subseteq_\omega 2^A$ in $\boldsymbol{Z}$. Let $\boldsymbol{X} \subseteq 2^A$. Then:*

$$\overline{apr}_{Z_0}(\boldsymbol{X}) = \overline{apr}_Z(\boldsymbol{X}) \;\; \textit{for all } Z \in \boldsymbol{Z} \quad \textit{implies} \quad \overline{apr}_{Z_0}(\boldsymbol{X}) = \overline{apr}_{\bigcap \boldsymbol{Z}}(\boldsymbol{X})\,.$$

Proof: Assume for all $Z \in \boldsymbol{Z}$: $\overline{apr}_{Z_0}(\boldsymbol{X}) = \overline{apr}_Z(\boldsymbol{X})$. Since $\bigcap \boldsymbol{Z} \subseteq Z_0$, also $\overline{apr}_{Z_0}(\boldsymbol{X}) \subseteq \overline{apr}_{\bigcap \boldsymbol{Z}}(\boldsymbol{X})$, in virtue of Fact 4.1.7. So consider an arbitrary $s \in \overline{apr}_{\bigcap \boldsymbol{Z}}(\boldsymbol{X})$. We prove that $s \in \overline{apr}_{Z_0}(\boldsymbol{X})$. Then, for some $s_0 \in \boldsymbol{X}$, $s_0 \sim_{\bigcap \boldsymbol{Z}} s$. Since Z_0 is finite, so are $\bigcap \boldsymbol{Z}$ and $Z_0 - \bigcap \boldsymbol{Z}$. Let $Z_0 - \bigcap \boldsymbol{Z} = \{z_0, \ldots, z_n\}$. Observe that for each $z \in \{z_0, \ldots, z_n\}$, there is some $Z \in \boldsymbol{Z}$ such that $z \notin Z$. Assuming the axiom of choice, let $\{Z'_0, \ldots, Z'_n\} \subseteq \boldsymbol{Z}$ be such that $z_i \notin Z'_i$, for each $i \leqslant n$. For each $0 \leqslant i \leqslant n+1$, define s^*_i as follows:

$$\begin{aligned} s^*_0 \quad &=_{df.} \quad s_0 \\ s^*_{i+1} \quad &=_{df.} \quad \left(s^*_i - \{z_i\}\right) \cup \left(s \cap \{z_i\}\right) \end{aligned}$$

Since by definition $z_i \notin Z'_i$, and with s^*_i and s^*_{i+1} differing at most at z_i, it follows that for each $0 \leqslant i \leqslant n+1$, $s^*_i \sim_{Z'_i} s^*_{i+1}$. As a consequence $s^*_{i+1} \in \overline{apr}_{Z_0}(\boldsymbol{X})$, for each $i \leqslant n+1$. To appreciate this, observe that by assumption $s^*_0 \in \boldsymbol{X}$ and hence also $s^*_0 \in \overline{apr}_{Z_0}(\boldsymbol{X})$. Now assume $s^*_i \in \overline{apr}_{Z_0}(\boldsymbol{X})$. By the initial assumption $\overline{apr}_{Z_0}(\boldsymbol{X}) = \overline{apr}_{Z'_i}(\boldsymbol{X})$, so $s^*_i \in \overline{apr}_{Z'_i}(\boldsymbol{X})$. Since, moreover, $s^*_i \sim_{Z'_i} s^*_{i+1}$, $s^*_{i+1} \in \overline{apr}_{Z'_i}\left(\overline{apr}_{Z'_i}(\boldsymbol{X})\right)$. Then also $s^*_{i+1} \in \overline{apr}_{Z'_i}(\boldsymbol{X})$ and eventually $s^*_{i+1} \in \overline{apr}_{Z_0}(\boldsymbol{X})$.

We now prove by induction on i that for all $i \leqslant n+1$ (letting $\{z_k, \ldots, z_n\} = \varnothing$ if $k > n$):

$$s^*_i \sim_{Z_0 - \{z_i, \ldots, z_n\}} s.$$

For $i = 0$, recall that by assumption $s^*_0 \sim_{\bigcap \boldsymbol{Z}} s$, which is exactly what we have to prove considering that $\{z_0, \ldots, z_n\} = Z_0 - \bigcap \boldsymbol{Z}$. For the induction step, we may assume that $s^*_i \sim_{Z_0 - \{z_i, \ldots, z_n\}} s$. Now consider s_{i+1} as well as an arbitrary $z \in Z_0 - \{z_{i+1}, \ldots, z_n\}$. If $z \in Z_0 - \{z_i, \ldots, z_n\}$, just observe that $s^*_{i+1} \sim_{Z_0 - \{z_i, \ldots, z_n\}} s^*_i \sim_{Z_0 - \{z_i, \ldots, z_n\}} s$. If, however, the only remaining possibility obtains and $z = z_i$, then also $s^*_{i+1} \sim_{\{z_i\}} s$. Hence, we may conclude that $s^*_{i+1} \sim_{Z_0 - \{z_{i+1}, \ldots, z_n\}} s$.

In particular it holds that $s^*_{n+1} \in \overline{apr}_{Z_0}(\boldsymbol{X})$ and that $s^*_{n+1} \sim_{Z_0} s$. Hence, $s \in \overline{apr}_{Z_0}\big(\overline{apr}_{Z_0}(\boldsymbol{X})\big)$, which is so much as to say that $s \in \overline{apr}_{Z_0}(\boldsymbol{X})$. Wrapping things up, we recall that s had been chosen arbitrarily such that $s_0 \sim_{\bigcap \boldsymbol{Z}} s$. So, we may conclude that $\overline{apr}_{\bigcap \boldsymbol{Z}}(\boldsymbol{X}) \subseteq \overline{apr}_{Z_0}(\boldsymbol{X})$. ⊣

Observe that Lemma 4.2.1 *does not* hold in general if $\boldsymbol{Z}$ does not contain at least one finite element. For a counterexample, consider a countably infinite set A and let $\boldsymbol{X}$ be the set of infinite subsets of A. Hence $\boldsymbol{X} \neq 2^A$. Let further $a_0, \ldots, a_n, \ldots$ be an enumeration of A and set $\boldsymbol{Z} =_{df.} \big\{A - \{a_0, \ldots, a_n\} : n \in \omega\big\}$. Clearly, $\bigcap \boldsymbol{Z} = \varnothing$ and so $\overline{apr}_{\bigcap \boldsymbol{Z}}(\boldsymbol{X}) = 2^A$. However, since every $X \in \boldsymbol{X}$ has an infinite intersection with any $Z \in \boldsymbol{Z}$, we also have $\overline{apr}_Z(\boldsymbol{X}) = \overline{apr}_{Z'}(\boldsymbol{X})$ for all $Z, Z' \in \boldsymbol{Z}$. However, if $\boldsymbol{Z}$ is itself finite, the restriction of it containing a finite element can be dropped. Just recall that $\overline{apr}_Z(\overline{apr}_{Z'}(\boldsymbol{X})) = \overline{apr}_{Z \cap Z'}(\boldsymbol{X})$. We are now in a position to prove the following proposition.

Proposition 4.2.2 *For every finite semantics for $L(A)$ and each formula φ of $L(A)$ there is a (unique) smallest $X \subseteq A$ such that $[\![\varphi]\!] = \overline{apr}_X([\![\varphi]\!])$.*

Proof: Consider a finite semantics for $L(A)$ along with an arbitrary formula φ. Now consider the set $\boldsymbol{Z} =_{df.} \big\{Z \subseteq A : \overline{apr}_Z([\![\varphi]\!]) = [\![\varphi]\!]\big\}$. By finiteness, there is also a finite $Z_0 \in \boldsymbol{Z}$. Hence for all $Z \in \boldsymbol{Z}$, $\overline{apr}_Z([\![\varphi]\!]) = \overline{apr}_{Z_0}([\![\varphi]\!])$. By Lemma 4.2.1, $\overline{apr}_{\bigcap \boldsymbol{Z}}([\![\varphi]\!]) = \overline{apr}_{Z_0}([\![\varphi]\!]) = [\![\varphi]\!]$. Hence, by definition of $\boldsymbol{Z}$, also

$\bigcap \boldsymbol{Z} \in \boldsymbol{Z}$, which proves the proposition. ⊣

This result warrants the definition of $\boldsymbol{A}(\varphi)$ as the *smallest* subset of propositional variables in A for which $[\![\varphi]\!] = \overline{apr}_X([\![\varphi]\!])$. More in general we set $\boldsymbol{A}(\Gamma) =_{df.} \bigcup_{\gamma \in \Gamma} \boldsymbol{A}(\gamma)$. Moreover we employ the notation $\varphi(a_0, \ldots, a_n)$ to indicate that $[\![\varphi]\!] = \overline{apr}_{\{a_0,\ldots,a_n\}}([\![\varphi]\!])$. Observe that $\boldsymbol{A}(\Gamma)$ does not in general denote the smallest subset X of propositional variables such that $[\![\Gamma]\!] = \overline{apr}_X([\![\Gamma]\!])$. We have the following two facts.

Fact 4.2.3 *Let φ be a formula of a propositional language $L(A)$. Then for all subsets Δ of A such that $\boldsymbol{A}(\varphi) \subseteq \Delta$:*

$$\overline{apr}_\Delta([\![\varphi]\!]) \;=\; \underline{apr}_\Delta([\![\varphi]\!]) \;=\; [\![\varphi]\!].$$

Proof: Merely consider the following equalities:

$$\overline{apr}_\Delta([\![\varphi]\!]) \;=\; \overline{apr}_\Delta\left(\overline{apr}_{\boldsymbol{A}(\varphi)}([\![\varphi]\!])\right) \;=_{\text{Fact 4.1.8}}$$
$$\overline{apr}_{\Delta \cap \boldsymbol{A}(\varphi)}([\![\varphi]\!]) \;=_{\boldsymbol{A}(\varphi) \subseteq \Delta}\; \overline{apr}_{\boldsymbol{A}(\varphi)}([\![\varphi]\!]) \;=\; [\![\varphi]\!].$$

That then also $\underline{apr}_\Delta([\![\varphi]\!]) = [\![\varphi]\!]$, follows immediately from Fact 4.1.1. ⊣

Fact 4.2.4 *Let Γ be a theory in $L(A)$ and Δ a subset of A such that $\boldsymbol{A}(\Gamma) \subseteq \Delta$. Then:*

$$\underline{apr}_\Delta([\![\Gamma]\!]) \;=\; \overline{apr}_\Delta([\![\Gamma]\!]) \;=\; [\![\Gamma]\!].$$
$$\underline{apr}_\Delta(\langle\!\langle\Gamma\rangle\!\rangle) \;=\; \overline{apr}_\Delta(\langle\!\langle\Gamma\rangle\!\rangle) \;=\; \langle\!\langle\Gamma\rangle\!\rangle.$$

Proof: Consider the following equalities:

$$\underline{apr}_\Delta([\![\Gamma]\!]) \;=\; \bigcap_{\gamma\in\Gamma} \underline{apr}_\Delta([\![\gamma]\!]) \;=_{\text{Fact 4.2.3}}\; \bigcap_{\gamma\in\Gamma} [\![\gamma]\!] \;=\; [\![\Gamma]\!],$$
$$\overline{apr}_\Delta([\![\Gamma]\!]) \;=\; \bigcup_{\gamma\in\Gamma} \overline{apr}_\Delta([\![\gamma]\!]) \;=_{\text{Fact 4.2.3}}\; \bigcup_{\gamma\in\Gamma} [\![\gamma]\!] \;=\; [\![\Gamma]\!].$$

That also $\overline{apr}_\Delta([\![\Gamma]\!]) = [\![\Gamma]\!]$ and $\underline{apr}_\Delta([\![\Gamma]\!]) = [\![\Gamma]\!]$ then follows by Fact 4.1.1. ⊣

A *classical propositional language* $L(A)$ over a set of propositional variables A is a minimal set containing A as well as $\bot$ and for each formula φ contains another formula denoted by $(\neg\varphi)$ as well as for each pair of formulas φ and ψ a formula denoted by $(\varphi \vee \psi)$ and allows for a classical semantics. A *classical semantics* for a classical language $L(A)$ is such that for each propositional variable $a \in A$ and all formulas φ and ψ:

$$\begin{aligned} [\![a]\!] &= \{s \in S : \ a \in s\} \\ [\![\bot]\!] &= \emptyset \\ [\![(\neg\varphi)]\!] &= S - [\![\varphi]\!] \\ [\![(\varphi \vee \psi)]\!] &= [\![\varphi]\!] \cup [\![\psi]\!] . \end{aligned}$$

The resulting logic we will refer to as *classical propositional logic* (CPC). Where possible we omit parentheses. We also have the usual abbreviations $\top$, $\varphi \wedge \psi$, $\varphi \to \psi$, $\bigwedge \Theta$ and $\bigvee \Theta$, where φ and ψ are formulas of $L(A)$ and Θ a finite and possibly empty sequences of formulas in $L(A)$. For each formula φ, the set of propositional variables *occurring* in φ is defined as usual and depicted by $A(\varphi)$. We use $A(\Gamma)$ to denote $\bigcup_{\gamma \in \Gamma} A(\gamma)$. Classical semantics is finite in the sense that for each formula φ there is a finite set Δ such that $[\![\varphi]\!] = \overline{apr}_\Delta([\![\varphi]\!])$.

Fact 4.2.5 *Classical semantics is finite.*

Proof: Consider an arbitrary classical propositional language $L(A)$ along with an equally arbitrary formula φ of $L(A)$. The proof is then by induction on φ.

First assume $\varphi = a$. Obviously $[\![a]\!] \subseteq \overline{apr}_{\{a\}}([\![a]\!])$; so, it suffices to show that $\overline{apr}_{\{a\}}([\![a]\!]) \subseteq [\![a]\!]$. Observe that for any $s \in \overline{apr}_{\{a\}}([\![a]\!])$, there is some $s' \in [\![a]\!]$ such that $s \sim_{\{a\}} s'$. Since by definition $a \in s'$, also $a \in s$. Hence, $s \in [\![a]\!]$.

Let $\varphi = \neg\psi$. In virtue of the induction hypothesis we may assume there to be a finite $\Delta \subseteq_\omega A$ such that $[\![\psi]\!] = \overline{apr}_\Delta([\![\psi]\!])$. Now consider the following equalities:

$$[\![\neg\psi]\!] \quad = \quad \overline{[\![\psi]\!]} \quad =_{\text{i.h.}} \quad \overline{\overline{apr}_\Delta([\![\psi]\!])} \quad = \quad \overline{\overline{\underline{apr}_\Delta\left(\overline{[\![\psi]\!]}\right)}} \quad =_{\text{i.h.}} \quad \underline{apr}_\Delta([\![\neg\psi]\!]) \, .$$

With Fact 4.1.1, we may conclude that $[\![\neg\psi]\!] = \overline{apr}_\Delta(\neg\psi)$.

In case $\varphi = \psi \vee \psi$, the induction hypothesis grants us there to be finite $\Delta, \Delta' \subseteq A$ such that $[\![\psi]\!] = \overline{apr}_\Delta([\![\psi]\!])$ and $[\![\chi]\!] = \overline{apr}_{\Delta'}([\![\chi]\!])$. Now consider the following equalities:

$$\begin{aligned}
\overline{apr}_{\Delta\cup\Delta'}(\llbracket\psi\vee\chi\rrbracket) &= \overline{apr}_{\Delta\cup\Delta'}(\llbracket\varphi\rrbracket\cup\llbracket\psi\rrbracket)\\
&= \overline{apr}_{\Delta\cup\Delta'}(\llbracket\psi\rrbracket)\cup\overline{apr}_{\Delta\cup\Delta'}(\llbracket\chi\rrbracket)\\
&=_{\text{i.h.}} \overline{apr}_{\Delta\cup\Delta'}(\overline{apr}_{\Delta}(\llbracket\psi\rrbracket))\cup\overline{apr}_{\Delta\cup\Delta'}(\overline{apr}_{\Delta'}(\llbracket\chi\rrbracket))\\
&= \overline{apr}_{(\Delta\cup\Delta')\cap\Delta}(\llbracket\psi\rrbracket)\cup\overline{apr}_{(\Delta\cup\Delta')\cap\Delta'}(\llbracket\chi\rrbracket)\\
&= \overline{apr}_{\Delta}(\llbracket\psi\rrbracket)\cup\overline{apr}_{\Delta'}(\llbracket\chi\rrbracket)\\
&=_{\text{i.h.}} \llbracket\psi\rrbracket\cup\llbracket\chi\rrbracket\\
&= \llbracket\psi\vee\chi\rrbracket.
\end{aligned}$$

This concludes the proof. ⊣

For classical propositional logic we have in general $\boldsymbol{A}(\varphi)\subseteq A(\varphi)$. Hence the following fact.

Fact 4.2.6 *Let $L(A)$ be a classical propositional language, $\Delta\subseteq A$ and φ a formula of $L(A)$. Then:*

$$\begin{aligned}
\underline{apr}_{\Delta}(\llbracket\varphi\rrbracket) &= \underline{apr}_{\Delta\cap\boldsymbol{A}(\varphi)}(\llbracket\varphi\rrbracket) = \underline{apr}_{\Delta\cap A(\varphi)}(\llbracket\varphi\rrbracket)\\
\overline{apr}_{\Delta}(\llbracket\varphi\rrbracket) &= \overline{apr}_{\Delta\cap\boldsymbol{A}(\varphi)}(\llbracket\varphi\rrbracket) = \overline{apr}_{\Delta\cap A(\varphi)}(\llbracket\varphi\rrbracket)
\end{aligned}$$

Proof: Immediately from the Facts 4.1.8 and Fact 4.2.3 and the fact that for classical propositional in general $\boldsymbol{A}(\varphi)\subseteq A(\varphi)$. ⊣

In the previous section we argued that $\overline{apr}_{\pi}$ can be construed as a cylindrification operation and $\underline{apr}_{\pi}$ as its dual. In first-order logic the quantifiers can be thought of in a similar manner. Quantification increases the expressive power of first-order logic tremendously and is responsible for Church's famous undecidability result. These phenomena connected with cylindrification, however, do not occur at all in classical propositional logic. As a matter of fact, the set of extensions $\mathscr{E}$ in classical propositional logic is closed under taking lower and upper approximations. Hence for each subset X of propositional variables we may assume the existence of formulas $\langle X\rangle\,\varphi$ and $[X]\,\varphi$ with the respective extensions $\overline{apr}_X(\llbracket\varphi\rrbracket)$ and $\underline{apr}_X(\llbracket\varphi\rrbracket)$. This we prove. Moreover we characterize the set of extensions of a classical propositional language as the set of fixed points of all operations $\overline{apr}_X$ and $\underline{apr}_X$ with X finite. Recall that $\mathscr{E}(A)$ denotes the set $\{\llbracket\varphi\rrbracket:\ \varphi\in\Phi(A)\}$.

Theorem 4.2.7 *Let $L(A)$ be a classical propositional language with a classical semantics and let $Fix\,(\overline{apr}_B) =_{df.} \{X \subseteq S : \; X = \overline{apr}_B(X)\}$. Then:*

$$\mathscr{E} \;=\; \bigcup_{B \subseteq_\omega A} Fix\,(\overline{apr}_B)\,.$$

Proof: The left-to-right direction is immediate by finiteness of classical semantics.

For the opposite direction consider an arbitrary $X \subseteq S$ such that $X = \overline{apr}_B(X)$, for some finite $B \subseteq A$. Now define for each $s \subseteq B$:

$$\beta_s \;\;=_{df.}\;\; \bigwedge \{b, \neg b' : \; b, b' \in B \text{ and } b \in s \text{ and } b' \notin s\}.$$

Obviously, $s \Vdash \beta_s$, for each $s \subseteq B$. Now set:

$$\beta \;\;=_{df.}\;\; \bigvee \{\beta_s : \; s \subseteq B \text{ and } s \in X\}.$$

We prove that $X = [\![\beta]\!]$. First assume, for an arbitrary $s \in S$, that $s \in [\![\beta]\!]$. Then $s \Vdash \beta_{s'}$, for some $s' \in X$ with $s' \subseteq B$. Some reflection reveals that $s \sim_B s'$ and subsequently $s \in \overline{apr}_A(X)$. With the assumption that $X = \overline{apr}_B(X)$, the latter is equivalent with $s \in X$.

Conversely, assume $s \in X$. Define $s^* =_{df.} s \cap B$; then $s^* \Vdash \beta_{s^*}$. Since, moreover, $s \sim_B s^*$ and β_{s^*} only depends on B we also have that $s \Vdash \beta_{s^*}$. It is equally clear that $s^* \subseteq B$. Since $s^* \sim_B s$, also $s^* \in \overline{apr}_B(X)$, *i.e.*, $s^* \in X$ by the assumption. *A fortiori*, also $s \Vdash \beta$, and we may conclude that $s \in [\![\beta]\!]$. ⊣

This result is very close to the more syntactically flavored fact of classical propositional logic that each formula is equivalent to a complete disjunctive normal form. Observe that if Δ comprises precisely the propositional variables occurring in a formula φ, then each disjunct of its complete disjunctive normal form characterizes a block in the partition π_Δ.

Corollary 4.2.8 *Let $L(A)$ be a classical propositional language. Then:*

$$\mathscr{E} \;\;=\;\; \{\overline{apr}_B(X) : \; B \subseteq_\omega A \;\; and \;\; X \subseteq 2^A\}.$$

Proof: The inclusion of $\mathscr{E}$ in $\{\overline{apr}_B(X) : \; B \subseteq_\omega A \text{ and } X \subseteq 2^A\}$ is an immediate consequence of Theorem 4.2.7. For the opposite inclusion just observe that $\overline{apr}\,(\overline{apr}\,(X)) = \overline{apr}\,(X)$ is a law of rough set theory and again Theorem 4.2.7. ⊣

This corollary establishes classical propositional logic as the most expressive one with a finite semantics, in the sense that the extensions of its formulas exhaust the set of valuations that can be finitely approximated.

So, the set of extensions $\mathscr{E}(A)$ of formulas of a propositional language $L(A)$ is given by the fixed points of the approximation operations $\overline{apr}_B$ on sets of valuations with B finite. Obviously, $\mathscr{E}(A)$ does not exhaust in general the powerset of valuations 2^A. If A is countably infinite, so is the set of formulas of $L(A)$. The set of valuations, not to mention the set of sets of valuations, however, is uncountably infinite and so there can impossibly be a formula for each subset of valuations, or even for each valuation.

In virtue of Theorem 4.2.7 and Corollary 4.2.8 we also have that the set of extensions is closed under taking approximations. Hence the following fact.

Fact 4.2.9 *Let $L(A)$ be a classical propositional language with a classical semantics. Then for all $B \subseteq A$:*

$$X \in \mathscr{E} \quad \textit{implies} \quad \overline{apr}_B(X) \in \mathscr{E}$$

Proof: At page 25 it was stated as a law of rough set theory that for any $B, C \subseteq A$, $\overline{apr}_B(\overline{apr}_C(X)) = \overline{apr}_{B\cap C}(X)$. Now consider an arbitrary $X \in \mathscr{E}$. By Theorem 4.2.7 there is a finite $C \subseteq A$ such that $X = \overline{apr}_C(X)$. Now consider an arbitrary $B \subseteq A$ and reason as follows:

$$\overline{apr}_B(X) \quad = \quad \overline{apr}_B(\overline{apr}_C(X)) \quad = \quad \overline{apr}_{B\cap C}(X)\,.$$

Since obviously $B \cap C$ is finite, with Corollary 4.2.8, $\overline{apr}_{B\cap C}(X) \in \mathscr{E}$ and we may conclude that $\overline{apr}_B(X) \in \mathscr{E}$. $\dashv$

The set of theories of a propositional language $L(A)$ will be uncountable if A is countably infinite. Nevertheless, there will still be subsets of valuations that are not the extension of some theory. For an example consider $2^A - \{\varnothing\}$. For a *reductio ad absurdum* assume that $[\![\Gamma]\!] = 2^A - \{\varnothing\}$. Then there is at least one γ in Γ such that $[\![\gamma]\!]$ does not contain the empty set $\varnothing$. In virtue of Corollary 4.2.8, there is a finite subset B of propositional variables and some subset of valuations X such that $[\![\gamma]\!] = \overline{apr}_B(X)$. Now consider the *valuation* $\overline{B}$. Observe that with B finite and A infinite, $\overline{B}$ is not empty. Hence, $\overline{B} \in [\![\Gamma]\!]$ and *a fortiori* also $\overline{B} \in [\![\gamma]\!]$. Therefore, $\overline{B} \in \overline{apr}_B(X)$. Evidently, $\overline{B} \sim_B \varnothing$ and so $\varnothing \in \overline{apr}_B(\overline{apr}_B(X)) = \overline{apr}_{B\cap B}(X) = \overline{apr}_B(X) = [\![\gamma]\!]$, which is at variance with the assumption that $\varnothing \notin [\![\gamma]\!]$.

4.3 Characterizing Set Induced Relations

In purpose of this section is to demarcate the classes of relations over the valuations as defined by formulas and theories.

The relations $\rho(X)$ as defined by subsets of a set S over S have a neat characterization in terms of general properties of relations. We say that a relation ρ on a set S is *bisective* if it is transitive and moreover satisfies the following condition:

$$(*) \qquad \text{for all } x, x', x'' \in S: \quad x \leqslant x' \ \text{ implies } \ x'' \leqslant x \text{ or } x' \leqslant x''.$$

Observe that the empty relation qualifies as bisective, as in that case both transitivity and $(*)$ are satisfied trivially in virtue of vacuous quantification. Any other bisective relation, however, is both reflexive in addition to being transitive.

Fact 4.3.1 *Any non-empty bisective relation over a set S is reflexive.*

Proof: Let ρ be a non-empty bisective relation over a set S. Then $x \leqslant x'$, for some $x, x' \in S$. Consider an arbitrary y in S. In virtue of ρ satisfying $(*)$, then, $y \leqslant x$ or $x' \leqslant y$. In both cases the reasoning runs along similar lines; here we deal with the former case only. If $y \leqslant x$, again in virtue of $(*)$, either $y \leqslant y$ or $x \leqslant y$. If $y \leqslant y$ we are done immediately. Otherwise, we have $y \leqslant x \leqslant y$ and by transitivity of ρ also $y \leqslant y$. ⊣

The following proposition characterizes a bisective relation over a universe S as one which coincides with $\rho(X)$ for some subset X of S. In its proof, as elsewhere in this section, $\uparrow_\rho x$ denotes the set $\{y \in S: \ (x,y) \in \rho\}$, for all elements x of and all relations ρ on a set S.

Proposition 4.3.2 *Let S be a set. Then the set $\left\{\rho(X) \subseteq S \times S: \ X \subseteq S\right\}$ coincides with the set of bisective relations on S.*

Proof: In case S is empty, the empty relation is the only (bisective) relation on S. Also, $\varnothing$ is the only subset of S. Now, observe that $\rho(\varnothing)$ is the empty relation as well. So, for the remainder of the proof we may assume S to be non-empty.

First consider an arbitrary $X \subseteq S$ along with equally arbitrary $x, x', x'' \in S$. We prove that $\rho(X)$ is bisective. For transitivity first assume that both (x,x') and (x',x'') are in $\rho(X)$ as well as that $x \in X$. Since $(x,x') \in \rho(X)$, also $x' \in X$ and because $(x',x'') \in \rho(X)$, moreover, $x'' \in X$. We may conclude that $(x,x'') \in \rho(X)$. To show that $\rho(X)$ satisfies condition $(*)$ as well, assume $(x,x') \in \rho(X)$. Either $x'' \in X$ or $x'' \notin X$. If the former, $(x',x'') \in \rho(X)$; if the latter $(x'',x) \in \rho(X)$. In both cases we are done.

To prove that for an arbitrary bisective relation ρ on S there is a subset X such that $\rho = \rho(X)$, assume ρ to be bisective and consider the set $\bigcap_{x \in S} \uparrow_\rho x$. Suppressing the subscript ρ in $\uparrow_\rho x$, we prove that $\rho = \rho\left(\bigcap_{x \in S} \uparrow x\right)$. In case ρ is empty, $\uparrow x$

is equally empty, for any $x \in S$. Having assumed S to be non-empty, $\bigcap_{x\in S} \uparrow x = \emptyset$. Hence, $\rho\left(\bigcap_{x\in S} \uparrow x\right) = \rho(\emptyset) = \emptyset$. For the remainder of the proof, we may accordingly assume ρ to be non-empty. In virtue of Fact 4.3.1, the relation ρ may be assumed to be reflexive as well.

For the $\subseteq$-inclusion, assume for arbitrary $y, y' \in S$ that $(y, y') \in \rho$. Assume further that $y \in \bigcap_{x\in S} \uparrow x$ and consider an arbitrary $x \in S$. Then $y \in \uparrow x$, *i.e.*, $(x, y) \in \rho$. By transitivity, also $(x, y') \in \rho$, *i.e.*, $y' \in \uparrow x$. With x having been chosen arbitrarily, $y \in \bigcap_{x\in S} \uparrow x$, and we are done.

For the $\supseteq$-inclusion, assume for arbitrary $y, y' \in S$ that $(y, y') \notin \rho$. Then, $y' \notin \uparrow y$ and, therefore, $y' \notin \bigcap_{x\in S} \uparrow x$. It suffices now to prove that $y \in \bigcap_{x\in S} \uparrow x$. So, consider an arbitrary $x \in S$; we prove that $y \in \uparrow x$. By reflexivity, $(y, y) \in \rho$. In virtue of $(y, y') \notin \rho$ and $(*)$, then $(y', y) \in \rho$. Again because of $(*)$, either (x, y') or $(y, x) \in \rho$. In the former case, $(x, y) \in \rho$ since ρ is transitive and $(y', y) \in \rho$. Also in the latter case we have $(x, y) \in \rho$, because $(y, y') \notin \rho$ and $(*)$. With x having been chosen arbitrarily, $y \in \bigcap_{x\in S} \uparrow x$. We may conclude that $(y, y') \notin \rho\left(\bigcap_{x\in S} \uparrow x\right)$. $\dashv$

For a classical propositional language with a countably infinite number of propositional variables, the relations $\rho(\varphi)$ for formulas φ of the language, exhaust the set of bisective relations on the valuations just as little as the extensions of the formulas exhaust the set of subsets of the valuations. Recall that the number of relations over the valuations of a countably infinite propositional language is uncountable, whereas the number of formulas remains countable. Corollary 4.2.8 on page 33 characterizes the set of extensions of a language as the approximations of the subsets of valuations by means of a finite subset of propositional variables. Proposition 4.3.5, gives a similar result for the relational semantics for classical propositional logic. Before getting there, however, we make some more general remarks concerning approximations of relations.

The approximation operators $\overline{apr}$ and $\underline{apr}$ on the powerset of some set S are relative to an equivalence relation ε on S. The coordinate-wise square[5] of an equivalence relation over S is again an equivalence relation over the Cartesian product of S. The coordinate-wise square of ε — denoted by $\varepsilon \otimes \varepsilon$ — can in turn be used to

[5]In Preliminaries. Let $\{S_i\}_{i\in I}$ be a family of sets. Let further for each $i \in I$, ρ_i be a relation on S_i. We define the *coordinate-wise product,* or *the product relation, of* $\{\rho_i\}_{i\in I}$ as the relation ρ^* on the generalized Cartesian order over the S_i such that for all $\vec{x}, \vec{y} \in \prod_{i\in I} S_i$:

$$(\vec{x}, \vec{y}) \in \rho^* \quad \text{iff} \quad \text{for all } i \in I: (x_i, y_i) \in \rho_i.$$

The *coordinate-wise square* $\rho \otimes \rho$ of a relation ρ on S is the coordinate-wise product relation of ρ with itself.

approximate *relations* on S by means of rough sets. Thus, we have for ρ a relation over a set S:

$(x, x') \in \overline{apr}_{\varepsilon\otimes\varepsilon}(\rho)$

iff for some $(y, y') \in S \times S$: $\big((x, x'), (y, y')\big) \in \varepsilon \otimes \varepsilon$ and $(y, y') \in \rho$

iff for some $y, y' \in S$: $(x, y), (x', y') \in \varepsilon$ and $(y, y') \in \rho$.

When no confusion is likely, we will denote the approximation operations on the relations of S relative to the squares of an equivalence relation ε_X also by $\overline{apr}_X$ and $\underline{apr}_X$. *I.e.*, we will write $\overline{apr}_X(\rho)$ for $\overline{apr}_{\varepsilon_X\otimes\varepsilon_X}(\rho)$.

The approximation operation $\overline{apr}_X$ on relations does not in general preserve transitivity. For a counterexample one consider a base set of three elements $\{a, b, c\}$. Let ρ be the smallest reflexive transitive relation on $2^{\{a,b,c\}}$ containing $(\varnothing, \{a\})$ and $(\{a, c\}, \{a, b\})$. Transitivity fails for the relation $\overline{apr}_{\{a,b\}}(\rho)$. Observe in this respect that both $(\varnothing, \{a, c\})$ and $(\{a, c\}, \{a, b\})$ are in $\overline{apr}_{\{a,b\}}(\rho)$. The latter because $\rho \subseteq \overline{apr}_{\{a,b\}}(\rho)$. For the former, observe that both $\varnothing \sim_{\{a,b\}}$ and $\{a\} \sim_{\{a,b\}} \{a, c\}$. As a consequence also $\big((\varnothing, \{a\}), (\varnothing, \{a, c\})\big) \in \varepsilon_{\{a,c\}} \otimes \varepsilon_{\{a,c\}}$. Nevertheless, $\big(\varnothing, \{a, b\}\big) \notin \overline{apr}_{\{a,b\}}(\rho)$. In a similar fashion it can be shown that the upper approximation operation does not preserve reflexivity.

Since, every relation $\rho(X)$ *is* transitive, witness Proposition 4.3.2, the set of bisective relations is not closed under taking approximations. The following proposition, however, establishes a general connection between bisective relations and their approximations.

Proposition 4.3.3 *Let ε be an equivalence relation on some set S and $\varepsilon \otimes \varepsilon$ its coordinatewise square. Then for each $X \subseteq S$:*

$$\rho\big(\overline{apr}_{\varepsilon}(X)\big) \subseteq \overline{apr}_{\varepsilon\otimes\varepsilon}\big(\rho(X)\big)$$

Proof: In case X is empty, in general, both $\overline{apr}(X) = \overline{apr}(\varnothing) = \varnothing$ and $\rho(\overline{apr}(X)) = \rho(\overline{apr}(\varnothing)) = \rho(\varnothing) = \varnothing$. Therefore, in particular, both $\rho\big(\overline{apr}_{\varepsilon}(X)\big)$ and $\overline{apr}_{\varepsilon\otimes\varepsilon}\big(\rho(X)\big)$ are empty, and we are done immediately. Hence, for the remainder of the proof we may assume X to be non-empty.

Assume $(x, x') \in \rho(\overline{apr}_{\varepsilon}(X))$. If also $(x, x') \in \rho(X)$, then $(x, x') \in \overline{apr}_{\varepsilon\otimes\varepsilon}(\rho(X))$ follows immediately. So, assume that $(x, x') \notin \rho(X)$. Then $x \in X$ and $x' \notin X$. Hence, $x \in \overline{apr}_{\varepsilon}(X)$, and having assumed that $(x, x') \in \rho(\overline{apr}_{\varepsilon}(X))$, also $x' \in \overline{apr}_{\varepsilon}(X)$. Consequently there is an $x'' \in X$ such that $x' \sim_{\varepsilon} x''$. Then also $(x, x'') \in \rho(X)$. Since, both $x \sim x$ and $x' \sim x''$, that $(x, x') \in \overline{apr}_{\varepsilon\otimes\varepsilon}(\rho(X))$ follows, and we are done. $\dashv$

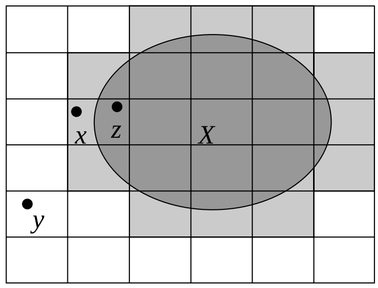

Figure 4.2. Counterexample against $\overline{apr}_{\varepsilon\otimes\varepsilon}\left(\rho\,(X)\right) \subseteq \rho\left(\overline{apr}_{\varepsilon}(X)\right)$. Let the partition be given by ε. Because $(x,y) \in \rho\,(X)$, immediately also $(x,y) \in \overline{apr}_{\varepsilon}(\rho\,(X))$. However, $x \in \overline{apr}_{\varepsilon}(X)$ and $y \notin \overline{apr}_{\varepsilon}(X)$. Therefore, $(x,y) \notin \rho\,(\overline{apr}_{\varepsilon}(X))$.

The opposite inclusion, however, does not hold in general. For a counterexample, consider a situation as pictured in Figure 4.2, in which there is a subset X of S and an equivalence relation ε such that there are some elements x and y of S such that neither x nor y are in X. Let there further be an equivalence relation ε with $(x,z) \in \varepsilon$ for some z in X and $(y,z) \in \varepsilon$ for *no* z in X. Accordingly, $x \in \overline{apr}_{\varepsilon}(X)$ and $Y \notin \overline{apr}_{\varepsilon}(X)$. Then, $(x,y) \in \overline{apr}_{\varepsilon\otimes\varepsilon}(\rho\,(X))$, because $(x,y) \in \rho\,(X)$ and $\varepsilon \otimes \varepsilon$ is reflexive. However, x is in $\overline{apr}_{\varepsilon}(X)$, whereas y is not and, therefore, $(x,y) \notin \rho\,(\overline{apr}_{\varepsilon}(X))$.

We say a relation ρ on 2^A is *of finite character* if and only if ρ is a fixed point of the operation $\overline{apr}_X$ on relations for some *finite* subset X of A, *i.e.*, if there is some finite $X \subseteq A$ such that $\rho = \overline{apr}_X(\rho)$. We find that the bisective relations of finite character on the set of valuations of a classical propositional language $L(A)$ coincide with the relations $\rho\,(\varphi)$, for formulas φ of $L(A)$. In analogy with Theorem 4.2.7 on page 33 we have the following theorem. Recall that $\mathscr{R}\,(A)$ denotes $\left\{\rho\,(\varphi) : \ \varphi \text{ a formula in } L(A)\right\}$.

Theorem 4.3.4 *Let $L(A)$ be a propositional language with A as propositional variables and S denote 2^A. For each finite subset $B \subseteq_\omega A$, let further $Fix\,(\overline{apr}_B)$ be defined as the set $\left\{\rho \in S \times S : \ \rho \text{ is bisective and } \rho = \overline{apr}_B(\rho)\,\right\}$. Then:*

$$\mathscr{R}\,(A) \quad = \quad \bigcup_{B\subseteq_\omega A} Fix\,(\overline{apr}_B)\,.$$

Proof: First consider an arbitrary relation ρ in $\mathscr{R}\,(A)$. Then there is some formula φ of $L(A)$ such that $\rho = \rho\,(\varphi)$. By Proposition 4.3.2, $\rho\,(\varphi)$ is bisective; we show that $\overline{apr}_{\boldsymbol{A}(\varphi)}\big(\rho(\varphi)\big) = \rho(\varphi)$ proving that it is of finite character as well. As $\rho(\varphi) \subseteq \overline{apr}_{\boldsymbol{A}(\varphi)}\big(\rho(\varphi)\big)$ is immediate, assume $(x,y) \in \overline{apr}_{\boldsymbol{A}(\varphi)}\big(\rho(\varphi)\big)$. Then, there are $x',y' \in 2^A$ such that $x \sim_{\boldsymbol{A}(\varphi)} x'$, $y \sim_{\boldsymbol{A}(\varphi)} y'$ and $(x',y') \in \rho(\varphi)$. Now assume $x \in [\![\varphi]\!]$. Then, $x' \in \overline{apr}_{\boldsymbol{A}(\varphi)}([\![\varphi]\!]) = [\![\varphi]\!]$. It follows that $y' \in [\![\varphi]\!]$ as well, and, hence, $y \in \overline{apr}_{\boldsymbol{A}(\varphi)}([\![\varphi]\!]) = [\![\varphi]\!]$. Therefore, $(x,y) \in \rho(\varphi)$.

For the opposite direction, consider an arbitrary bisective relation ρ of finite character. We may thus assume there to be a finite subset $B \subseteq_\omega A$ such that $\rho = \overline{apr}_B(\rho)$ as well as a subset $X \subseteq 2^A$ such that $\rho = \rho(X)$. If X is empty, we have $\rho = \rho(\varnothing) = \varnothing = \rho(\bot)$. So, for the remainder of the proof, we may assume X to be non-empty. We prove that $X = \overline{apr}_B(X)$. Then $\rho(X) = \rho(\overline{apr}_B(X))$. In virtue of Corollary 4.2.8 on page 33 and B being finite, there is also a formula φ such that $[\![\varphi]\!] = \overline{apr}_B(X)$ and, hence, $\rho(X) = \rho(\overline{apr}_B(X)) = \rho(\varphi)$.

Because $X \subseteq \overline{apr}_B(X)$ is immediate, we assume $s \in \overline{apr}_B(X)$, for an arbitrary s and prove that $s \in X$. Then there is some s' such that $s \sim_B s'$ and $s' \in X$. It follows that $(s,s') \in \rho(X)$. Moreover, since $s \sim_B s'$ and, trivially, both $s' \sim_B s'$ and $(s',s') \in \rho(X)$, also $(s',s) \in \overline{apr}_B(\rho(X))$. By the initial assumptions, $\overline{apr}_B(\rho(X)) = \rho(X)$ and therefore $(s',s) \in \rho(X)$. With $s' \in X$, finally, we may conclude that $s \in X$ as well. ⊣

The following corollary has a certain likeness with Corollary 4.2.8 on page 33 above, which characterized the extensions of the formulas of $L(A)$ in a much similar way.

Corollary 4.3.5 *Let $L(A)$ be a classical propositional language. Then $\mathscr{R}(A)$ coincides with the set of bisective relations of finite character on 2^A, the set of valuations, i.e.:*

$$\mathscr{R}(A) \quad = \quad \left\{ \overline{apr}_B(\rho(X)) : \; X \subseteq 2^A \textit{ and } B \subseteq_\omega A \right\}.$$

Proof: The inclusion of $\mathscr{R}(A)$ in $\left\{ \overline{apr}_B(\rho(X)) : \; X \subseteq 2^A \text{ and } B \subseteq_\omega A \right\}$ follows immediate from Theorem 4.3.4. The inclusion in the opposite direction is an immediate consequence of $\overline{apr}_B(\rho(X)) = \overline{apr}_B(\overline{apr}_B(\rho(X)))$, which is an instance of a rough set law, and again Theorem 4.3.4. ⊣

As another corollary we find that, for propositional languages $L(A)$ on a *finite* set of propositional variables, $\mathscr{R}(A)$ is complete with respect to all bisective relations on 2^A.

Corollary 4.3.6 *Let A be a finite set of propositional variables on which $L(A)$ is defined. Then, $\mathscr{R}(A)$ is complete with respect to the bisective relations on 2^A.*

Proof: Immediate Theorem 4.3.5. ⊣

The relation a *theory* defines over the valuations, however, can be characterized as the limit of the *finite* approximations of a proto-order (*i.e.*, an empty or reflexive and transitive relation) over the valuations. With a finite approximation of a relation over the valuations we mean here the approximation of that relation relative to the equivalence relation defined over the valuations by a finite set of propositional variables, ie, relative to the coordinate square of a relation ε_B where B is a finite subset of propositional variables. First, we prove two preliminary facts.

Fact 4.3.7 *Let ρ be a proto-order over a set S. Then,* $\rho = \boldsymbol{\rho}\left(\left\{\uparrow_\rho x : \ x \in S\right\}\right)$.

Proof: First assume that ρ be empty. Then, $\uparrow_\rho x = \varnothing$, for each $x \in S$. Hence, $\boldsymbol{\rho}\left(\left\{\uparrow_\rho x : \ x \in S\right\}\right) = \{\varnothing\}$ and $\boldsymbol{\rho}(\{\varnothing\}) = \rho(\varnothing) = \varnothing$. So, for the remainder of the proof we may assume ρ to be both reflexive and transitive.

First assume that $(y, y') \in \boldsymbol{\rho}(\{\uparrow x : \ x \in S\})$, for arbitrary $y, y' \in S$. Then, $y \in \uparrow x$ implies $y' \in \uparrow x$, for all $x \in S$. By reflexivity of ρ, we have $(y, y) \in \rho$, *i.e.*, $y \in \uparrow y$. With $y \in S$, then also $y' \in \uparrow y$, *i.e.*, $(y, y') \in \rho$.

For the opposite direction, assume that $(y, y') \in \rho$ as well as that $y \in \uparrow x$, for an arbitrary $x \in S$. Then, $(x, y) \in \rho$. By transitivity then also $(x, y') \in \rho$, *i.e.*, $y' \in \uparrow x$. Therefore, $(y, y') \in \rho(\uparrow x)$, and with x having been chosen arbitrarily, eventually, $(y, y') \in \boldsymbol{\rho}(\{\uparrow x : \ x \in S\})$. ⊣

This fact has the following corollary, which says that the class consisting of the reflexive and transitive relations over a set together with the empty relation can be characterized as intersections of bisective relations.

Corollary 4.3.8 *Let ρ be a relation over some set S. Then:*

$$\rho \textit{ is a proto-order} \quad \textit{iff} \quad \textit{for some } \boldsymbol{X} \subseteq 2^S, \ \ \rho = \boldsymbol{\rho}(\boldsymbol{X}).$$

Proof: The left-to-right direction is immediate by Fact 4.3.7. For the opposite direction, assume $\rho = \boldsymbol{\rho}(\boldsymbol{X})$, for some $\boldsymbol{X} \subseteq 2^S$. If $\boldsymbol{X}$ contains the empty set $\varnothing$, then $\left\{\rho(X) : \ X \in \boldsymbol{X}\right\}$ contains $\rho(\varnothing)$, *i.e.*, the empty relation. Hence, $\boldsymbol{\rho}(\boldsymbol{X}) = \bigcap_{X \in \boldsymbol{X}} \rho(X) = \varnothing$ and by the initial assumption, ρ is empty as well. So, henceforth $\boldsymbol{X}$ may be assumed not to contain the empty set. By Fact 4.3.1, $\rho(X)$ is both reflexive and transitive, for each $X \in \boldsymbol{X}$. An easy check then establishes that $\boldsymbol{\rho}(\boldsymbol{X}) = \bigcap_{X \in \boldsymbol{X}} \rho(X)$ is reflexive and transitive as well. ⊣

This result has as an immediate corollary that for a language $L(A)$ on a *finite* set of propositional variables, the class consisting of the relations $\boldsymbol{\rho}(\Gamma)$ for all theories Γ

in $L(A)$ is also complete with respect to the reflexive and transitive and otherwise empty relations over the valuations.

Corollary 4.3.9 *Let $L(A)$ be propositional language on a* finite *set A of propositional variables and let ρ be a relation over 2^A. Then:*

$$\rho \textit{ is a proto-order} \quad \textit{iff} \quad \textit{for some } \Gamma \textit{ of } L(A), \quad \rho = \boldsymbol{\rho}\,(\Gamma)\,.$$

Proof: Immediately by the Corollaries 4.3.6 and 4.3.9. $\dashv$

The relations $\boldsymbol{\rho}\,(\Gamma)$, as defined by the theories of a language $L(A)$, however fail to exhaust the set of reflexive and transitive, or otherwise empty relations over the valuations of $L(A)$, if the set A of propositional variables is infinite. For, in any such case, the subsets of valuations of language $L(A)$ outnumber its formulas and for some $X \subseteq 2^A$ there is no formula φ in $L(A)$ such that $[\![\varphi]\!] = X$. We find that for any such X, there is no theory Γ of $L(A)$ such that relation $\boldsymbol{\rho}\,(\{X\})$ equals the relation $\boldsymbol{\rho}\,(\Gamma)$. To appreciate this observe that $\boldsymbol{\rho}\,(\{X\}) = \rho\,(X)$ and assume for a *reductio ad absurdum* that there be some Γ such that $\rho\,(X) = \boldsymbol{\rho}\,(\Gamma)$. Consider an arbitrary $\gamma \in \Gamma$. By choice of X, then $X \neq [\![\gamma]\!]$. Moreover, $\rho\,(X) \subseteq \rho\,(\gamma)$, for otherwise $\rho\,(X) \nsubseteq \rho\,(\gamma)$, which would be absurd because $\boldsymbol{\rho}\,(\Gamma) \subseteq \rho\,(\gamma)$. By Fact 2.1.5, then either $X = \varnothing$ or $[\![\gamma]\!] = 2^A$. Since $[\![\bot]\!] = \varnothing$ and $\bot$ is a formula of $L(A)$, the former cannot obtain by choice of X. Hence, $[\![\gamma]\!] = 2^A$ and, consequently, $\rho\,(\gamma)$ is the universal relation over the valuations. With γ having been chosen arbitrarily and the initial assumption, it follows both $\boldsymbol{\rho}\,(\Gamma)$ and $\rho\,(X)$ coincide with the universal relation as well. Hence, $\rho\,(X) = \rho\,(\top)$. This however yields a contradiction, because, by Fact 2.1.2, $X = [\![\top]\!]$ would follow, which is absurd with $\top$ being a formula of $L(A)$.

As the next best thing, the following theorem characterizes the set of relations defined over the valuations by the theories of a propositional language in the general case.

Theorem 4.3.10 *Let ρ be a reflexive and transitive relation or the empty relation over S, with $S = 2^A$ and A a set of propositional variables. Then:*

$$\rho = \bigcap_{B \subseteq_\omega A} \overline{apr}_B(\rho) \quad \textit{iff} \quad \textit{for some theory } \Gamma \textit{ in } L(A){:} \quad \rho = \boldsymbol{\rho}\,(\Gamma).$$

Proof: If ρ is the empty relation, then so is any relation $\overline{apr}\,(\rho)$. Hence, $\rho = \varnothing = \bigcap_{B \subseteq_\omega A} \overline{apr}_B(\rho)$. Now observe that for the theory $\{\bot\}$, the relation $\boldsymbol{\rho}\,(\{\bot\})$ is empty as well. Thus, for the remainder of the proof, we may assume ρ to be transitive and reflexive.

For the left-to-right direction, assume $\rho = \bigcap_{B \subseteq_\omega A} \overline{apr}_B(\rho)$ and let $\boldsymbol{X} =_{df.} \{\uparrow_\rho s : \; s \in S\}$. (Henceforth in this proof we omit the subscript ρ in $\uparrow_\rho s$.) By Corollary 4.2.8 on page 33 above, there is a formula φ such that $[\![\varphi]\!] = \overline{apr}_B(X)$, for each $X \in \boldsymbol{X}$ and each finite $B \subseteq_\omega A$. Now let:

$$\Gamma^* \;\; =_{df.} \bigcup_{X \in \boldsymbol{X}} \{\varphi : \; \text{for some } B \subseteq_\omega A, \; [\![\varphi]\!] = \overline{apr}_B(X)\,\}.$$

We prove that $\rho = \boldsymbol{\rho}\,(\Gamma^*)$.

For the $\supseteq$-direction, assume an arbitrary pair (s, s') to be in $\boldsymbol{\rho}\,(\Gamma)$. Consider an arbitrary finite subset B of A; we show that $(s, s') \in \overline{apr}_B(\rho)$. Also consider $\uparrow_\rho s$. Then, there is some γ in Γ^* such that $[\![\gamma]\!] = \overline{apr}_B(\uparrow s)$. Since $(s, s') \in \boldsymbol{\rho}\,(\Theta)$, in particular $(s, s') \in \rho\,(\gamma)$ and so $s \in [\![\gamma]\!]$ implies $s' \in [\![\gamma]\!]$. By reflexivity of ρ, trivially, $s \in \uparrow s$ and *a fortiori* $s \in \overline{apr}_B(\uparrow s) = [\![\gamma]\!]$. Hence, also $s' \in [\![\gamma]\!] = \overline{apr}_B(\uparrow s)$. *I.e.*, for some $s'' \in S$, both $s' \sim_B s''$ and $s'' \in \uparrow s$, *i.e.*, $(s, s'') \in \rho$. As trivially, $s \sim_B s$, we may conclude that $(s, s') \in \overline{apr}_B(\rho)$.

For the $\subseteq$-direction, observe that with ρ reflexive and transitive and Fact 4.3.7 we have $\rho = \boldsymbol{\rho}\left(\{\uparrow s : \; s \in S\}\right)$. Hence:

$$\rho \;=\; \bigcap_{B \subseteq_\omega A} \overline{apr}_B(\rho) \;=\; \bigcap_{B \subseteq_\omega A} \overline{apr}_B\big(\boldsymbol{\rho}\left(\{\uparrow s : \; s \in S\}\right)\big)\,.$$

Assume for arbitrary valuations s and s' that $(s, s') \notin \boldsymbol{\rho}\,(\Gamma^*)$. Hence, for some $\gamma \in \Gamma^*$, both $s \in [\![\gamma]\!]$ and $s' \notin [\![\gamma]\!]$. By definition of Γ^*, there is some finite subset $B \subseteq_\omega A$ and some s_0 in S such that $[\![\gamma]\!] = \overline{apr}_B(\uparrow s_0)$. Then, for all valuations s'' such that $s' \sim_B s''$, $s'' \notin \uparrow s_0$. Also, by transitivity of ε_B, $s''' \in \uparrow s_0$, for all valuations s''' with $s \sim_B s'''$. Hence, $(s, s') \notin \overline{apr}_B(\rho\,(\uparrow s_0))$. Since $\boldsymbol{\rho}\left(\{\uparrow s : \; s \in S\}\right) \subseteq \rho\,(\{\uparrow s_0\})$, we obtain $(s, s') \notin \overline{apr}_B\big(\boldsymbol{\rho}\left(\{\uparrow s : \; s \in S\}\right)\big)$. Accordingly, $(s, s') \notin \bigcap_{B \subseteq_\omega A} \overline{apr}_B\big(\boldsymbol{\rho}\left(\{\uparrow s : \; s \in S\}\right)\big)$, *i.e.*, $(s, s') \notin \rho$.

For the opposite direction assume $\rho = \boldsymbol{\rho}\,(\Gamma)$ for some theory Γ of $L(A)$. Then $\rho \subseteq \bigcap_{B \subseteq_\omega A} \overline{apr}_B(\rho)$ is immediate. Assume that $(s, s') \notin \rho$; we prove that $(s, s') \notin \bigcap_{B \subseteq_\omega A} \overline{apr}_B(\rho)$. In virtue of the assumption, there is some γ such that $s \in [\![\gamma]\!]$ and $s' \notin [\![\gamma]\!]$. Now consider the finite set $\boldsymbol{A}(\gamma)$. For arbitrary valuations s'' and s''' such that $s \sim_{\boldsymbol{A}(\gamma)} s''$ and $s' \sim_{\boldsymbol{A}(\gamma)} s'''$, we have $s'' \in \overline{apr}_{\boldsymbol{A}(\gamma)}([\![\gamma]\!]) = [\![\gamma]\!]$ and $s''' \notin \overline{apr}_{\boldsymbol{A}(\gamma)}([\![\gamma]\!]) = [\![\gamma]\!]$. Therefore, $(s'', s''') \notin \rho\,(\gamma)$. It follows that $(s, s') \notin \overline{apr}_{\boldsymbol{A}(\gamma)}(\rho\,(\gamma))$. As $\boldsymbol{\rho}\,(\Gamma) \subseteq \rho\,(\gamma)$, also $\overline{apr}_{\boldsymbol{A}(\gamma)}(\boldsymbol{\rho}\,(\Gamma)) \subseteq \overline{apr}_{\boldsymbol{A}(\gamma)}(\rho\,(\gamma))$. Therefore, $(s, s') \notin \overline{apr}_{\boldsymbol{A}(\gamma)}(\boldsymbol{\rho}\,(\Gamma))$ and *a fortiori* $(s, s') \notin \bigcap_{B \subseteq_\omega A} \overline{apr}_B(\boldsymbol{\rho}\,(\Gamma))$. Having assumed that $\rho = \boldsymbol{\rho}\,(\Gamma)$, eventually $(s, s') \notin \bigcap_{B \subseteq_\omega A} \overline{apr}_B(\rho)$. ⊣

Thus we find that the class of relations induced by theories of a propositional language contains the empty relation as well as those partial preorders over the valuations that can be considered the limit of their own finite approximations. Theorem 4.3.10 sets a bound on the relations that can be expressed as a relation $\boldsymbol{\rho}(\Gamma)$ for some propositional theory Γ.

Chapter 5

Closure Conditions for Set-Induced Relations

In classical logic a theory may be closed under its consequences without affecting its deductive properties. At a semantical level, this fact is reflected in that the extension of a theory Γ is identical to the extension of its closure under logical consequence, *i.e.*, in general $[\![\Gamma]\!] = [\![Cn(\Gamma)]\!]$.

The relations over the valuations induced by theories, however, are more sensitive in this respect. In particular, it is not in general the case that the relations $\boldsymbol{\rho}(\Gamma)$ and $\boldsymbol{\rho}(Cn(\Gamma))$ are identical. For an easy counterexample consider a propositional language containing the propositional variables a and b. Obviously we have $a \vee b \in Cn(\{a\})$. For the valuations $\varnothing$ and $\{b\}$ clearly $(\{b\}, \varnothing) \in \boldsymbol{\rho}(\{a\})$. However, $(\{b\}, \varnothing) \notin \boldsymbol{\rho}(\{a, a \vee b\})$, because $\{b\} \Vdash a \vee b$ but $\varnothing \nVdash a \vee b$. Hence, $(\{b\}, \varnothing) \notin \rho(a \vee b)$. The same argument holds for $\boldsymbol{\rho}_0$. It is obvious, however, that a theory Γ may generally be closed under formulas that are *logically equivalent* in the classical sense without affecting $\boldsymbol{\rho}(\Gamma)$.

At a set-theoretic level, a set of sets $\boldsymbol{X}$ cannot in general be closed under supersets without affecting the relation $\boldsymbol{\rho}(\boldsymbol{X})$. On the other hand, different sets of sets may very well induce the same relation on a universe, *i.e.*, $\boldsymbol{\rho}(\boldsymbol{X})$ and $\boldsymbol{\rho}(\boldsymbol{Y})$ may be identical even if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are distinct. This chapter aims at making precise the conditions on sets of sets $\boldsymbol{X}$ and $\boldsymbol{Y}$ that have to be satisfied for the relations $\boldsymbol{\rho}(\boldsymbol{X})$ and $\boldsymbol{\rho}(\boldsymbol{Y})$ to be identical. Closure conditions on theories that preserve relations induced by theories then follow as a matter of course. We find that relations $\boldsymbol{\rho}_0(\boldsymbol{X})$ are slightly better behaved than relations $\boldsymbol{\rho}(\boldsymbol{X})$ and, therefore, we will focus on the former first.

As an example of monotonicity, $\boldsymbol{\rho}_0(\{X, Y\})$ includes $\boldsymbol{\rho}_0(\{X, Y, X \cap Y, X \cup Y\})$.

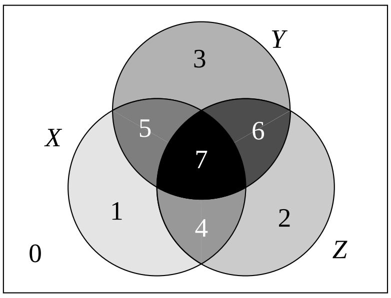

Figure 5.1. Three intersecting sets, X, Y and Z. A pair (x,y) is in the relation $\boldsymbol{\rho}(\{X,Y,Z\})$ the set of sets $\{X,Y,Z\}$ defines over the universe S whenever one can reach y from y without ever moving from an area to an area that is colored lighter. *E.g.*, $(x,y) \in \boldsymbol{\rho}(\{X,Y,Z\})$, for all elements x in area 1 and y in the darker colored area 5. But any element in area 2 and any element in area 5 are incomparable with respect to $\boldsymbol{\rho}(\{X,Y,Z\})$. Some reflection reveals that closing the set $\{X,Y,Z\}$ under intersections and unions would not distort this relation.

The opposite inclusion, however, also holds in general. For a generalization of this fact, define for $\boldsymbol{X}$ a set of subsets of a set S:

$$\boldsymbol{X}^{\cup} =_{df.} \left\{\bigcup \boldsymbol{X}' : \boldsymbol{X}' \subseteq \boldsymbol{X}\right\} \qquad \boldsymbol{X}^{\cap} =_{df.} \left\{\bigcap \boldsymbol{X}' : \boldsymbol{X}' \subseteq \boldsymbol{X}\right\}.$$

The following proposition says in effect that, for any set of sets $\boldsymbol{X}$, relation $\boldsymbol{\rho}_0(\boldsymbol{X})$ is invariant under taking arbitrary intersections as well as under taking arbitrary unions.

Proposition 5.0.11 *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be sets of subsets of a set S such that $\boldsymbol{X} \subseteq \boldsymbol{Y} \subseteq \boldsymbol{X}^{\cup} \cup \boldsymbol{X}^{\cap}$. Then $\boldsymbol{\rho}_0(\boldsymbol{X}) = \boldsymbol{\rho}_0(\boldsymbol{Y})$.*

Proof: By monotonicity immediately $\boldsymbol{\rho}_0(\boldsymbol{Y}) \subseteq \boldsymbol{\rho}_0(\boldsymbol{X})$. Therefore, it suffices to prove that $\boldsymbol{\rho}_0(\boldsymbol{X}) \subseteq \boldsymbol{\rho}_0(\boldsymbol{Y})$. Consider arbitrary $x, x' \in S$ such that $(x,x') \in \boldsymbol{\rho}_0(\boldsymbol{X})$. Consider, furthermore, an arbitrary $Y \in \boldsymbol{Y}$ and assume that $x \in Y$. We prove that $x' \in Y$. Either $Y = \bigcap \boldsymbol{X}'$ or $Y = \bigcup \boldsymbol{X}'$, for some $\boldsymbol{X}' \subseteq \boldsymbol{X}$. Consider this $\boldsymbol{X}'$. In the former case, consider an arbitrary $X \in \boldsymbol{X}'$. Then both $X \in \boldsymbol{X}$ and $x \in X$. Since $(x,x') \in \boldsymbol{\rho}_0(\boldsymbol{X})$, in particular $(x,x') \in \rho_0(X)$. Hence, $x' \in X$. With X having been chosen as an arbitrary element of $\boldsymbol{X}'$, finally $x \in \bigcap \boldsymbol{X}'$, and we may conclude that $x \in Y$.

In the latter case — *i.e.*, if $Y = \bigcup \boldsymbol{X}'$ — we have $x \in X$, for some $X \in \boldsymbol{X}'$. As $\boldsymbol{X}' \subseteq \boldsymbol{X}$, also $X \in \boldsymbol{X}$ and with $(x,x') \in \boldsymbol{\rho}_0(\boldsymbol{X})$, in particular $(x,x') \in \rho_0(X)$. It follows that $x' \in X$ and, subsequently, $x' \in \bigcup \boldsymbol{X}'$, *i.e.*, $x' \in Y$. ⊣

Corollary 5.0.12 *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be sets of subsets of a set S. Then $\boldsymbol{X} \subseteq \boldsymbol{Y} \subseteq \boldsymbol{X}^{\cup\cap}$ implies $\boldsymbol{\rho}_0(\boldsymbol{X}) = \boldsymbol{\rho}_0(\boldsymbol{Y})$. Similarly, if $\boldsymbol{X} \subseteq \boldsymbol{Y} \subseteq \boldsymbol{X}^{\cap\cup}$ then $\boldsymbol{\rho}_0(\boldsymbol{X}) = \boldsymbol{\rho}_0(\boldsymbol{Y})$.*

Proof: In virtue of monotonicity it suffices to show that $\boldsymbol{\rho}_0(\boldsymbol{X}) = \boldsymbol{\rho}_0(\boldsymbol{X}^{\cup\cap})$ and that $\boldsymbol{\rho}_0(\boldsymbol{X}) = \boldsymbol{\rho}_0(\boldsymbol{X}^{\cap\cup})$. Evidently, $\boldsymbol{X} \subseteq \boldsymbol{X}^{\cap}$ as well as $\boldsymbol{X}^{\cap} \subseteq \boldsymbol{X}^{\cap\cup}$. With Proposition 5.0.11, $\boldsymbol{\rho}_0(\boldsymbol{X}) = \boldsymbol{\rho}_0(\boldsymbol{X}^{\cap})$. And again with Proposition 5.0.11, also, $\boldsymbol{\rho}_0(\boldsymbol{X}^{\cap}) = \boldsymbol{\rho}_0(\boldsymbol{X}^{\cap\cup})$. Hence, $\boldsymbol{\rho}_0(\boldsymbol{X}) = \boldsymbol{\rho}_0(\boldsymbol{X}^{\cap\cup})$. The proof for $\boldsymbol{\rho}_0(\boldsymbol{X}) = \boldsymbol{\rho}_0(\boldsymbol{X}^{\cap\cup})$ is fully analogous. ⊣

For relations induced by a theory over a set of valuations, this corollary means that a theory Γ may be closed under arbitrary conjunctions and disjunctions without affecting the relation $\rho_0(\Gamma)$.

The ground has now been cleared to formulate exact conditions under which the relations $\boldsymbol{\rho}_0(\boldsymbol{X})$ and $\boldsymbol{\rho}_0(\boldsymbol{Y})$ are identical, for possibly distinct sets of sets $\boldsymbol{X}$ and $\boldsymbol{Y}$.

Proposition 5.0.13 *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be sets of subsets of a set S. Then:*

$$\boldsymbol{\rho}_0(\boldsymbol{X}) = \boldsymbol{\rho}_0(\boldsymbol{Y}) \quad \textit{iff} \quad \boldsymbol{X}^{\cap\cup} = \boldsymbol{Y}^{\cap\cup}.$$

Proof: For the right-to-left direction, observe that $\boldsymbol{X}^{\cap\cup} = \boldsymbol{Y}^{\cap\cup}$ immediately implies $\boldsymbol{\rho}_0(\boldsymbol{X}^{\cap\cup}) = \boldsymbol{\rho}_0(\boldsymbol{Y}^{\cap\cup})$. Since, by Corollary 5.0.12, both $\boldsymbol{\rho}_0(\boldsymbol{X}^{\cap\cup}) = \boldsymbol{\rho}_0(\boldsymbol{X})$ and $\boldsymbol{\rho}_0(\boldsymbol{X}^{\cap\cup}) = \boldsymbol{\rho}_0(\boldsymbol{X})$, we are done.

The left-to-right direction is less straightforward. Assume the contrapositive $\boldsymbol{X}^{\cap\cup} \neq \boldsymbol{Y}^{\cap\cup}$. Without loss of generality we may assume there be an $X \subseteq S$ such that $X \in \boldsymbol{X}^{\cap\cup}$ and $X \notin \boldsymbol{Y}^{\cap\cup}$. Consider this X. Observe that trivially $\varnothing \subseteq \boldsymbol{Y}^{\cap}$ and $\bigcup \varnothing = \varnothing$. Hence, $\varnothing \in \boldsymbol{Y}^{\cap\cup}$. Moreover, since $\varnothing \subseteq \boldsymbol{Y}$ and $\bigcap \varnothing = S$, both $S \in \boldsymbol{Y}^{\cap}$ and $\{S\} \subseteq \boldsymbol{Y}^{\cap}$. Since $\bigcup \{S\} = S$, also $S \in \boldsymbol{Y}^{\cap\cup}$. It follows that $X \neq \varnothing$ and $X \neq S$. Now consider the set $\boldsymbol{Y}^*$, defined as:

$$\boldsymbol{Y}^* =_{df.} \left\{ Y \in \boldsymbol{Y}^{\cap} : \; Y \subseteq X \right\}.$$

Clearly, $\bigcup \boldsymbol{Y}^* \in \boldsymbol{Y}^{\cap\cup}$ and $\bigcup \boldsymbol{Y}^* \subseteq X$. Due to the assumption that $X \notin \boldsymbol{Y}^{\cap\cup}$, however, $X \neq \bigcup \boldsymbol{Y}^*$. Hence, there is some $x^* \in X$ such that $x^* \notin \bigcup \boldsymbol{Y}^*$. Consider this x^*. We prove that an x in S exists such that is not contained in X and for which it is moreover the case that, for all $Y \in \boldsymbol{Y}$, $x \in Y$, if $x^* \in Y$ as well. *I.e.*:

($*$) there is an $x \notin X$ such that for all $Y \in \boldsymbol{Y}$: $x^* \in Y$ implies $x \in Y$.

This suffices because, with $x^* \in X$ then $(x^*, x) \notin \rho_0(X)$. Then $(x^*, x) \notin \boldsymbol{\rho}_0(\boldsymbol{X}^{\cap\cup})$, because X had been assumed to be in $\boldsymbol{\rho}_0(\boldsymbol{X}^{\cap\cup})$. With Corollary 5.0.12, then

$\boldsymbol{\rho}_0(\boldsymbol{X}^{\cap\cup}) = \boldsymbol{\rho}_0(\boldsymbol{X})$ and $(x^*, x) \notin \boldsymbol{\rho}_0(\boldsymbol{X})$ follows. Moreover, also $(x^*, x) \in \rho_0(Y)$, for each $Y \in \boldsymbol{X}$. Hence $(x^*, x) \in \boldsymbol{\rho}_0(\boldsymbol{Y})$, which would prove the proposition.

We prove $(*)$ by a *reductio ad absurdum*. So assume:

$(**)$ for all $x \notin X$ there is a $Y \in \boldsymbol{Y}$ such that both $x \notin Y$ and $x^* \in Y$.

Then, consider the set $\boldsymbol{Y}^{**}$, defined as:

$$\boldsymbol{Y}^{**} =_{df.} \bigcup_{x \notin X} \left\{ Y \in \boldsymbol{Y} : \ x \notin Y \text{ and } x^* \in Y \right\}.$$

By $(**)$ and the fact that $X \neq S$, we have $\boldsymbol{Y}^{**} \neq \varnothing$. Obviously, $\boldsymbol{Y}^{**} \subseteq \boldsymbol{Y}$ and so $\bigcap \boldsymbol{Y}^{**} \in \boldsymbol{Y}^{\cap}$. By construction, $x^* \in \bigcap \boldsymbol{Y}^{**}$. Moreover, by construction and $(**)$, also $\bigcap \boldsymbol{Z}^{**} \subseteq X$. It would follow that $\bigcap \boldsymbol{Y}^{**} \in \boldsymbol{Y}^*$ as well as that $x^* \in \bigcup \boldsymbol{Y}^*$, *quod non*. ⊣

Corollary 5.0.12 has as a special case that $\boldsymbol{\rho}_0(\boldsymbol{X}) = \boldsymbol{\rho}_0(\boldsymbol{X}^{\cap\cup})$, which signifies that closing a set of subsets $\boldsymbol{X}$ under arbitrary intersections and then arbitrary unions does not affect the relation induced on the universe. As a corollary of Proposition 5.0.13 we now find, moreover, that $\boldsymbol{X}^{\cap\cup}$ is also maximal in this respect, *i.e.*, that $\boldsymbol{X}$ can not be extended beyond $\boldsymbol{X}^{\cap\cup}$ without distorting the relation $\boldsymbol{\rho}_0(\boldsymbol{X})$.

Corollary 5.0.14 *Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be sets of subsets of* S. *Then:*

$$\boldsymbol{\rho}_0\left(\boldsymbol{X}^{\cap\cup}\right) = \boldsymbol{\rho}_0\left(\boldsymbol{X}^{\cap\cup} \cup \boldsymbol{Y}\right) \quad \textit{iff} \quad \boldsymbol{Y} \subseteq \boldsymbol{X}^{\cap\cup}.$$

Proof: From right to left the proof is trivial. So assume that $\boldsymbol{\rho}_0(\boldsymbol{X}^{\cap\cup}) = \boldsymbol{\rho}_0(\boldsymbol{X}^{\cap\cup} \cup \boldsymbol{Y})$. It can easily be verified that $\boldsymbol{X} \cup \boldsymbol{Y} \subseteq \boldsymbol{X}^{\cap\cup} \cup \boldsymbol{Y} \subseteq (\boldsymbol{X} \cup \boldsymbol{Y})^{\cap\cup}$. By Proposition 5.0.11, then, $\boldsymbol{\rho}_0(\boldsymbol{X} \cup \boldsymbol{Y}) = \boldsymbol{\rho}_0(\boldsymbol{X}^{\cap\cup} \cup \boldsymbol{Y})$. In virtue of the same proposition, also $\boldsymbol{\rho}_0(\boldsymbol{X}) = \boldsymbol{\rho}_0(\boldsymbol{X}^{\cap\cup})$. With the initial assumption then it follows that $\boldsymbol{\rho}_0(\boldsymbol{X}) = \boldsymbol{\rho}_0(\boldsymbol{X} \cup \boldsymbol{Y})$. Proposition 5.0.13 then gives $\boldsymbol{X}^{\cap\cup} = (\boldsymbol{X} \cup \boldsymbol{Y})^{\cap\cup}$. Because, $\boldsymbol{X}^{\cap\cup} \cup \boldsymbol{Y} \subseteq (\boldsymbol{X} \cup \boldsymbol{Y})^{\cap\cup}$, then $\boldsymbol{X}^{\cap\cup} \cup \boldsymbol{Y} \subseteq \boldsymbol{X}^{\cap\cup}$. We may conclude that $\boldsymbol{Y} \subseteq \boldsymbol{X}^{\cap\cup}$. ⊣

Similar results can be obtained for relations $\boldsymbol{\rho}(\boldsymbol{X})$ induced by sets of sets $\boldsymbol{X}$. Unfortunately, things are not as neat as for $\boldsymbol{\rho}_0(\boldsymbol{X})$. Because $\boldsymbol{\rho}(\boldsymbol{X})$ and $\boldsymbol{\rho}_0(\boldsymbol{X})$ are distinct only if $\boldsymbol{X}$ contains the empty set (Proposition 2.1.3), Proposition 5.0.13 also has the following corollary for $\boldsymbol{\rho}(\boldsymbol{X})$.

Corollary 5.0.15 *Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be sets of subsets of a set* S. *Then:*

$$\boldsymbol{\rho}(\boldsymbol{X}) = \boldsymbol{\rho}(\boldsymbol{Y}) \quad \textit{iff} \quad \boldsymbol{X}^{\cap\cup} = \boldsymbol{Y}^{\cap\cup} \ \textit{ or } \ \varnothing \in \boldsymbol{X} \cap \boldsymbol{Y}.$$

Proof: Immediately by the Facts 2.1.3 and 2.1.4 together with Proposition 5.0.13.

It is, however, not in general the case that for $\boldsymbol{X}$ not containing the empty set the relations $\boldsymbol{\rho}(\boldsymbol{X})$ and $\boldsymbol{\rho}(\boldsymbol{X}^{\cap\cup})$ coincide. Observe in this respect that $\boldsymbol{X}^{\cap}$ always contains the empty set, since $\bigcup\varnothing = \varnothing$ and $\varnothing \subseteq \boldsymbol{X}$. Moreover, $\boldsymbol{\rho}(\boldsymbol{X})$ need not be the empty relation, not even if $\boldsymbol{X}$ contains disjoint sets. In any such case, however, $\boldsymbol{X}^{\cap}$ will contain the empty set and $\boldsymbol{\rho}(\boldsymbol{X}^{\cap})$ will also end up empty. In order to obtain the desired closure properties, define for $\boldsymbol{X}$ a set of subsets of a set S:

$$\boldsymbol{X}^{\sqcup} =_{df.} \left\{\bigcup\boldsymbol{X}' : \ \boldsymbol{X}' \subseteq \boldsymbol{X} \text{ and } \boldsymbol{X}' \neq \varnothing\right\}$$
$$\boldsymbol{X}^{\sqcap} =_{df.} \boldsymbol{X} \cup \left\{\bigcap\boldsymbol{X}' : \ \boldsymbol{X}' \subseteq \boldsymbol{X} \text{ and } \bigcap\boldsymbol{X}' \neq \varnothing\right\}.$$

The idea behind these definitions is essentially the same as those of $\boldsymbol{X}^{\cup}$ and $\boldsymbol{X}^{\cap}$, be it that they prevent the empty set to be included in $\boldsymbol{X}^{\sqcup}$ or $\boldsymbol{Y}^{\sqcap}$ if, and only if, $\boldsymbol{X}$ does not conclude the empty set. It is therefore not surprising that $\boldsymbol{X}^{\sqcup}$ and $\boldsymbol{X}^{\sqcap}$ are in extension very similar to $\boldsymbol{X}^{\cup}$ and $\boldsymbol{X}^{\cap}$, respectively.

Fact 5.0.16 *For $\boldsymbol{X}$ a set of subsets of some set S:*

$$\boldsymbol{X}^{\sqcup} = \begin{cases} \boldsymbol{X}^{\cup} - \{\varnothing\} & \text{if } \varnothing \notin \boldsymbol{X}, \\ \boldsymbol{X}^{\cup} & \text{otherwise} \end{cases} \qquad \boldsymbol{X}^{\sqcap} = \begin{cases} \boldsymbol{X}^{\cap} - \{\varnothing\} & \text{if } \varnothing \notin \boldsymbol{X}, \\ \boldsymbol{X}^{\cap} & \text{otherwise}. \end{cases}$$

Proof: For the first case, first assume $\varnothing \notin \boldsymbol{X}$. Observe that $\varnothing \notin \boldsymbol{X}^{\sqcup}$. For, assuming otherwise would that $\varnothing = \bigcup\boldsymbol{X}'$ for some non-empty $\boldsymbol{X}' \subseteq \boldsymbol{X}$. This would imply that $\boldsymbol{X}' = \{\varnothing\}$ and hence $\varnothing \in \boldsymbol{X}$, *quod non*. Hence, $\boldsymbol{X}^{\sqcup} \subseteq \boldsymbol{X}^{\cup} - \{\varnothing\}$ and it suffices to prove the opposite inclusion. Consider an arbitrary $X \in \boldsymbol{X}^{\cup} - \{\varnothing\}$. Then, $X \neq \varnothing$ and $X = \bigcup\boldsymbol{X}'$ for some $\boldsymbol{X}' \subseteq \boldsymbol{X}$. Moreover, $\boldsymbol{X}' \neq \varnothing$ by assumption, and so $\bigcap\boldsymbol{X}' = X \in \boldsymbol{X}^{\sqcup}$. Second, assume that $\varnothing \in \boldsymbol{X}$. Observe that trivially, $\boldsymbol{X}^{\sqcup} \subseteq \boldsymbol{X}^{\cup}$. Hence it suffices to prove the opposite inclusion. Consider an arbitrary $X \in \boldsymbol{X}^{\cup}$. If $X = \varnothing$, observe that by the assumption $\{\varnothing\} \subseteq \boldsymbol{X}$. Since $\{\varnothing\} \neq \varnothing$, it follows that $\bigcup\{\varnothing\} = \varnothing \in \boldsymbol{X}^{\sqcup}$. In case $X \neq \varnothing$ the proof is like the case in which $\varnothing \notin \boldsymbol{X}$.

For the second case, first assume that $\varnothing \notin \boldsymbol{X}$. Some reflection on the definitions reveals that then $\varnothing \notin \boldsymbol{X}^{\sqcap}$ and also $\boldsymbol{X}^{\sqcap} \subseteq \boldsymbol{X}^{\cap} - \{\varnothing\}$. Proving the opposite inclusion, consider an arbitrary $X \in \boldsymbol{X}^{\cap} - \{\varnothing\}$. Then, $X = \bigcap\boldsymbol{X}'$ for some $X' \subseteq \boldsymbol{X}$. It follows that $\bigcap\boldsymbol{X}' \neq \varnothing$ and so $\bigcap\boldsymbol{X}' = X \in \boldsymbol{X}^{\sqcap}$. Finally, let $\varnothing \in \boldsymbol{X}$ and observe that $\boldsymbol{X}^{\sqcap} \subseteq \boldsymbol{X}^{\cap}$. Hence, consider an arbitrary $X \in \boldsymbol{X}^{\cap}$. If $X = \varnothing$, we are done immediately by the assumption that $\varnothing \in \boldsymbol{X}$. Otherwise, the reasoning is like the case in which $\varnothing \notin \boldsymbol{X}$. ⊣

On basis of this fact and employing Proposition 5.0.11 the following closure properties for $\boldsymbol{\rho}(\boldsymbol{X})$ are obtained.

Proposition 5.0.17 *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be sets of subsets of a set S such that $\boldsymbol{X} \subseteq \boldsymbol{Y} \subseteq \boldsymbol{X}^{\sqcup} \cup \boldsymbol{X}^{\sqcap}$. Then, $\boldsymbol{\rho}(\boldsymbol{X}) = \boldsymbol{\rho}(\boldsymbol{Y})$.*

Proof: First assume $\varnothing \in \boldsymbol{Y}$. Then, $\varnothing \in \boldsymbol{X}^{\sqcap}$ or $\varnothing \in \boldsymbol{X}^{\sqcup}$. In either case, $\varnothing \in \boldsymbol{X}$, by Fact 5.0.16. Then, $\boldsymbol{\rho}(\boldsymbol{X}) = \varnothing = \boldsymbol{\rho}(\boldsymbol{Y})$. So, for the remainder of the proof we may assume that $\varnothing \notin \boldsymbol{Y}$. Then, $\varnothing \notin \boldsymbol{X}$ and in virtue of Fact 5.0.16 both $\boldsymbol{X}^{\sqcup} = \boldsymbol{X}^{\cup}$ and $\boldsymbol{X}^{\sqcap} = \boldsymbol{X}^{\cap}$. Hence $\boldsymbol{X} \subseteq \boldsymbol{Y} \subseteq \boldsymbol{X}^{\cup} \cup \boldsymbol{X}^{\cap}$ and by Proposition 5.0.11, then $\boldsymbol{\rho}_0(\boldsymbol{X}) = \boldsymbol{\rho}_0(\boldsymbol{Y})$. With the assumption that $\varnothing \notin \boldsymbol{X}$ and Fact 2.1.3, $\boldsymbol{\rho}(\boldsymbol{X}) = \boldsymbol{\rho}_0(\boldsymbol{X})$ and $\boldsymbol{\rho}(\boldsymbol{Y}) = \boldsymbol{\rho}_0(\boldsymbol{Y})$ and we may conclude that $\boldsymbol{\rho}(\boldsymbol{Y}) = \boldsymbol{\rho}(\boldsymbol{X})$. ⊣

As corollaries of Proposition 5.0.17 we find that a theory Γ may be closed under disjunctions and *consistent* conjunctions without this having consequences for the relation induced on the valuations.

Corollary 5.0.18 *Let Γ be a theory in a propositional language $L(A)$ Let φ and ψ be formulas in Γ. Then, $\boldsymbol{\rho}(\Gamma) = \boldsymbol{\rho}(\Gamma \cup \{\varphi \vee \psi\})$. Moreover, if $\{\varphi, \psi\}$ is classically satisfiable, then also $\boldsymbol{\rho}(\Gamma) = \boldsymbol{\rho}\big(\Gamma \cup \{\varphi \wedge \psi\}\big)$.*

Proof: Immediately by Proposition 5.0.17. ⊣

Chapter 6

Compositionality of Formula Induced Relations

Introduction

On page 8 it was argued that the set of relations of the form $\rho_0(\varphi)$ does not allow for a compositional definition in φ. We found that $\rho_0(\top \wedge a)$ is not identical to $\rho_0(\bot \wedge a)$, but since $\rho_0(\top) = \rho_0(\bot)$, this difference could not be made on the basis of the relations of $\rho_0(a)$, $\rho_0(\top)$ and $\rho_0(\bot)$ alone. We stated that the relations $\rho(\varphi)$ defined for propositional formulas φ fared better in this respect. This chapter gives an, admittedly monstrous, proof of this claim.

We have to prove that, for each formula φ with sub-formulas $\psi_0, \ldots, \psi_n$, the relation $\rho(\varphi)$ can be viewed as a construct in the relations $\rho(\psi_0), \ldots, \rho(\psi_n)$. In any such construct we may use the operations from relational algebra, *i.e.*, the identity relation (*Id*), relational composition ($\rho \circ \sigma$) and relational inverse ($\rho^{\smile}$), along with the usual Boolean operators. As we may assume the relations $\rho(a)$ over the valuations for propositional variables a to be given, it suffices for our purposes to define operations $-$, $+$ and $\cdot$ on relations on a set S in general such that in particular all subsets X and Y we have:

$$
\begin{aligned}
-\rho(X) &= \rho\left(\overline{X}\right), \\
\rho(X) + \rho(Y) &= \rho(X \cup Y), \\
\rho(X) \cdot \rho(Y) &= \rho(X \cap Y).
\end{aligned}
$$

The following definitions and propositions in the next section are all auxiliary to this one result.

6.1 The Proof of Compositionality

For easy reference let us first summarize the non-Boolean laws of relation algebra:[1]

$$\begin{array}{rclrcl}
\rho \circ Id & = & \rho & Id \circ \rho & = & \rho \\
\rho^{\smile\smile} & = & \rho & (\overline{\rho})^{\smile} & = & \overline{\rho^{\smile}} \\
(\rho \cup \sigma)^{\smile} & = & \rho^{\smile} \cup \sigma^{\smile} & (\rho \circ \sigma)^{\smile} & = & \sigma^{\smile} \circ \rho^{\smile} \\
\rho \circ (\sigma \cup \tau) & = & (\rho \circ \sigma) \cup (\rho \circ \tau) & (\rho \cup \sigma) \circ \tau & = & (\rho \circ \tau) \cup (\sigma \circ \tau) \\
(\rho \circ \sigma) \circ \tau & = & \rho \circ (\sigma \circ \tau) & \left(\rho^{\smile} \circ \overline{(\rho \circ \sigma)}\right) & \leqslant & \overline{\sigma}
\end{array}$$

The universal relation is denoted by U. We further introduce $\dot{\rho}$ as a shorthand notation for $\rho \cap \overline{\rho^{\smile}}$, *i.e.*, the *strict subrelation* of ρ. Thus we have:

$$\left(x, x'\right) \in \dot{\rho} \quad \text{iff} \quad \left(x, x'\right) \in \rho \text{ and } \left(x', x\right) \notin \rho.$$

We usually denote $\overset{\cdot}{\rho(X)}$ by $\dot{\rho}(X)$. It can easily be established that $\dot{U}$ and $\dot{Id}$ are both empty. Also, $(\dot{\rho^{\smile}})$ equals $(\dot{\rho})^{\smile}$, as witness the following relation algebraic equations:

$$(\dot{\rho^{\smile}}) \;=\; \rho^{\smile} \cap \overline{\rho^{\smile\smile}} \;=\; \rho^{\smile} \cap \left(\overline{\rho^{\smile}}\right)^{\smile} \;=\; \left(\rho \cap \overline{\rho^{\smile}}\right)^{\smile} \;=\; (\dot{\rho})^{\smile}.$$

Hence, we can use $\dot{\rho}^{\smile}$ and $\dot{\rho}(X)^{\smile}$ without ambiguity. We also have the following easy, but useful, facts.

Fact 6.1.1 *Let X be a non-empty subset of some set S. Then, $\rho(X)^{\smile} = \rho\left(\overline{X}\right)$.*

Proof: For arbitrary $x, x' \in S$ we have $(x, x') \in \rho(X)^{\smile}$, if and only if $(x', x) \in \rho(X)$, if and only if $x' \notin X$ or $x \in X$, if and only if $x \notin \overline{X}$ or $x' \in \overline{X}$, if and only if $(x, x') \in \rho\left(\overline{X}\right)$. ⊣

Fact 6.1.2 *Let X be a non-empty subset of some set S. Then,* $\overline{\rho(X)} \subseteq \rho(X)^{\smile}$.

Proof: For arbitrary $x, x' \in S$ we have $(x, x') \in \overline{\rho(X)}$ imply $(x, x') \notin \rho(X)$. Then, $x \in X$. Therefore, subsequently, $(x', x) \in \rho(X)$ and $(x, x') \in \rho(X)^{\smile}$. ⊣

Fact 6.1.3 *Let X and Y be a subsets of set S such that $X \cap Y \neq \emptyset$. Then:*

$$\rho(X) \cap \rho(Y) \quad \subseteq \quad \rho(X \cap Y).$$

[1] These inequalities are taken from van Benthem (1996).

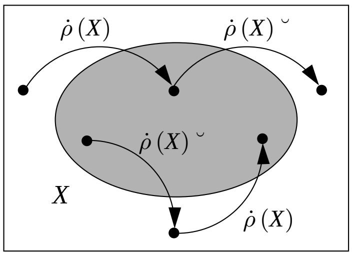

Figure 6.1. All elements outside x are connected with one another *via* $\dot{\rho}(X) \circ \dot{\rho}(X)^{\smile}$. All elements in X are related to one another by $\dot{\rho}(X)^{\smile} \circ \dot{\rho}(X)$.

Proof: Assume $(x, x) \in \rho(X) \cap \rho(Y)$, *i.e.*, both $(x, x) \in \rho(X)$ and $(x, x) \in \rho(Y)$. Now assume also that $x \in X \cap Y$. Then both $x \in X$ and $x \in Y$. Therefore, also $x' \in X$ and $x' \in Y$, from which in turn $x' \in X \cap X$. We may conclude that $(x, x') \in \rho(X \cap Y)$. ⊣

Fact 6.1.4 *Let X be a non-empty subset of some set S. Then:*

$$(x, x') \in \overline{\rho(X)^{\smile}} \quad \textit{iff} \quad x \notin X \ \textit{and} \ x' \in X.$$

Proof: For arbitrary $x, x' \in S$ we have $(x, x') \in \overline{\rho(X)^{\smile}}$, if and only if $(x, x') \notin \rho(X)^{\smile}$, if and only if $(x', x) \notin \rho(X)$, if and only if $x \notin X$ and $x' \in X$. ⊣

Fact 6.1.1 suggests relational inverse as a candidate for the unary operation $-$. It holds, however, only for relations $\rho(X)$ in case X is not empty. For the empty set we have $\rho(\varnothing)^{\smile} = \varnothing^{\smile} = \varnothing$ but also $\rho(\overline{\varnothing}) = \rho(S) = S \times S$. The following lemma establishes the rather more complicated $\overline{\rho} \cup (\dot{\rho}^{\smile} \circ \dot{\rho}) \cup (\dot{\rho} \circ \dot{\rho}^{\smile})$ as a proper definition for $-\rho$. It takes an easy check to establish that defined thus $-\rho(\overline{X})$ is the empty relation if X coincides with S, and the universal relation if X is empty. Figure 6.1 explains informally the workings of the definition for a relation $\rho(X)$ with X a non-empty proper subset of S. To appreciate the underlying idea observe that the relation $\dot{\rho}(\overline{X})$ holds exactly between those pairs (x, x') such that $x \notin X$ and $x' \in X$, *i.e.*, for which $(x, x') \in \overline{\rho(X)}$. In $\rho(\overline{X})$ also connects any two elements that are both inside or both outside X. The relations $(\dot{\rho}(X)^{\smile} \circ \dot{\rho}(X))$ and $(\dot{\rho}(X) \circ \dot{\rho}(X)^{\smile})$ take care of this. Formally, we have the following lemma.

Lemma 6.1.5 *Let X be a subset of some set S. Then:*

$$\rho(\overline{X}) \;=\; \overline{\rho(X)} \;\cup\; \left(\dot{\rho}(X)^{\smile} \circ \dot{\rho}(X)\right) \;\cup\; \left(\dot{\rho}(X) \circ \dot{\rho}(X)^{\smile}\right).$$

Proof: We distinguish the cases in which X is empty, in which X is S and in which X is neither of the previous. First assume $X = \varnothing$. Then observe:

$$\rho\left(\overline{\varnothing}\right) \;=\; \rho\left(S\right) \;=\; U \;=\; U - \varnothing \;=\; U - \rho\left(\varnothing\right) \;=\; \overline{\rho\left(\varnothing\right)}$$

Hence certainly, $\overline{\rho\left(\varnothing\right)} \cup \left(\dot{\rho}\left(\varnothing\right)^{\smile} \circ \dot{\rho}\left(\varnothing\right)\right) \cup \left(\dot{\rho}\left(\varnothing\right) \circ \dot{\rho}\left(\varnothing\right)^{\smile}\right) = U$. Secondly, assume $X = S$. Then $\rho\left(\overline{S}\right) = \rho\left(\varnothing\right) = \varnothing$. Now observe that $\overline{\rho\left(S\right)} = \overline{U} = \varnothing$ and that $\dot{\rho}\left(S\right) = \dot{U} = \varnothing$. Consequently, also $\dot{\rho}\left(S\right)^{\smile} = \varnothing$ and the following equations hold:

$$\overline{\rho\left(S\right)} \;\cup\; \left(\dot{\rho}\left(S\right)^{\smile} \circ \dot{\rho}\left(S\right)\right) \;\cup\; \left(\dot{\rho}\left(S\right) \circ \dot{\rho}\left(S\right)^{\smile}\right) \;=\; \varnothing \cup \varnothing \cup \varnothing \;=\; \varnothing.$$

Finally, assume $X \neq \varnothing$ and $X \neq S$, in which case $\rho\left(X\right) = \rho\left(X\right)$. By Fact 6.1.1, now, $\rho\left(\overline{X}\right) = \rho\left(X\right)^{\smile}$ and it suffices to prove that:

$$\rho\left(X\right)^{\smile} \;=\; \overline{\rho\left(X\right)} \cup \left(\dot{\rho}\left(X\right)^{\smile} \circ \dot{\rho}\left(X\right)\right) \cup \left(\dot{\rho}\left(X\right) \circ \dot{\rho}\left(X\right)^{\smile}\right).$$

For the $\subseteq$-direction suppose $(x, x') \in \rho\left(X\right)^{\smile}$, *i.e.*, $(x', x) \in \rho\left(X\right)$ and distinguish the following three cases:

(a) $x' \in X$ and $x \in X$ (b) $x' \notin X$ and $x \in X$ (c) $x' \notin X$ and $x \notin X$

If (b), $(x, x') \notin \rho\left(X\right)$ and immediately $(x, x') \in \overline{\rho\left(X\right)}$. Now assume ($a$) to be the case. Since $X \neq S$ there is some $y \notin X$. For this y both $(x, y) \in \dot{\rho}\left(X\right)^{\smile}$ and $(y, x') \in \dot{\rho}\left(X\right)$. Hence, $(x, x') \in \dot{\rho}\left(X\right)^{\smile} \circ \dot{\rho}\left(X\right)$. Similarly, if ($c$), observe that since $X \neq \varnothing$, there is some $y \in X$. Now again both $(x, y) \in \dot{\rho}\left(X\right)$ and $(y, x') \in \dot{\rho}\left(X\right)^{\smile}$. Hence, in this case $(x, x') \in \dot{\rho}\left(X\right) \circ \dot{\rho}\left(X\right)^{\smile}$. So, in each of the three cases we have that $(x, x') \in \overline{\rho\left(X\right)} \cup \left(\dot{\rho}\left(X\right)^{\smile} \circ \dot{\rho}\left(X\right)\right) \cup \left(\dot{\rho}\left(X\right) \circ \dot{\rho}\left(X\right)^{\smile}\right)$.

Finally, for the $\supseteq$-direction, observe that $\overline{\rho\left(X\right)} \subseteq \rho\left(X\right)^{\smile}$ by Fact 6.1.2. So, assume for arbitrary $x, x' \in S$ such that $(x, x') \in \dot{\rho}\left(X\right)^{\smile} \circ \dot{\rho}\left(X\right)$. Then there is some y such that $(x, y) \in \dot{\rho}\left(X\right)^{\smile}$ and $(y, x') \in \dot{\rho}\left(X\right)$. Consequently, $x \in X$, $y \notin X$ and $x' \in X$ and we may conclude that $(x, x') \in \rho\left(X\right)^{\smile}$. Similarly if $(x, x') \in \dot{\rho}\left(X\right)^{\smile} \circ \dot{\rho}\left(X\right)$. Then there is some y such that $(x, y) \in \dot{\rho}\left(X\right)$ and $(y, x') \in \dot{\rho}\left(X\right)^{\smile}$. From this follows that $x, x' \notin X$ and $y \in X$ and again we may conclude that $(x, x') \in \rho\left(X\right)^{\smile}$. $\dashv$

A proper definition of an operation $\cdot$ such that $\rho\left(X\right) \cdot \rho\left(Y\right) = \rho\left(X \cap Y\right)$ has to give rise to the correct relation in a number of different cases.

First, we consider the case in which one of X or Y is a subset of the other. For an illustration of this situation, consider Figure 6.1, assuming without loss of generality that $Y \subseteq X$. Barring the borderline cases in which either X equals S or Y

is empty, the intersection of $\rho(X)$ and $\rho(Y)$ gives us exactly the relation we want, except that also all pairs (x, x') with $x \in X - Y$ and $x' \in \overline{X}$ should be in $\rho(X \cap Y)$ as well. Define for arbitrary relations ρ and ρ' over some set S:

$$\rho \circledast \sigma \quad =_{df.} \quad \big(\rho \cap \sigma\big) \cup \Big((\dot{\rho} \cup \dot{\sigma}) \circ (\dot{\rho}^{\smile} \cap \dot{\sigma}^{\smile}) \Big).$$

We abbreviate $\rho(X) \circledast \rho(Y)$ to $\rho^*(X, Y)$. This definition, when applied to bisective relations, gives precisely the relation we were after. Figure 6.1 illustrates its workings informally, whereas formally we have the following lemma.

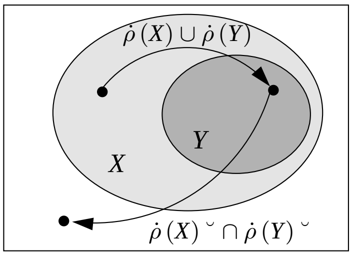

Figure 6.2. Each element in $X - Y$ is related to all elements outside X by $\big(\dot{\rho}(X) \cup \dot{\rho}(Y)\big) \circ \big(\dot{\rho}(X)^{\smile} \cap \dot{\rho}(Y)^{\smile}\big)$, if $Y \subseteq X$.

Lemma 6.1.6 *Let X, Y be subsets of some set S such that $X \subseteq Y$ or $Y \subseteq X$. Then,*

$$\rho(X \cap Y) \quad = \quad \rho^*(X, Y).$$

Proof: First assume either X or Y to be empty. Then $X \cap Y = \emptyset$ and consequently $\rho(X \cap Y) = \rho(\emptyset) = \emptyset$. It also follows that $\rho(X) = \emptyset$ or $\rho(Y) = \emptyset$. In either case, $\rho(X) \cap \rho(X) = \emptyset$ and, moreover, $\dot{\rho}(X)^{\smile} \cap \dot{\rho}(Y)^{\smile} = \emptyset$, which renders $\big(\dot{\rho}(X) \cup \dot{\rho}(Y)\big) \circ \big(\dot{\rho}(X)^{\smile} \cap \dot{\rho}(Y)^{\smile}\big)$ to be empty. Hence, $\rho^*(X, Y) = \emptyset \cup \emptyset = \emptyset$, as well.

So, for the remainder we may assume that $X \neq \emptyset$ and $Y \neq \emptyset$. Moreover, since either $X \subseteq Y$ or $Y \subseteq X$, we also have that $X \cap Y \neq \emptyset$ and $\rho(X \cap Y) = \rho(X \cap Y)$. Without loss of generality we may assume that $X \subseteq Y$. Now, either $Y = S$ or $Y \neq S$.

In the former case, $\rho(X \cap Y) = \rho(X \cap S) = \rho(X)$. Observe that $\rho(Y) = \rho(S) = U$ and hence $\rho(X) \cap \rho(Y) = \rho(X) \cap U = \rho(X)$. Moreover, $\dot{\rho}(Y) = \dot{U} = \emptyset$, which makes that $\big(\dot{\rho}(X) \cup \dot{\rho}(Y)\big) \circ \big(\dot{\rho}(X)^{\smile} \cap \dot{\rho}(Y)^{\smile}\big)$ is empty. Hence, $\rho^*(X, Y) = \rho(X) \cup \emptyset = \rho(X) = \rho(X \cap Y)$.

This leaves us with the case in which $Y \neq S$. So for the $\supseteq$-direction, consider arbitrary $x, x' \in S$ such that $(x, x') \in \rho^* (X, Y)$. If $(x, x') \in \rho (X) \cap \rho (Y)$, by Fact 6.1.3, immediately $(x, x') \in \rho (X \cap Y)$. So assume $(x, x') \in \big(\dot{\rho} (X) \cup \dot{\rho} (Y)\big) \circ \big(\dot{\rho} (X)^{\smile} \cap \dot{\rho} (Y)^{\smile}\big)$. Then there is some $x'' \in S$ such that $(x, x'') \in \dot{\rho} (X) \cup \dot{\rho} (Y)$ and $(x'', x') \in \dot{\rho} (X)^{\smile} \cap \dot{\rho} (Y)^{\smile}$. From the former follows that $x \notin X$ or $x \notin Y$; in either case $x \notin X \cap Y$. Hence, $(x, x') \in \rho (X \cap Y)$.

For the $\subseteq$-direction, assume $(x, x') \in \rho (X \cap Y)$. So either $x \notin X \cap Y$ or $x' \in X \cap Y$. If the latter, then both $(x, x') \in \rho (X)$ and $(x, x') \in \rho (Y)$ and so $(x, x') \in \rho (X) \cap \rho (Y)$ and *a fortiori* $(x, x') \in \rho^* (X, Y)$. Hence, in the former case we may additionally assume that $x' \notin X \cap Y$. With $X \subseteq Y$, then $x \notin X$ and $x' \notin X$. Accordingly, $(x, x') \in \rho (X)$. If now $x' \in Y$, then $(x, x') \in \rho (Y)$. Accordingly $(x, x') \in \rho (X) \cap \rho (Y)$ and $(x, x') \in \rho^* (X, Y)$. So, finally, consider the case in which $x' \notin Y$. Since $X \cap Y \neq \varnothing$ there is some $x'' \in X \cap Y$. With $x \notin X$ and $x'' \in X$, we have $(x, x'') \in \dot{\rho} (X)$ and so also $(x, x'') \in \dot{\rho} (X) \cup \dot{\rho} (Y)$. Moreover, with $x' \notin X$ and $x' \notin Y$, also both $(x'', x') \in \dot{\rho} (X)^{\smile}$ and $(x'', x') \in \dot{\rho} (Y)^{\smile}$. We conclude the proof by observing that now $(x'', x') \in \dot{\rho} (X)^{\smile} \cap \dot{\rho} (Y)^{\smile}$, and subsequently that $(x, x') \in \big(\dot{\rho} (X) \cup \dot{\rho} (Y)\big) \circ \big(\dot{\rho} (X)^{\smile} \cap \dot{\rho} (Y)^{\smile}\big)$. Hence, $(x, x') \in \rho^* (X, Y)$. ⊣

As an auxiliary relation construct, define for arbitrary relations ρ and σ over some set S the binary operator $\circledcirc$ as follows:

$$\rho \circledcirc \sigma \quad =_{df.} \quad \big(\rho \circ (\dot{\rho}^{\smile} \cap \dot{\sigma}^{\smile}) \circ \rho^{\smile}\big) \cup \big(\sigma \circ (\dot{\sigma}^{\smile} \cap \dot{\rho}^{\smile}) \circ \sigma^{\smile}\big).$$

We denote $\rho (X) \circledcirc \rho (Y)$ by $\rho^{\circ} (X, Y)$. For subsets X and Y the relation $\rho^{\circ} (X, Y)$ connects any elements of S with any one *outside* $X \cap Y$, provided that the relative differences between X and Y are not empty. Otherwise, $\rho^{\circ} (X, Y)$ is the empty relation. The machinations of $\rho^{\circ} (X, Y)$ if $X - Y$ and $Y - X$ are non-empty are illustrated in Figure 6.1. Formally, we have the following lemma.

Lemma 6.1.7 *Let S be a set, $X, Y \subseteq S$. Then:*

$$\rho^{\circ} (X, Y) \quad = \quad \begin{cases} \big\{(x, x') : \; x' \notin X \cap Y\big\}, & \textit{if } X - Y, Y - X \neq \varnothing \\ \varnothing & \textit{otherwise}. \end{cases}$$

Proof: First assume that either $X - Y = \varnothing$ or $Y - X = \varnothing$. Then either $X \subseteq Y$ or $Y \subseteq X$. Without loss of generality, we may assume the former. It suffices to demonstrate that both $\dot{\rho} (X)^{\smile} \cap \dot{\rho} (Y) = \varnothing$ and $\dot{\rho} (Y)^{\smile} \cap \dot{\rho} (X) = \varnothing$. In case $X = \varnothing$, both $\dot{\rho} (X) = \varnothing$ and $\dot{\rho} (X)^{\smile} = \varnothing$ and we are done immediately. If however $X \neq \varnothing$, then also $Y \neq \varnothing$. Now consider arbitrary $x, x' \in S$. We first prove that

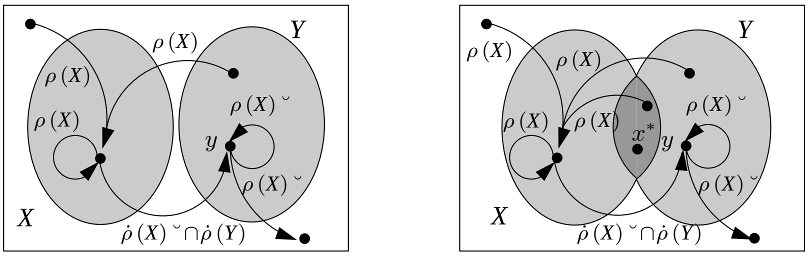

Figure 6.3. The relation $\dot\rho(X)^{\smile} \cap \dot\rho(Y)$ holds precisely between those elements in $X - Y$ and those in $Y - X$. Each element in $Y - X$ is connected to any other element outside X and to none therein; consider, *e.g.*, the element x^*. Provided that both $X - Y$ and $Y - X$ are non-empty, all elements are related to an element in $X - Y$ by $\rho(X)$ and also to precisely all elements outside X *via* $\rho(X) \circ \big(\dot\rho(X)^{\smile} \cap \dot\rho(Y)\big) \circ \rho(X)^{\smile}$. A similar argument shows that $\rho(Y) \circ \big(\dot\rho(Y)^{\smile} \cap \dot\rho(X)\big) \circ \rho(Y)^{\smile}$ connects all elements to all elements outside Y.

$\dot\rho(X) \cap \dot\rho(Y)^{\smile} = \varnothing$. If $(x, x') \notin \dot\rho(X)$ we have immediately $(x, x') \notin \dot\rho(X) \cap \dot\rho(Y)^{\smile}$. So assume $(x, x') \in \dot\rho(X)$. Then, $x \notin X$ and $x' \in X$. By the inclusion of X in Y, also $x' \in Y$. It follows that $(x, x') \notin \dot\rho(Y)^{\smile}$. Now again $(x, x') \notin \dot\rho(X) \cap \dot\rho(Y)^{\smile}$. With x and y having been chosen arbitrarily, we have that $\dot\rho(X) \cap \dot\rho(Y)^{\smile} = \varnothing$.

We now prove that $\dot\rho(Y) \cap \dot\rho(X)^{\smile} = \varnothing$. Similarly, for arbitrary $x, x' \in S$, if $(x, x') \notin \dot\rho(Y)$ immediately $(x, x') \notin \dot\rho(Y) \cap \dot\rho(X)^{\smile}$. So assume $(x, x') \in \dot\rho(Y)$; then $x' \in Y$. Also $Y \neq \varnothing$ and $x \notin X$. From the latter, $(x', x) \notin \dot\rho(X)$, *i.e.*, $(x, x') \notin \dot\rho(X)^{\smile}$. It now follows that $(x, x') \notin \dot\rho(X)^{\smile} \cap \dot\rho(Y)$. With x and x' having been chosen arbitrarily, $\dot\rho(Y)^{\smile} \cap \dot\rho(X) = \varnothing$.

Second, assume that both $X - Y \neq \varnothing$ and $Y - X \neq \varnothing$. Then both $X \neq \varnothing$ and $Y \neq \varnothing$. We prove subsequently that $\rho^\circ(X, Y) \subseteq \big\{(x, x') : \; x' \notin X \cap Y\big\}$ and that $\rho^\circ(X, Y) \supseteq \big\{(x, x') : \; x' \notin X \cap Y\big\}$.

For the $\subseteq$-direction, assume for arbitrary $x, x' \in S$ first that $(x, x') \in \rho(X) \circ (\dot\rho(X)^{\smile} \cap \dot\rho(Y)) \circ \rho(X)^{\smile}$. Then there are y, y'' such that $(x, y) \in \rho(X)$, $(y, y') \in \dot\rho(X)^{\smile} \cap \dot\rho(Y)$ and $(y', x') \in \rho(X)^{\smile}$. Hence, in particular $(y, y') \in \dot\rho(X)^{\smile}$ and so $y' \notin X$. Since also $(y', x') \in \rho(X)^{\smile}$, we have $x' \notin X$ and *a fortiori* $x' \notin X \cap Y$. For $(x, x') \in \rho(Y) \circ (\dot\rho(Y)^{\smile} \cap \dot\rho(X)) \circ \rho(Y)^{\smile}$ a similar argument shows that $x' \notin Y$ and, accordingly, $x' \notin X \cap Y$.

For the $\supseteq$-direction assume for arbitrary $x, x' \in S$ that $x' \notin X \cap Y$. Hence, either $x' \notin X$ or $x' \notin Y$. For symmetry reasons, we may assume without loss of generality that the former holds. Observe that since $X - Y \neq \varnothing$, there is some $x'' \in X$ such that $x'' \notin Y$. Similarly, in virtue of $Y - X \neq \varnothing$, there is some $y \in Y$ such that $y \notin X$.

Now, obviously, $(x, x'') \in \rho(X)$. Moreover, with $x'' \in X$ and $y \notin X$, $(y, x'') \in \dot{\rho}(X)$ and so $(x'', y) \in \dot{\rho}(X)^{\smile}$. With $x'' \notin Y$ and $y \in Y$ also $(x'', y) \in \dot{\rho}(Y)$. Hence $(x'', y) \in \dot{\rho}(X)^{\smile} \cap \dot{\rho}(Y)$. With $x' \notin X$, immediately $(y, x') \in \rho(X)^{\smile}$. Putting things together, we may conclude that $(x, x') \in \rho(X) \circ (\dot{\rho}(X)^{\smile} \cap \dot{\rho}(Y)) \circ \rho(X)^{\smile}$, which suffices for a proof. $\dashv$

The ground has now been cleared for a definition of $\rho(X \cap Y)$ in terms of $\rho(X)$, $\rho(Y)$ and the relation algebraic operators alone. Little intuitive motivation can unfortunately be given apart from that it works.

Lemma 6.1.8 *Let X and Y be subsets of some set S. Then:*

$$\rho(X \cap Y) \;=\; \big(\rho^{*}(X, Y) - \rho^{\circ}(X, Y)\big) \cup \big(\dot{\overline{\rho^{\circ}(X, Y)}} \circ \rho^{\circ}(X, Y)\big).$$

Proof: If $X - Y = \varnothing$ or $Y - X = \varnothing$, either $X \subseteq Y$ or $Y \subseteq X$. In this case, by Lemma 6.1.7, $\rho^{\circ}(X, Y) = \varnothing$ and, hence, also $\dot{\overline{\rho^{\circ}(X, Y)}} = \dot{\overline{\rho^{\circ}(X, Y)}} \circ \rho^{\circ}(X, Y) = \varnothing$.

we must prove that $\rho(X \cap Y) = \rho^{*}(X, Y)$. This, however, is immediate by Lemma 6.1.6.

For the remainder of the proof we may assume that $X - Y \neq \varnothing$ and $Y - X \neq \varnothing$. Hence we may also assume that $\rho^{\circ}(X, Y) = \{(x, x') : \; x' \notin X \cap Y\}$ and, moreover, that $(x, x') \in \dot{\overline{\rho^{\circ}(X, Y)}}$ if and only if $x \notin X \cap Y$ and $x' \in X \cap Y$. There are two cases; either $X \cap Y = \varnothing$ or $X \cap Y \neq \varnothing$. If the former, $\rho(X \cap Y) = \rho(\varnothing) = \varnothing$. Also observe that $\rho^{\circ}(X, Y) = U$. Hence, $\rho^{*}(X, Y) - \rho^{\circ}(X, Y) = \rho^{*}(X, Y) - U = \varnothing$ and $\dot{\overline{\rho^{\circ}(X, Y)}} \circ \rho^{\circ}(X, Y) = \dot{\overline{U}} \circ U = \dot{\varnothing} \circ U = \varnothing \circ U = \varnothing$.

So, assume $X \cap Y \neq \varnothing$. Then neither X nor Y is empty and the facts 6.1.1 through 6.1.4 may be invoked. We prove, subsequently, both $\rho(X \cap Y) \subseteq \big(\rho^{*}(X, Y) - \rho^{\circ}(X, Y)\big) \cup \big(\dot{\overline{\rho^{\circ}(X, Y)}} \circ \rho^{\circ}(X, Y)\big)$ and $\rho(X \cap Y) \supseteq \big(\rho^{*}(X, Y) - \rho^{\circ}(X, Y)\big) \cup \big(\dot{\overline{\rho^{\circ}(X, Y)}} \circ \rho^{\circ}(X, Y)\big)$.

For the $\subseteq$-direction, consider arbitrary arbitrary $x, x' \in S$ such that $(x, x') \in \rho(X \cap Y)$. Either $x' \notin X \cap Y$ or $x' \in X \cap Y$. If the latter, both $x' \in X$ and $x' \in Y$. Hence $(x, x') \in \rho(X)$ and $(x, x') \in \rho(Y)$ and so $(x, x') \in \rho(X) \cap \rho(Y)$. Consequently also $(x, x') \in \rho^{*}(X, Y)$. Since $x' \in X \cap Y$, $(x, x') \notin \rho^{\circ}(X, Y)$. Accordingly, $(x, x') \in \rho^{*}(X, Y) - \rho^{\circ}(X, Y)$, and we are done.

In the former case, *i.e.*, if $x' \notin X \cap Y$, because $(x, x) \in \rho(X \cap Y)$, also $x \notin X \cap Y$. With $X \cap Y$ having been assumed to be not empty, there is some $x'' \in X \cap Y$. So, $(x, x'') \in \dot{\overline{\rho^{\circ}(X, Y)}}$. Moreover, since $x' \notin X \cap Y$, also $(x'', x') \in \rho^{\circ}(X, Y)$. Hence, $(x, x') \in \dot{\overline{\rho^{\circ}(X, Y)}} \circ \rho^{\circ}(X, Y)$, and again we are done.

For the $\supseteq$-direction, consider arbitrary elements x and x' of S such that $(x,x') \in \left(\rho^*(X,Y) - \rho^\circ(X,Y)\right) \cup \left(\dot{\overline{\rho^\circ(X,Y)}} \circ \rho^\circ(X,Y)\right)$. If $(x,x') \in \rho^*(X,Y) - \rho^\circ(X,Y)$, obviously $(x,x') \notin \rho^\circ(X,Y)$. Under the present assumptions this means that $x' \in X \cap Y$, and so, immediately, $(x,x') \in \rho(X \cap Y)$.

Finally assume $(x,x') \in \dot{\overline{\rho^\circ(X,Y)}} \circ \rho^\circ(X,Y)$. Then there is an $x'' \in S$ such that $(x,x'') \in \dot{\overline{\rho^\circ(X,Y)}}$ and $(x'',x') \in \rho^\circ(X,Y)$. From the former we obtain that $x \notin X \cap Y$ and consequently $(x,x') \in \rho(X \cap Y)$. ⊣

Now we are in a position to define the relation constructs $-$, $\cdot$ and $+$. For ρ and σ be relations on some set S, define:

$$\begin{aligned} -\rho &=_{df.} \overline{\rho} \cup (\dot{\rho}^{\smile} \circ \dot{\rho}) \cup (\dot{\rho} \circ \dot{\rho}^{\smile}) \\ \rho \cdot \sigma &=_{df.} \left(\rho \circledast \sigma - \rho \circledcirc \sigma\right) \cup \left(\dot{\overline{\rho \circledcirc \sigma}} \circ \rho \circledcirc \sigma\right) \\ \rho + \sigma &=_{df.} -(-\rho \cdot -\sigma). \end{aligned}$$

The following proposition merely wraps things up.

Proposition 6.1.9 *Let X and Y be relations on a set S. Then:*

$$\begin{aligned} -\rho(X) &= \rho\left(\overline{X}\right), \\ \rho(X) \cdot \rho(Y) &= \rho(X \cap Y), \\ \rho(X) + \rho(Y) &= \rho(X \cup Y). \end{aligned}$$

Proof: The first two claims are immediate from Lemmas 6.1.5 and 6.1.8. The last claim follows from the first two and duality of $\cup$ and $\cap$. ⊣

As an almost immediate consequence of Proposition 6.1.9 we have that $\rho(\varphi)$ can be given a compositional definition in φ. To appreciate this define for each formula φ of a propositional language $L(A)$ a relation $\rho_2(\varphi)$ over the valuations inductively as follows:

$$\begin{aligned} \rho_2(a) &=_{df.} \left\{(s,s') : \ a \in s \text{ implies } a \in s'\right\} \\ \rho_2(\bot) &=_{df.} \emptyset \\ \rho_2(\neg\varphi) &=_{df.} -\rho_2(\varphi) \\ \rho_2(\varphi \wedge \psi) &=_{df.} \rho_2(\varphi) \cdot \rho_2(\psi) \end{aligned}$$

The following theorem formulates the result we were after in this section.

Theorem 6.1.10 *Let φ be a formula of a propositional language $L(A)$. Then,* $\rho(\varphi) = \rho_2(\varphi)$.

Proof: By induction to the complexity of φ. The first atomic case is almost trivial, merely observe that $[\![a]\!]$ is never empty. The second atomic case is trivial. The inductive cases follow almost immediately from Proposition 6.1.9. $\dashv$

Bibliography

Benthem, J. F. A. K. van (1996), *Exploring Logical Dynamics*. Studies in Logic, Language and Information. CSLI, Stanford, Ca.

Boutilier, C. (1994), "Towards a Logic for Qualitative Decision Theory." in: *Proceedings of the Fourth International Conference on Knowledge Representation and Reasoning (KR'94)*, pp. 75–86. Morgan Kaufmann, San Francisco.

Chang, C. C. and Keisler, J. K. (1973), *Model Theory*. Studies in Logic and the Foundations of Mathematics. North-Holland, Amsterdam.

Düntsch, I. (1999), *Rough Sets and Algebras of Relations*. University of Ulster.

——— (no date), *A Logic for Rough Sets*. University of Ulster.

Gärdenfors, P. (1988), *Knowledge in Flux*. The MIT Press, Cambridge, Mass.

Grätzer, G. (1968), *Universal Algebra*. D. Van Nostrand, Princeton, N.J.

Harrenstein, B. P. (2004), "Logic in Conflict. Logical Explorations in Strategic Equilibrium." Ph.D. thesis, Utrecht University, Utrecht.

——— (to appear), "Logical Consequence and the Theory of Games." *Philosophia Scientiae* **8** (2), 183–197. to appear in the special issue on game theory and logic in 2004.

Kraus, S., Lehmann, D. and Magidor, M. (1990), "Nonmonotonic Reasoning, Preferential Models and Cumulative Logics." *Artificial Intelligence* **44**, 167–207.

Makinson, D. (1994), "General Patterns in Nonmonotonic Reasoning." in: Gabbay, D., Hogger, C. J. and Robinson, J. A. (eds.), *Nonmonotonic Reasoning and Uncertain Reasoning*, Volume 3 of *Handbook of Logic in Artificial Intelligence and Logic Programming*, pp. 35–110. Oxford U.P., Oxford.

Pawlak, Z. (1991), *Rough Sets: Theoretical Aspects of Reasoning about Data*. Kluwer, Dordrecht.

Segerberg, K. (1982), *Classical Propositional Operators*. Volume 5 of *Oxford Logic Guides*. Oxford U.P., Oxford.

Shoham, Y. (1988), *Reasoning About Change*. MIT Press, Cambridge, Mass.

Smullyan, R. (1968), *First-Order Logic*. Springer Verlag, New York.

Tarski, A. (1936a), “Der Wahrheitsbegriff in den formalisierten Sprachen.” *Studia Philosophica* **1**, 261–405. Translated as ‘The concept of truth in formalized languages’ in: J.H. Woodger (ed.), *Logic, Semantics and Metamathematics*.

——— (1936b), “Über den Begriff der logischen Folgerung.” in: *Actes du Congrès International de Philosophie Scientifique VII*, Actualités Scientifiques et Industrielles 394, pp. 1–11. Hermann & Cie, Paris. Translated as ‘On the Concept of Logical Consequence’ in: J.H. Woodger (ed.), *Logic, Semantics and Metamathematics*.

——— (1944), “The Semantic Conception of Truth.” *Philosophy and Phenomenological Research* **4**, 341–375.

Veltman, F. (1996), “Defaults in Update Semantics.” *Journal of Philosophical Logic* **25**, 221–261.

Yao, Y. Y., Wong, M. and Lin, T. Y. (1997), “A Review of Rough Set Models.” in: Lin, T. Y. and Cercone, N. (eds.), *Rough Sets and Data Mining. Analysis of Imprecise Data*, pp. 47–75. Kluwer, Dordrecht.