Immobilizing Hinged Polygons

Jae-Sook Cheong* A. Frank van der Stappen* Ken Goldberg†
Mark H. Overmars* Elon Rimon‡

Abstract: The immobilization of *non-rigid* objects is a largely unaddressed subject. We explore the problem by studying the immobilization of a serial chain of polygons connected by rotational joints, or *hinges*, in a given placement with frictionless point contacts. We show that $n+2$ such contacts along edge interiors or at concave vertices suffice to immobilize any serial chain of $n \neq 3$ polygons without parallel edges; it remains open whether five contacts can immobilize three hinged polygons. At most $n+3$ contacts suffice to immobilize a serial chain of n polygons when the polygons are allowed to have parallel edges. We also study a robust version of immobility, comparable to the classical notion of form closure, which is insensitive to perturbations. The robustness is achieved at the cost of a small increase in the number of contacts: $\lceil \frac{6}{5}(n+2) \rceil$ and $\lceil \frac{5}{4}(n+2) \rceil$ frictionless point contacts suffice for a chain of n hinged polygons without and with parallel edges respectively.

1 Introduction

Many manufacturing operations, such as machining and assembly, require the objects, or *parts*, that are subjected to these operations to be held such that they can resist all external wrenches (i.e., forces and torques). Immobilization of parts, fundamental to the common tasks of grasping and fixturing [4, 6, 11, 36], is a problem that has been studied extensively, see e.g. [9, 15, 17, 19, 21, 27, 29, 30]. We consider a planar version of part immobilization, and therefore confine ourselves to 2D in the discussion of related work.

The concept of *form closure* was formulated by Reuleaux [27] in 1876. It provides a sufficient condition for constraining all finite and infinitesimal motions of a rigid part by a set of contacts along its boundary, despite the application of possible external wrenches. In other words, any finite or infinitesimal motion of a part in form closure violates the rigidity of the part or the contacts. Form closure is based on an analysis of instantaneous velocity centers. Markenscoff et al. [17] and Mishra et al. [19] independently showed that four frictionless point contacts are sufficient to put a polygon in form closure. In fact, their results apply to almost any planar rigid part.

Czyzowicz et al. [9] showed that three frictionless contacts suffice to immobilize a polygon without parallel edges. They identified necessary and sufficient conditions under which three frictionless point contacts immobilize polygon. These conditions depend only on the geometry of the part and the positions of the contacts. Rimon and Burdick [29, 30] showed that three contacts can immobilize

*Institute of Information and Computing Sciences, Utrecht University, P.O.Box 80089, 3508 TB Utrecht, the Netherlands; email: {jaesook,markov,frankst}@cs.uu.nl.

†Department of Industrial Engineering and Operations Research, University of California at Berkeley, Berkeley, CA 94720, USA; email: goldberg@ieor.berkeley.edu. Goldberg was supported in part by the National Science Foundation under DMI-0010069 and by a grant from Ford Motor Company.

‡Department of Mechanical Engineering, Technion – Israel Institute of Technology, Haifa 32000, Israel; email: elon@robby.technion.ac.il.

almost *any* rigid part. The analysis of their so-called *second-order immobility* takes place in the (three-dimensional) configuration space of the part and regards the contacts as obstacles that limit the part's ability to move. The fundamental difference between the equivalent notions (for polygonal parts) of Czyzowicz et al. [9] and Rimon and Burdick [29, 30] on the one hand, and form closure on the other hand is that the former take into account the curvature of the possible motions of the part, whereas the latter only considers the directions of these motions. (The difference explains why Rimon and Burdick [29, 30] refer to form closure as first-order immobility.) The possible motions of the part depend on the shapes of the part and the contacts. The inclusion of curvature effects is powerful enough to show that three frictionless point contacts (instead of the four for form closure) suffice to immobilize almost all planar parts, including polygons without parallel edges [28].

All aforementioned results on immobility analyses and on sufficient numbers of contacts for immobilization deal with rigid parts. Likewise, all papers on the computation of immobilizing grasps and fixtures focus on rigid parts (see e.g. [4, 13, 21, 24, 25, 26, 33, 34]). Parts subjected to manufacturing operations may, however, be non-rigid; they can be entire assemblies, have movable parts, and even be deformable. Deformable objects have been studied in areas such as graphics and virtual environments, but also in robotics. Gibson and Mirtich [10] survey modeling of deformable objects in graphics. Many of the results for deformable objects in robotics are in motion planning and manipulation. The motion planning problem has e.g. been addressed for flexible surface patches [12], elastic objects [14, 16], and deformable volumes [1], and for non-rigid structures like serpentine robots [23, 5]. Deformable objects such as flexible metal sheets [22], vibrating objects [31], and wires [20] have been considered in the context of manipulation. Closer to immobilization is the work on assemblies and stability by Baraff et al. [2] and Mattikalli et al. [18], but the parts in the assembly are not connected. Also somewhat related is the work on fixturing toleranced parts by Chen et al. [7].

As a first step in the direction of immobilizing non-rigid parts, we study the immobilization of a serial chain of n polygons connected by rotational joints, or hinges. This case study presents an initial step towards immobilization of complex structures of interconnected rigid bodies and of deformable objects. We shall assume that a hinge is a vertex of each of the two polygons it connects. A hinge allows both adjacent polygons to rotate around it. Our aim is to identify how many frictionless point contacts suffice to immobilize any serial chain of n hinged polygons in a priorly specified placement. For practical reasons it is undesirable to place point contacts at convex vertices since that may lead to damage to the part (see e.g. [4]), and the process of establishing such a contact is inherently unstable. As a result, we forbid the placement of contacts at convex vertices. Note that it is therefore impossible to have a contact at a hinge: a hinge is a convex vertex of at least one of the two incident polygons.

Our intuitive analysis of possible motions of serial chains resembles that of Czyzowicz et al. [9] in that it also takes place in the two-dimensional space of the object itself, and it also takes into account the curvature of the motions. Our analysis concentrates on local motions of the $n-1$ hinges and rotations of the two distal polygons. These ingredients suffice to determine whether the entire chain is immobilized.

We show that $n+2$ frictionless point contacts suffice to immobilize any serial chain of $n \neq 3$ polygons without parallel edges in a given placement; it is unclear whether five contacts can always achieve immobilization of a chain of three polygons. We observe that the number $n+2$ equals the number of degrees of freedom of a serial chain of n hinged polygons. If the individual polygons are allowed to have parallel edges, then we can show that $n+2$ point contacts are still sufficient if n is even and that $n+3$ point contacts suffice if n is odd. All the proofs are constructive in the sense that we give actual immobilizing point-contact arrangements.

An essential difference between a three-contact arrangement providing second-order immobility (or immobility according to Czyzowicz et al.) and a four-contact arrangement providing form closure of a rigid part lies in their sensitivity to slight perturbations. In the former case small perturbations of the contacts—caused e.g. by misplacement of the contacts along the part boundary—are highly unlikely to retain immobility.[1] In the latter case, immobility is definitely maintained for sufficiently small perturbations. Our immobilizations with $n+2$ point contacts behave consistently, as it will be easy to see that small perturbations of the contacts destroy the immobility. This has motivated us to also investigate the number of point contacts that is sufficient to obtain a more robust form of immobilization, which has the property—like form closure—that any contact can be perturbed slightly along the edges without destroying the immobility. We construct robustly immobilizing point contact arrangements for a serial chain of n polygons with $\lceil \frac{6}{5}(n+2) \rceil$ contacts if the polygons have no parallel edges, and with $\lceil \frac{5}{4}(n+2) \rceil$ contacts if the polygons are allowed to have parallel edges. Informally speaking, we achieve robustness at the cost of one additional contact per five or four polygons.

The paper is organized as follows. We discuss immobility and introduce robust immobility in Section 2, and give some key results for rigid polygonal parts. In Sections 3 and 4, we constructively show how many contacts suffice to immobilize or robustly immobilize a serial chain of n hinged polygons without and with parallel edges. Section 5 summarizes the results, identifies the open problems, and gives directions for future research.

2 Preliminaries

In this section we introduce some notation and discuss our notions of immobility and robust immobility. We briefly review the graphical analysis of instantaneous velocity centers that forms the basis to form closure; we will employ this conservative analysis of potential motions in some of our proofs. Finally we report some useful results on immobilization of a single polygon.

Let $(\mathcal{B}_1, ..., \mathcal{B}_n)$ be a serial chain of n hinged closed polygons. Each polygon $\mathcal{B}_i$ in the chain shares a vertex—the hinge—with its successor $\mathcal{B}_{i+1}$; we denote the hinge connecting $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$ by v_i. A hinge v_i allows the adjacent polygons $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$ to rotate relative to each other. We will denote by $\mathcal{E}_i$ a largest enclosed circle of $\mathcal{B}_i$.

It is our aim to study how many frictionless point fingers along interiors of edges or at concave vertices of the polygons in the chain are sufficient to immobilize a chain $(\mathcal{B}_1, ..., \mathcal{B}_n)$ in a given placement q. We assume that the two edges of polygon $\mathcal{B}_i$ incident to its hinge v_i are not collinear. We also assume that at the given placement q, the polygons are strictly disjoint unless they are neighbors in the chain, in which case their intersection equals the hinge connecting them. Note that this implies that no two hinges coincide at the placement q.

A set of point contacts immobilizes the chain $(\mathcal{B}_1, ..., \mathcal{B}_n)$ at a placement q, if these contacts prevent the chain from moving to a neighboring placement q'. In other words, the set of contacts *immobilizes* the chain if any motion of the chain violates the rigidity of a polygon or a contact, or the connectedness of the polygons. As argued in the introduction, the various notions of immobility differ in the way they analyze potential motions. Just like with second-order immobility and Czyzowicz et al.'s notion of immobility, our immobility analysis takes the curvature of potential motions (which are dictated by the curvatures of the object boundary and the fingers) into account. To show that a chain is immobilized, we will regularly use an intuitive two-step analysis. First we show that none of the hinges v_i ($1 \leqslant i \leqslant n-1$) can move. Then, clearly every $\mathcal{B}_i$ with $2 \leqslant i \leqslant n-1$ is immobilized because two of its vertices—the hinges v_i and v_{i+1}—are unable to move. As a result, it only remains to show that $\mathcal{B}_1$ cannot rotate around v_1 and that $\mathcal{B}_n$ cannot rotate around v_{n-1}.

[1] In practice small perturbations of the contacts may be compensated by friction and compliance contact effects. But these effects are beyond our frictionless rigid body model.

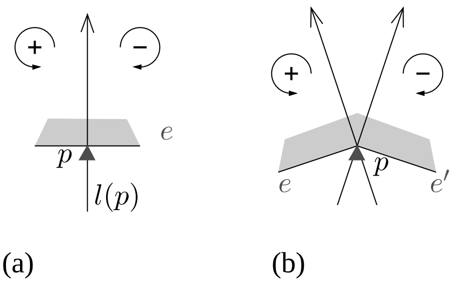


Figure 1: Motions allowed by (a) a point contact at an interior point p of an edge e, (b) a point contact at a concave vertex p with incident edges e and e'.

As announced in the introduction, our immobilizing grasps will be sensitive to perturbations: any perturbation of a contact will destroy the immobility. This is consistent with the situation for three-contact immobilizations of rigid parts. This motivates us to explore the price—in terms of an increase of the sufficient numbers of contacts—of insensitivity to small perturbations, which is also present in form-closure immobilizations. We define a *robust* form of immobility.

Definition 2.1 *A set of point contacts* robustly *immobilizes the chain* $(\mathcal{B}_1, ..., \mathcal{B}_n)$ *if these contacts immobilize* $(\mathcal{B}_1, ..., \mathcal{B}_n)$*, and if there exists a real number* $\epsilon > 0$*, such that any perturbation of a point contact in the interior of an edge by a distance at most* ϵ *along the edge maintains the immobility.*

Note that the definition does not allow for perturbations of contacts at concave vertices. We argue that misplacements at concave vertices are much less likely, as it is possible to exploit the concavity of the vertex to obtain and maintain exact contact location.

In some of our proofs it is not necessary to take into account the curvature of potential motions; instead we can settle for a conservative analysis of the instantaneous velocity centers [27]. We therefore briefly review the analysis of so-called half-plane constraints for a rigid part. It is based on the observation that every infinitesimal motion of a rigid part can be seen as either a clockwise or a counterclockwise infinitesimal rotation around a point in the plane; infinitesimal translations are rotations around a point at infinity. Hence, every infinitesimal rotation can be characterized by a point in the plane and a sign. Now let us assume that a point contact is placed at some interior point p of an edge e. Let the *normal line* of a contact at p, denoted $l(p)$, be the directed line through p perpendicular to e and directed towards the interior of the part. Clockwise rotations are possible around points in the closed half-plane right of the normal line $l(p)$ and impossible around points in the open half-plane left of that line (see Figure 1a). Similarly, counterclockwise rotations are possible around points in the closed half-plane left of the line $l(p)$ and impossible around points in the open half-plane right of the line. When a point contact is at a concave vertex, it induces two such half-plane constraints, as such a vertex is considered to belong to both incident edges (see Figure 1b). A part is in form closure if the half-plane constraints induced by the contacts along the part boundary jointly rule out all possible centers of clockwise and counterclockwise rotations. Normal lines with the same or with opposite directions are assumed to intersect at infinity. It is clear that four half-plane constraints are necessary to achieve form closure.

We end this section by reporting several useful results concerning the immobilization of a single polygon. Lemma 2.1 links form closure to our notion of robust immobility.

Lemma 2.1 *A two dimensional polygon* $\mathcal{B}_i$ *in form closure is robustly immobilized.*

Proof: This follows easily from the half-plane constraint analysis for form closure. The intersection of the closed half-planes left of a set of directed lines is empty, and the intersection of the closed half-planes right of the same set of directed lines is empty. It is easy to see that sufficiently small perturbations maintain the emptiness of the intersections of the closed half-planes. ◻

Several papers employ a largest enclosed circle of a part to generate one immobilizing arrangement of that part. The following results for a polygon with a largest enclosed circle that intersects the polygon boundary in three points of which no two are antipodal on that circle follow immediately from Lemma 2.1, a proof by Czyzowicz et al. [9], and results by Rimon and Burdick [28] and Van der Stappen [32].

Lemma 2.2 *Let $\mathcal{B}_i$ be a polygon. Assume a largest enclosed circle $\mathcal{E}_i$ of $\mathcal{B}_i$ intersects the boundary of $\mathcal{B}_i$ at three points p_1, p_2, and p_3 of which no two are antipodal on $\mathcal{E}_i$.*

- *Three frictionless point contacts at p_1, p_2, and p_3 suffice to immobilize $\mathcal{B}_i$.*
- *Three frictionless point contacts at or close to p_1, p_2, and p_3 suffice to robustly immobilize $\mathcal{B}_i$ if at least one of p_1, p_2, and p_3 is a (necessarily concave) vertex of $\mathcal{B}_i$.*

Contacts at p_1, p_2, and p_3 will nearly always robustly immobilize $\mathcal{B}_i$ if one of these points is a vertex. A perturbation of exactly one contact (covered by the addition of 'close to' in the lemma) is necessary in one particular situation; see [32] for details.

Lemma 2.3 constrains the motion of a polygon with a largest enclosed circle that intersects the polygon boundary at two points that are antipodal on the circle. It considers the four segments connecting these points to the adjacent vertices on the polygon. Notice that the two segments connecting an intersection point to its adjacent vertices can be two adjacent edges of the polygon (if the intersection point is the common vertex) or two parts of a single edge (if the intersection point is an interior point of that edge). The proof of the lemma is given in the appendix.

Lemma 2.3 *Let $\mathcal{B}_i$ be a polygon. Assume a largest enclosed circle $\mathcal{E}_i$ of $\mathcal{B}_i$ intersects the boundary of $\mathcal{B}_i$ at two points p_1 and p_2 that are antipodal on $\mathcal{E}_i$. Let l be the line through p_1 and p_2.*

- *Two frictionless point contacts at p_1 and p_2 suffice to immobilize $\mathcal{B}_i$ if no (part of an) edge adjacent to p_1 is parallel to (a part of) an edge adjacent to p_2 on the same side of l.*
- *Three frictionless point contacts at or close to p_1 and p_2 suffice to robustly immobilize $\mathcal{B}_i$ if no (part of an) edge adjacent to p_1 is parallel to (a part of) an edge adjacent to p_2 on the same side of l.*
- *Two frictionless point contacts at p_1 and p_2 suffice to constrain the local motions of $\mathcal{B}_i$ to translations along a single line segment if (the part of) an edge adjacent to p_1 is parallel to (the part of) the edge adjacent to p_2 on the same side of l.*

In general, a largest enclosed circle of a polygon without parallel edges intersects the boundary of that polygon at three points of which no two are antipodal. The single rare exception occurs when the circle intersects the polygon boundary in two pairs of antipodal points. The absence of parallel edges, however, still allows us to summarize Lemmas 2.2 and 2.3 as follows.

Corollary 2.1 *Let $\mathcal{B}_i$ be a polygon without parallel edges.*

- *Three frictionless point contacts suffice to immobilize $\mathcal{B}_i$.*
- *Three frictionless point contacts suffice to robustly immobilize $\mathcal{B}_i$ if a largest enclosed circle $\mathcal{E}_i$ of $\mathcal{B}_i$ intersects a (necessarily concave) vertex of $\mathcal{B}_i$.*

Three point contacts are insufficient to immobilize certain polygons with parallel edges, such a rectangles, parallelograms, and thin trapezoids. For such cases, Lemma 2.4 supplies a sufficient number of contacts. It follows from Lemma 2.1 and results by Markenscoff et al. [17], Mishra et al. [19], and van der Stappen et al. [33].

Lemma 2.4 *Any polygon can be robustly immobilized with four frictionless point contacts.*

3 Immobility of Serial Chains

We consider immobilization of serial chains of polygons without parallel edges in Section 3.1, and with parallel edges in Section 3.2. We obtain a difference of one in the respective sufficient numbers of point contacts if the number of polygons is odd.

3.1 Polygons without Parallel Edges

We first discuss the immobilization of serial chains of at most four polygons without parallel edges. The results serve as building blocks for the immobilization of longer chains.

3.1.1 Short Chains

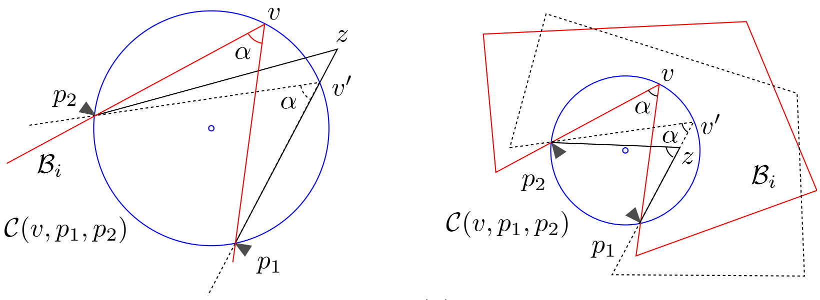
The construction of an immobilization of two hinged polygons relies on a bound on the area where the common endpoint of two edges can lie if these edges contain point contacts. Let $\mathcal{C}(p, p', p'')$ be the unique circle through three non-collinear points p, p' and p''. For a circle $\mathcal{C}$, we denote its interior including the boundary by $\mathcal{C}^+$, and the (unbounded) exterior including the boundary by $\mathcal{C}^-$. The following lemma is a generalization of a result in [3, pages 61–62].

Lemma 3.1 *Let e_1 and e_2 be adjacent edges of a polygon $\mathcal{B}_i$, and let v be their common endpoint. Assume that point contacts are placed at points p_1 and p_2 on e_1 and e_2 respectively. Any motion of $\mathcal{B}_i$ causes v to locally move into $\mathcal{C}^+(v, p_1, p_2)$ when v is convex, and $\mathcal{C}^-(v, p_1, p_2)$ when v is concave.*

Proof: Let α be the angle between e_1 and e_2 at v. Assume that v is convex. We assume for a contradiction that v can move locally to some point z strictly outside $\mathcal{C}^+(v, p_1, p_2)$ (see Figure 2a), under the restriction of the point contacts at p_1 and p_2. Let v' be the intersection point of the segment $\overline{p_1 z}$ and $\mathcal{C}(v, p_1, p_2)$. It is a well known geometric fact that the angle $\angle p_1 v' p_2 = \angle p_1 v p_2 = \alpha$. A simple trigonometric calculation shows that $\angle p_1 z p_2 < \angle p_1 v' p_2$, thus $\angle p_1 z p_2 < \alpha$, which is a contradiction. Therefore, v can only move locally in $\mathcal{C}^+(v, p_1, p_2)$.
Assume that v is concave. We assume for a contradiction that v can move locally to some point z strictly outside $\mathcal{C}^-(v, p_1, p_2)$ (see Figure 2b), under the restriction of the point contacts at p_1 and p_2. Let v' be the intersection point of $\mathcal{C}(v, p_1, p_2)$ and the extension of $\overline{p_1 z}$ past z. Again we have that $\angle p_1 v' p_2 = \angle p_1 v p_2 = \alpha$, but now the simple calculation shows that $\angle p_1 z p_2 > \angle p_1 v' p_2 = \alpha$, which is a contradiction. Therefore, v can only move locally in $\mathcal{C}^-(v, p_1, p_2)$. ◻

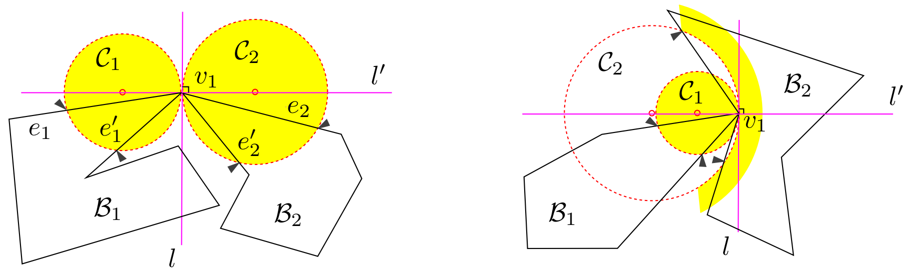
We are now ready to immobilize the chain $(\mathcal{B}_1, \mathcal{B}_2)$. Let e_1 and e_1', and e_2 and e_2' be the edges incident to v_1 of $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. The hinge v_1 must be a convex vertex of $\mathcal{B}_1$ or $\mathcal{B}_2$; we assume without loss of generality that v_1 is a convex vertex of $\mathcal{B}_1$.

- If v_1 is a convex vertex of $\mathcal{B}_2$ (as in Figure 3a) then there exists a line l through v_1 that strictly separates e_1 and e_1' from e_2 and e_2'. Let l' be the line through v_1 perpendicular to l. Let $\mathcal{C}_1$ be a circle centered on l', through v_1, and intersecting the interiors of both e_1 and e_1'. Let $\mathcal{C}_2$ be a similar circle intersecting the interiors of both e_2 and e_2'. Note that such circles can always be



(a) (b)

Figure 2: The area where v can locally move around under the restriction of the two contacts p_1 and p_2 equals (a) the closure of the interior of the circle $\mathcal{C}(v, p_1, p_2)$ when v is convex, and (b) the closure of the exterior of the circle $\mathcal{C}(v, p_1, p_2)$ when v is concave.



(a) (b)

Figure 3: Immobilization of a two-polygon chain $(\mathcal{B}_1, \mathcal{B}_2)$ with four contacts when the hinge v_1 is (a) a convex vertex of $\mathcal{B}_2$, and (b) a concave vertex of $\mathcal{B}_2$.

constructed by taking their centers on the appropriate side of l and sufficiently close to v_1; $\mathcal{C}_1$ and $\mathcal{C}_2$ are both tangent to l at v_1.

- If v_1 is a concave vertex of $\mathcal{B}_2$ (as in Figure 3b) then there exists a line l through v_1 that leaves e_1, e'_1, e_2, and e'_2 strictly on one side. Let l' be the line through v_1 perpendicular to l. Let $\mathcal{C}_2$ be a circle centered on l', through v_1, and intersecting the interiors of both e_2 and e'_2. Let $\mathcal{C}_1$ be a strictly smaller circle centered on l', through v_1, and intersecting the interiors of both e_2 and e'_2. Note that such circles can again always be constructed by properly selecting the locations of their centers; $\mathcal{C}_1$ and $\mathcal{C}_2$ are again both tangent to l at v_1.

In both cases, we place point contacts at the intersections p_1 and p'_1 of $\mathcal{C}_1$ with e_1 and e'_1, and at the intersections p_2 and p'_2 of $\mathcal{C}_2$ with e_2 and e'_2. The contacts at p_1, p'_1, p_2, and p'_2 immobilize $(\mathcal{B}_1, \mathcal{B}_2)$.

Lemma 3.2 *Four frictionless point contacts suffice to immobilize a serial chain of two polygons.*

Proof: Let $e_1, e'_1, e_2, e'_2, \mathcal{C}_1, \mathcal{C}_2, p_1, p'_1, p_2, p'_2$ be as described above. If v_1 is a convex vertex of $\mathcal{B}_2$ we observe that $\mathcal{C}_1^+ = \mathcal{C}^+(v_1, p_1, p'_1)$ and $\mathcal{C}_2^+ = \mathcal{C}^+(v_1, p_2, p'_2)$. According to Lemma 3.1, the contacts at p_1 and p'_1 force v_1 to stay inside $\mathcal{C}_1^+$ and those at p_2 and p'_2 force v_1 to stay inside $\mathcal{C}_2^+$. Since the intersection $\mathcal{C}_1^+ \cap \mathcal{C}_2^+$ contains just one point—the current location of v_1—the hinge v_1 cannot move. It remains to show that $\mathcal{B}_1$ and $\mathcal{B}_2$ cannot rotate around v_1. It is easy to see that the contacts at p_1 and p'_1 prevent clockwise and counterclockwise rotations of $\mathcal{B}_1$; p_2 and p'_2 do the same for $\mathcal{B}_2$.
If v_1 is a concave vertex of $\mathcal{B}_2$ we observe that $\mathcal{C}_1^+ = \mathcal{C}^+(v_1, p_1, p'_1)$ and $\mathcal{C}_2^- = \mathcal{C}^-(v_1, p_2, p'_2)$.

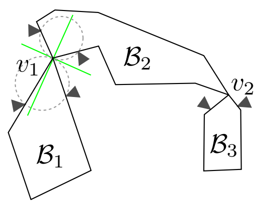


Figure 4: Immobilization of a three-polygon chain $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ with six contacts.

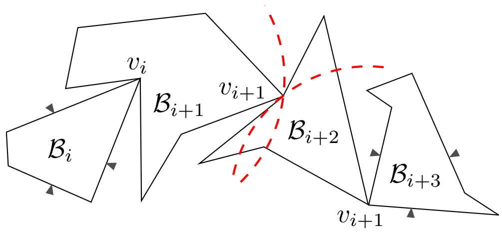


Figure 5: If subchains $(\mathcal{B}_1, ..., \mathcal{B}_i)$ and $(\mathcal{B}_{i+3}, ..., \mathcal{B}_n)$ are immobilized, the entire chain $(\mathcal{B}_1, ..., \mathcal{B}_n)$ is also immobilized.

According to Lemma 3.1, the contacts at p_1, p_1', p_2, and p_2' force v_1 to stay inside the intersection $\mathcal{C}_1^+ \cap \mathcal{C}_2^-$ which once again only contains the current location of v_1. Rotations of $\mathcal{B}_1$ and $\mathcal{B}_2$ are again prevented by the contacts at p_1, p_1', p_2, and p_2'. ◽

In general four frictionless contacts are also *necessary* to immobilize the chain $(\mathcal{B}_1, \mathcal{B}_2)$. Assume, for example, the easily obtained case in which no line through v_1 perpendicularly intersects an edge of the convex polygons $\mathcal{B}_1$ and $\mathcal{B}_2$. Any point contact can stop either the clockwise or the counter-clockwise rotations of either $\mathcal{B}_1$ or $\mathcal{B}_2$. As a result, three contacts are insufficient to immobilize this chain $(\mathcal{B}_1, \mathcal{B}_2)$.

The immobilization of the serial chain $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ poses a serious challenge. We have not succeeded in finding a generic way of immobilizing such chains with five frictionless contacts. Although this would mark a striking exception, we therefore conjecture that there exist chains of three hinged polygons that cannot be immobilized with five frictionless point contacts. It is straightforward to immobilize any chain $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ with six point contacts. One simply uses four contacts to immobilize the subchain $(\mathcal{B}_1, \mathcal{B}_2)$, and then add two contacts on the edges of $\mathcal{B}_3$ incident to v_2 to immobilize $\mathcal{B}_3$ as well (Figure 4).

Lemma 3.3 *Six frictionless point contacts suffice to immobilize a serial chain of three polygons.*

Proof: By immobilizing the subchain $(\mathcal{B}_1, \mathcal{B}_2)$ using Lemma 3.2, we fix the location of the hinge v_2. As a result, it remains to show that $\mathcal{B}_3$ cannot rotate around v_2. The contacts along the edges of $\mathcal{B}_3$ incident to v_2 prevent clockwise and counterclockwise rotations of $\mathcal{B}_3$. ◽

Finally, we address the immobilization of serial chains of four polygons. The immobilization is based on the following lemma.

Lemma 3.4 *A chain $(\mathcal{B}_1, ..., \mathcal{B}_n)$ with $n \geqslant 4$ is immobilized if, for some $1 \leqslant i \leqslant n-3$, the subchains $(\mathcal{B}_1, ..., \mathcal{B}_i)$ and $(\mathcal{B}_{i+3}, ..., \mathcal{B}_n)$ are both immobilized.*

Proof: We only need to show that the location of hinge v_{i+1} is fixed. As the locations of the hinges v_i and v_{i+2} are already fixed, the polygons $\mathcal{B}_{i+1}$ and $\mathcal{B}_{i+2}$ can only rotate around v_i and v_{i+2} respectively. The rigidity of $\mathcal{B}_{i+1}$ and $\mathcal{B}_{i+2}$ simultaneously constrains the motion of the connecting hinge v_{i+1} to a circular arc of radius $|v_i v_{i+1}|$ centered at v_i and to a circular arc of radius $|v_{i+1}v_{i+2}|$ centered at v_{i+2}, which is not allowed to coincide with v_i (see Figure 5). As the two circular arcs—which have different centers—have at most two isolated points in common, one of which is the current location of v_{i+1}, the location of v_{i+1} is fixed. ◻

Corollary 2.1 and Lemma 3.4 allow us to immobilize a chain $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4)$ by immobilizing $\mathcal{B}_1$ and $\mathcal{B}_4$, using three contacts on each (see Figure 5).

Corollary 3.1 *Six frictionless point contacts suffice to immobilize a serial chain of four polygons without parallel edges.*

3.1.2 Long Chains

Here we focus on the immobilization of serial chains of $n \geqslant 5$ hinged polygons without parallel edges. Before we provide a generic scheme for the immobilization of such long chains, we solve one more case explicitly, namely that of a serial chain of seven polygons. To immobilize $(\mathcal{B}_1, ..., \mathcal{B}_7)$ we immobilize $\mathcal{B}_1$, $\mathcal{B}_4$, and $\mathcal{B}_7$, using three contacts on each polygon.

Lemma 3.5 *Nine frictionless point contacts suffice to immobilize a serial chain of seven polygons without parallel edges.*

Proof: Apply Corollary 2.1 to $\mathcal{B}_1$, $\mathcal{B}_4$, and $\mathcal{B}_7$. Then apply Lemma 3.4 to the subchains $(\mathcal{B}_1)$ and $(\mathcal{B}_4)$ of $(\mathcal{B}_1, ..., \mathcal{B}_4)$. Finally, apply Lemma 3.4 to the subchains $(\mathcal{B}_1, ..., \mathcal{B}_4)$ and $(\mathcal{B}_7)$ of $(\mathcal{B}_1, ..., \mathcal{B}_7)$. ◻

Lemmas 3.2, 3.4, 3.5, and Corollaries 2.1 and 3.1 provide the tools for the immobilization of chains of any length $n \neq 3$. We observe that any $n \neq 3$ can be written as $n = k + 4j$, where $k \in \{1, 2, 4, 7\}$ and j is a non-negative integer. We immobilize a chain $(\mathcal{B}_1, ..., \mathcal{B}_n)$ by immobilizing the subchain $(\mathcal{B}_1, ..., \mathcal{B}_k)$ with $k+2$ point contacts, and every two-element subchain $(\mathcal{B}_{k+4h-1}, \mathcal{B}_{k+4h})$, with $1 \leqslant h \leqslant j$, with four contacts. The resulting number of contacts equals $k + 2 + 4j = k + 2 + 4 \cdot \frac{1}{4}(n-k) = n + 2$.

Theorem 3.1 *A serial chain of $n \neq 3$ hinged polygons without parallel edges can be immobilized with $n + 2$ frictionless point contacts. Six such contacts can immobilize three polygons without parallel edges.*

Proof: If $n = 3$, the result follows from Lemma 3.3. If $n = k + 4j \neq 3$, with $k \in \{1, 2, 4, 7\}$ and j a non-negative integer, apply Corollary 2.1, Lemma 3.2, Corollary 3.1, or Lemma 3.5 to the subchain $(\mathcal{B}_1, ..., \mathcal{B}_k)$. Then, for each integer $1 \leqslant h \leqslant j$, subsequently apply Lemma 3.2 to $(\mathcal{B}_{k+4h-1}, \mathcal{B}_{k+4h})$ and Lemma 3.4 to the subchains $(\mathcal{B}_1, ..., \mathcal{B}_{k+4h-4})$ and $(\mathcal{B}_{k+4h-1}, \mathcal{B}_{k+4h})$ of $(\mathcal{B}_1, ..., \mathcal{B}_{k+4h})$. ◻

3.2 Polygons with Parallel Edges

Three point contacts are sometimes insufficient to immobilize a polygon with parallel edges. Fortunately, Lemma 2.4 shows that four point contacts always suffice in such a case. As a result, we face the remarkable fact that four contacts may be necessary to immobilize a single polygon with parallel edges, but are also sufficient to immobilize any chain of two such polygons. Our results for larger n are also different for chains of even and odd length.

Besides Corollary 2.1, we are also unable to apply Corollary 3.1 if the polygons in the serial chain are allowed to have parallel edges. We first provide an alternative way of immobilizing a chain of four polygons that can have parallel edges, then use the result to deduce sufficient numbers of contacts for longer chains.

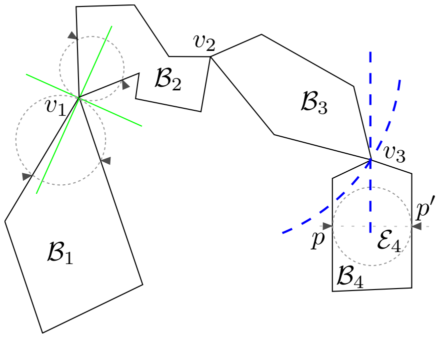


Figure 6: Immobilization of a four-polygon chain $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4)$ with six contacts when the largest enclosed circle $\mathcal{E}_4$ intersects $\mathcal{B}_4$ at two points that are antipodal on $\mathcal{B}_4$.

We concentrate on immobilization of the chain $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4)$. If the largest enclosed circles $\mathcal{E}_1$ and $\mathcal{E}_4$ intersect the boundaries of $\mathcal{B}_1$ and $\mathcal{B}_4$ respectively at three points of which no two are antipodal, then we immobilize $\mathcal{B}_1$ and $\mathcal{B}_4$ with three contacts each. Alternatively, let us assume without loss of generality that $\mathcal{E}_4$ intersects the boundary of $\mathcal{B}_4$ at two antipodal points p and p'. In that case we immobilize $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4)$ by immobilizing the subchain $(\mathcal{B}_1, \mathcal{B}_2)$ with four contacts and placing two additional contacts at p and p' (see Figure 6).

Lemma 3.6 *Six frictionless point contacts suffice to immobilize a serial chain of four polygons with parallel edges.*

Proof: Let p and p' be as defined above. If both $\mathcal{E}_1$ and $\mathcal{E}_4$ intersect the polygons $\mathcal{B}_1$ and $\mathcal{B}_4$ at three points of which not two are antipodal, then apply Lemma 2.2 to $\mathcal{B}_1$ and $\mathcal{B}_4$, and Lemma 3.4 to the subchains $(\mathcal{B}_1)$ and $(\mathcal{B}_4)$ of $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4)$.
If $\mathcal{E}_4$ intersects the boundary of $\mathcal{B}_4$ at two antipodal points p and p', then apply Lemma 3.2 to immobilize the subchain $(\mathcal{B}_1, \mathcal{B}_2)$, and thus fix the location of the hinges v_1 and v_2. We distinguish two cases according to Lemma 2.3. If points contacts at p and p' immobilize $\mathcal{B}_4$ and thus fix the location of the hinge v_3, then it follows immediately that $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4)$ is immobilized. If point contacts at p and p' force $\mathcal{B}_4$ and the hinge v_3 to translate along a line segment orthogonal to pp', then we observe that the fixed location of v_2 and the rigidity of $\mathcal{B}_3$ simultaneously constrains the motion of v_3 to a circular arc with radius $|v_2 v_3|$ centered at v_2. See Figure 6. As the line segment orthogonal to pp' and the circular arc have at most two isolated points in common, the location of v_3 is fixed. The fixed locations of v_1, v_2, and v_3, and the immobilization of $\mathcal{B}_1$ and $\mathcal{B}_4$ imply the immobilization of $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4)$. ◻

Lemmas 2.4, 3.2, 3.3, 3.4, and 3.6 provide the tools for the immobilization of chains of any length n. Any positive n can be written as $n = k + 4j$ with $1 \leqslant k \leqslant 4$ and j a non-negative integer. We can immobilize the subchain $(\mathcal{B}_1, ..., \mathcal{B}_k)$ with $k + 2$ point contacts if k is even and $k + 3$ point contacts if k is odd, and every two-element subchain $(\mathcal{B}_{k+4h-1}, \mathcal{B}_{k+4h})$, with $1 \leqslant h \leqslant j$, with four contacts. The resulting number of contacts equals $k + 2 + 4j = k + 2 + 4 \cdot \frac{1}{4}(n - k) = n + 2$ if k and n are even, and $k + 3 + 4j = k + 3 + 4 \cdot \frac{1}{4}(n - k) = n + 3$ if k and n are odd.

Theorem 3.2 *A serial chain of n hinged polygons with parallel edges can be immobilized with $n + 2$ frictionless point contacts if n is even, and with $n + 3$ frictionless point contacts if n is odd.*

Proof: Depending on the value of k, apply Lemma 2.4, Lemma 3.2, Lemma 3.3, or Lemma 3.6 to the subchain $(\mathcal{B}_1, ..., \mathcal{B}_k)$. Then, for each integer $1 \leqslant h \leqslant j$, subsequently apply Lemma 3.2 to $(\mathcal{B}_{k+4h-1}, \mathcal{B}_{k+4h})$ and Lemma 3.4 to the subchains $(\mathcal{B}_1, ..., \mathcal{B}_{k+4h-4})$ and $(\mathcal{B}_{k+4h-1}, \mathcal{B}_{k+4h})$ of $(\mathcal{B}_1, ..., \mathcal{B}_{k+4h})$. ◻

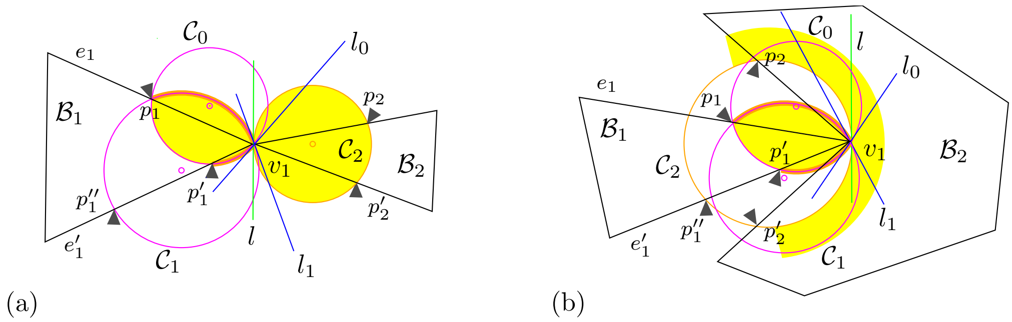


Figure 7: Robust immobilization of a two-polygon chain $(\mathcal{B}_1, \mathcal{B}_2)$ with five contacts when the hinge v_1 is (a) a convex vertex of $\mathcal{B}_2$, and (b) a concave vertex of $\mathcal{B}_2$.

4 Robust Immobility of Serial Chains

We study the robust immobilization of serial chains of polygons without parallel edges in Section 4.1 and with parallel edges in Section 4.2. As in the case of immobility, we have different results for chains of polygons with and without parallel edges.

4.1 Polygons without Parallel Edges

We first discuss the robust immobilization of chains consisting of at most five polygons, then use these chains as building blocks for longer chains.

4.1.1 Short Chains

The construction of a robust immobilization of a serial chain $(\mathcal{B}_1, \mathcal{B}_2)$ departs from the immobilization of that chain outlined in Subsection 3.1.1. Once more we assume without loss of generality that v_1 is a convex vertex of $\mathcal{B}_1$. Let e_1, e_1', e_2, e_2', l, $\mathcal{C}_2$, p_2, p_2' be as in Section 3.1.1, and recall that the edges e_1 and e_1' lie on the same side of l. Recall also that $\mathcal{C}_2$ is tangent to l at v_1. We rotate l by a clockwise angle around v_1 that is sufficiently small to keep e_1 and e_1' on the same side. Let l_0 be the resulting rotated copy of l. Similarly, we rotate l by a counterclockwise angle to obtain a line l_1. Let l_0' and l_1' be the lines through v_1 perpendicular to l_0 and l_1. We construct circles $\mathcal{C}_0$ and $\mathcal{C}_1$ centered on l_0' and l_1' respectively that pass through v_1, through the same point in the interior of e_1, and through (different points in) the interior of e_1'. (See Figure 7a and 7b for the circles in the case that v_1 is a convex and concave vertex of $\mathcal{B}_2$ respectively.) Note that such circles can always be constructed by properly choosing their centers in the immediate vicinity of v_1.

We place point contacts at the common intersection p_1 of $\mathcal{C}_0$ and $\mathcal{C}_1$ with e_1, at the distinct intersections p_1' and p_1'' of $\mathcal{C}_0$ and $\mathcal{C}_1$ with e_1', and at the intersections p_2 and p_2' of $\mathcal{C}_2$ with e_2 and e_2'. See Figure 7. The contacts at p_1, p_1', p_1'', p_2, and p_2' robustly immobilize $(\mathcal{B}_1, \mathcal{B}_2)$.

Lemma 4.1 *Five frictionless point contacts suffice to robustly immobilize a serial chain of two polygons.*

Proof: Let e_1, e_1', e_2, e_2', $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$, p_1, p_1', p_1'', p_2, p_2' be as described above. We observe that $\mathcal{C}_0^+ = \mathcal{C}^+(v_1, p_1, p_1')$, $\mathcal{C}_1^+ = \mathcal{C}^+(v_1, p_1, p_1'')$, $\mathcal{C}_2^+ = \mathcal{C}^+(v_1, p_2, p_2')$, and $\mathcal{C}_2^- = \mathcal{C}^-(v_1, p_2, p_2')$. In the spirit of the proof of Lemma 3.2, we argue that the current location of v_1 is the only point in $\mathcal{C}_0^+ \cap \mathcal{C}_1^+ \cap \mathcal{C}_2^+$ if v_1 is a convex vertex of $\mathcal{B}_2$ and in $\mathcal{C}_0^+ \cap \mathcal{C}_1^+ \cap \mathcal{C}_2^-$ if v_1 is a concave vertex of $\mathcal{B}_2$. Lemma 3.1 then says that the contacts at p_1, p_1', p_1'', p_2, and p_2' prohibit any motion of v_1. It is easy to

see that the contacts at p_1 and p_1' (or p_1'') prevent the clockwise and counterclockwise rotations of $\mathcal{B}_1$; p_2 and p_2' do the same for $\mathcal{B}_2$.
We notice that the three circles $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$—which are tangent to l_0, l_1, and l respectively—properly intersect in v_1. This implies that the intersection of $\mathcal{C}_0^+ \cap \mathcal{C}_1^+$ on the one hand and $\mathcal{C}_2^+$ or $\mathcal{C}_2^-$ on the other hand continues to consist of only the current location of v_1 when the point contacts at p_1, p_1', p_1'', p_2, or p_2' are perturbed along a sufficiently small but finite distance ϵ along their respective edges. Hence, the contacts provide robust immobility. ◻

We move on to consider the robust immobilization of a chain $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$. Let m and m' be the centers of the largest enclosed circles $\mathcal{E}_1$ and $\mathcal{E}_3$ respectively. Let l be the line through the hinges v_1 and v_2. For ease of discussion we assume without loss of generality that l is horizontal and v_1 lies to the left of v_2.

We focus on $\mathcal{B}_1$. If $\mathcal{E}_1$ intersects at least one vertex of $\mathcal{B}_1$ then we robustly immobilize $\mathcal{B}_1$ with at most three contacts at the intersection points p_1, p_1' (and p_1''). Alternatively, the circle $\mathcal{E}_1$ intersects the boundary of $\mathcal{B}_1$ at three points p_1, p_1', and p_1'' in the interior of edges. The normal lines at p_1, p_1', and p_1'' intersect at m, and at most one of them may coincide with l, because no two of p_1, p_1', and p_1'' are antipodal on $\mathcal{E}_1$. In the particular case that the intersection point m lies on the line l we shall properly perturb exactly two of p_1, p_1', and p_1'' along their respective edges. We select one of the points whose normal line does not coincide with l to remain stationary, and slide the other two along their edges in such a way that the intersection of their normal lines slides along the normal line of the stationary point. This will cause the intersection of the three normal lines to move away from l (see Figure 8b). Let q be the resulting intersection point of the normal lines at p_1, p_1', and p_1''.

Before placing point contacts, we perturb one of p_1, p_1', and p_1''. If the intersection point q of the normal lines at p_1, p_1', and p_1'' lies below l we perturb p_1, p_1', or p_1'' to obtain a small triangle entirely below l and lying to the left of the normal lines. If q lies above l we perturb p_1, p_1', or p_1'' to create a triangle that lies entirely above l and to the right of the normal lines. We place contacts at the resulting points p_1, p_1', and p_1'' (see Figure 8a).

We treat $\mathcal{B}_3$ similarly to $\mathcal{B}_1$. The single difference lies in the final perturbation if $\mathcal{E}_3$ intersects the boundary of $\mathcal{B}_3$ at three interior points p_3, p_3', and p_3''. If the intersection point q' of the normal lines at p_3, p_3', and p_3'' lies below l we perturb p_3, p_3', or p_3'' to obtain a small triangle entirely below l and lying to the right of the normal lines. If q lies above l we perturb p_1, p_1', or p_1'' to create a triangle that lies entirely above l and to the left of the normal lines.

We place contacts at the resulting points p_1, p_1', p_1'', p_3, p_3', and p_3''. The frictionless point contacts robustly immobilize $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$.

Lemma 4.2 *Six frictionless point contacts suffice to robustly immobilize a serial chain of three polygons without parallel edges.*

Proof: Let p_1, p_1', p_1'', p_3, p_3', p_3'', l be as described above. Let τ and τ' be the triangles defined by the normal lines at p_1, p_1', and p_1'', and at p_3, p_3', and p_3'' respectively. Denote by l_1 and l_2 the lines perpendicular to l and through v_1 and v_2 respectively (see Figure 8a). Let H_1 be the union of the current location of v_1 and the open half-plane bounded by l_1 not containing v_2, and let H_2 be the union of the current location of v_2 and the open half-plane bounded by l_2 not containing v_1. We establish first that the contacts on $\mathcal{B}_1$ constrain v_1's motion to H_1, and those on $\mathcal{B}_3$ constrain v_2's motion to H_2.
Let us focus on $\mathcal{B}_1$, and neglect the fact that it is connected to $\mathcal{B}_2$ and $\mathcal{B}_3$ for the time being. If the contacts at p_1, p_1', p_1'' immobilize $\mathcal{B}_1$, then clearly the hinge v_1 stays inside H_1. Alternatively, consider the triangle τ, and denote by c the cone of all lines through a point in τ and v_1. Let γ be the cone of all lines through v_1 that are perpendicular to a line in c, and define $\gamma_1 = \gamma \cap H_1$. Half-plane analysis

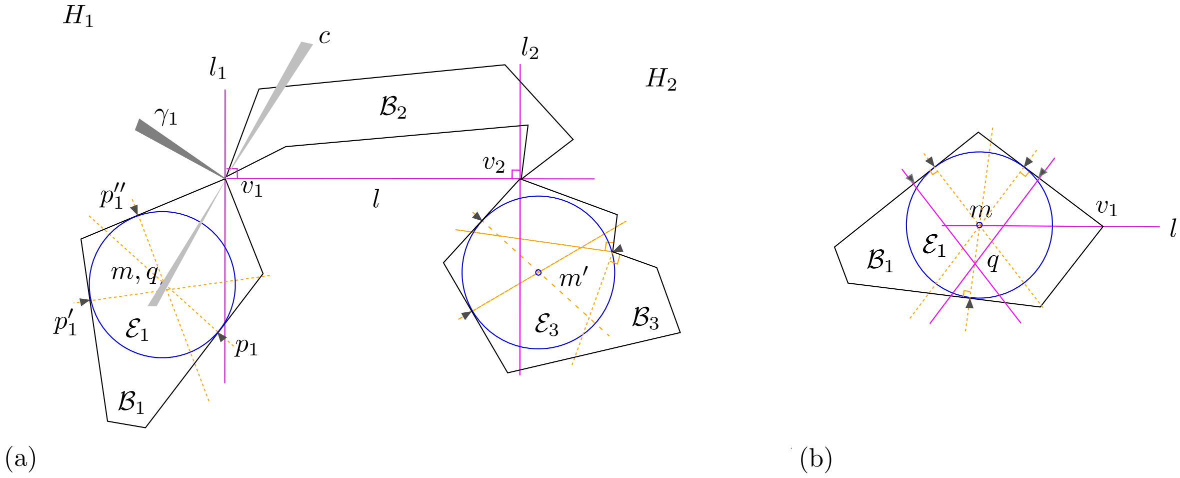


Figure 8: (a) Robust immobilization of a three-polygon chain $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ with six contacts in the case that $\mathcal{E}_3$ intersects a vertex of $\mathcal{B}_3$, and provided that the intersection point m does not lie on the line l. (b) When m lies on the line l, a perturbation is applied to move m off l.

shows that counterclockwise rotations about points in τ are the only possible motions of $\mathcal{B}_1$ when τ is below l. Similarly, clockwise rotations about points in l are the only possible motions when τ is above l. A rotation of a point moves that point in a direction perpendicular to the line connecting the point and the rotation center. Any rotation about a point in τ in clockwise direction if τ is above l and in counterclockwise direction if τ is below l forces v_1 to move into γ_1 and hence to stay inside H_1. A similar analysis shows that v_2 cannot leave $\gamma_2 = \gamma' \cap H_2$, where γ' is the cone of lines through v_2 perpendicular to a line through v_2 and a point in τ'. The hinge v_2 must therefore stay inside H_2.
As v_1 and v_2 are both vertices of the rigid polygon $\mathcal{B}_2$ their distance is fixed. The only two points in H_1 and H_2 that are at most $|v_1 v_2|$ apart are the current locations of v_1 and v_2. As a consequence, the hinges v_1 and v_2 cannot move.
It remains to show that $\mathcal{B}_1$ and $\mathcal{B}_3$ cannot rotate around v_1 and v_3 respectively. These facts follow immediately from the observations that the location of v_1 lies outside τ and that the location of v_2 lies outside τ'. Hence, the contacts at p_1, p_1', p_1'', p_3, p_3', and p_3'' immobilize $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$.
It is clear that sufficiently small perturbations of the contacts maintain the properties of the normal lines, triangles, and half-cones with respect to each other as well as the lines l, l_1, and l_2. So, the immobilization is robust. ◽

The following lemma is the key to the robust immobilization of a chain $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4)$.

Lemma 4.3 *A chain $(\mathcal{B}_1, ..., \mathcal{B}_n)$, with $n \geqslant 4$, is robustly immobilized if, for some $1 \leqslant i \leqslant n-3$, the subchains $(\mathcal{B}_1, ..., \mathcal{B}_i)$ and $(\mathcal{B}_{i+3}, ..., \mathcal{B}_n)$ are both robustly immobilized.*

Proof: The chain $(\mathcal{B}_1, ..., \mathcal{B}_n)$ is immobilized by Lemma 3.4, so it suffices to show that the immobilization is robust. The immobilizing contacts on the subchains $(\mathcal{B}_1, ..., \mathcal{B}_i)$ and $(\mathcal{B}_{i+3}, ..., \mathcal{B}_n)$ can be perturbed because of the robustness of the immobilization. All other contacts on $(\mathcal{B}_1, ..., \mathcal{B}_n)$ are redundant for the immobilization and can therefore be perturbed anyway. ◽

Lemmas 2.4 and 4.3 allow us to robustly immobilize a chain $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4)$ by robustly immobilizing $\mathcal{B}_1$ and $\mathcal{B}_4$, using four contacts on each.

Corollary 4.1 *Eight frictionless point contacts suffice to robustly immobilize a serial chain of four polygons.*

As a final separate case we robustly immobilize $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5)$ by robustly immobilizing $\mathcal{B}_1$ with four contacts, and the subchain $(\mathcal{B}_4, \mathcal{B}_5)$ with five contacts.

Lemma 4.4 *Nine frictionless contacts suffice to robustly immobilize a serial chain of five polygons without parallel edges.*

Proof: Apply Lemma 2.4 to $\mathcal{B}_1$ and Lemma 4.1 to $(\mathcal{B}_4, \mathcal{B}_5)$. Then apply Lemma 4.3 to the subchains $(\mathcal{B}_1)$ and $(\mathcal{B}_4, \mathcal{B}_5)$ of $(\mathcal{B}_1, ..., \mathcal{B}_5)$. ◻

4.1.2 Long Chains

We are ready to deal with robust immobilization of serial chains of $n \geqslant 6$ hinged polygons without parallel edges. Lemmas 2.4, 4.1, 4.2, 4.3, 4.4, and Corollary 4.1 provide the tools for the robust immobilization of chains of any positive length. Any positive n can be written as $n = k + 5j$, where $1 \leqslant k \leqslant 5$ and j is a non-negative integer. We robustly immobilize a chain $(\mathcal{B}_1, ..., \mathcal{B}_n)$ by robustly immobilizing the subchain $(\mathcal{B}_1, ..., \mathcal{B}_k)$ with $\lceil \frac{6}{5}(k+2) \rceil$ point contacts, and every three-element subchain $(\mathcal{B}_{k+5h-2}, \mathcal{B}_{k+5h-1}, \mathcal{B}_{k+5h})$, with $1 \leqslant h \leqslant j$, with six contacts. The resulting number of contacts equals $\lceil \frac{6}{5}(k+2) \rceil + 6j = \lceil \frac{6}{5}(k+2) \rceil + 6 \cdot \frac{1}{5}(n-k) = \lceil \frac{6}{5}(n+2) \rceil$.

Theorem 4.1 *A serial chain of n hinged polygons without parallel edges can be immobilized robustly with $\lceil \frac{6}{5}(n+2) \rceil$ contacts.*

Proof: Depending on the value of k, apply Lemma 2.4, Lemma 4.1, Lemma 4.2, Corollary 4.1, or Lemma 4.4 to the subchain $(\mathcal{B}_1, ..., \mathcal{B}_k)$. Then, for each integer $1 \leqslant h \leqslant j$, subsequently apply Lemma 4.2 to $(\mathcal{B}_{k+5h-2}, \mathcal{B}_{k+5h-1}, \mathcal{B}_{k+5h})$ and Lemma 4.3 to the subchains $(\mathcal{B}_1, ..., \mathcal{B}_{k+5h-5})$ and $(\mathcal{B}_{k+5h-2}, \mathcal{B}_{k+5h-1}, \mathcal{B}_{k+5h})$ of $(\mathcal{B}_1, ..., \mathcal{B}_{k+5h})$. ◻

4.2 Polygons with Parallel Edges

We are unable to apply Lemma 4.2 if the polygons in the chain have parallel edges. Besides the immobilization of a chain of three polygons, this also affects the immobilization of all chains of at least six polygons. We provide a way of immobilizing a chain of three polygons with parallel edges with seven instead of the aforementioned six contacts. As a consequence, it is better to use the robust immobilization of two polygons by five contacts provided by Lemma 4.1 as a building block for the immobilization of long chains.

We obtain a robust immobilization of the chain $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ by robustly immobilizing the subchain $(\mathcal{B}_1, \mathcal{B}_2)$ with five contacts and placing two additional contacts along the interiors of the edges e_3 and e'_3 of $\mathcal{B}_3$.

Lemma 4.5 *Seven frictionless point contacts suffice to robustly immobilize a serial chain of three polygons with parallel edges.*

Proof: Let e_3 and e'_3 as defined above. Apply Lemma 4.1 to robustly immobilize the subchain $(\mathcal{B}_1, \mathcal{B}_2)$, thus fixing the location of the hinges v_1 and v_2. The only motions these contacts do not exclude are clockwise and counterclockwise rotations of $\mathcal{B}_3$ around v_2; these are ruled out by the contacts along e_3 and e'_3.
By Lemma 4.1 we can perturb the contacts on $\mathcal{B}_1$ and $\mathcal{B}_2$. It is easy to see that the contacts on e_3 and e'_3 can be perturbed arbitrarily along these edges. The immobilization by the seven contacts is robust. ◻

Lemmas 2.4, 4.1, 4.5, 4.3, and Corollary 4.1 provide the tools for the robust immobilization of chains of any positive length. Any positive n can be written as $n = k + 4j$, where $1 \leqslant k \leqslant 4$

and j is a non-negative integer. We robustly immobilize a chain $(\mathcal{B}_1, ..., \mathcal{B}_n)$ by robustly immobilizing the subchain $(\mathcal{B}_1, ..., \mathcal{B}_k)$ with $\lceil \frac{5}{4}(k+2) \rceil$ point contacts, and every two-element subchain $(\mathcal{B}_{k+4h-1}, \mathcal{B}_{k+4h})$, with $1 \leqslant h \leqslant j$, with five contacts. The resulting number of contacts equals $\lceil \frac{5}{4}(k+2) \rceil + 5j\lceil \frac{5}{4}(k+2) \rceil + 5 \cdot \frac{1}{4}(n-k) = \lceil \frac{5}{4}(n+2) \rceil$.

Theorem 4.2 *A serial chain of n hinged polygons with parallel edges can be immobilized robustly with $\lceil \frac{5}{4}(n+2) \rceil$ contacts.*

Proof: Depending on the value of k, apply Lemma 2.4, Lemma 4.1, Lemma 4.5, or Corollary 4.1 to the subchain $(\mathcal{B}_1, ..., \mathcal{B}_k)$. Then, for each integer $1 \leqslant h \leqslant j$, subsequently apply Lemma 4.1 to $(\mathcal{B}_{k+4h-1}, \mathcal{B}_{k+4h})$ and Lemma 4.3 to the subchains $(\mathcal{B}_1, ..., \mathcal{B}_{k+4h-4})$ and $(\mathcal{B}_{k+4h-1}, \mathcal{B}_{k+4h})$ of $(\mathcal{B}_1, ..., \mathcal{B}_{k+4h})$. ◻

5 Discussion

We have studied the immobilization of n polygons serially connected by rotational joints, or hinges, as a first step in the virtually unexplored area of immobilization of non-rigid objects and structures. We have found that $n+2$ frictionless point contacts can immobilize a serial chain of polygons without parallel edges for all $n \neq 3$. If the individual polygons in the chain are allowed to have parallel edges, the bound remains $n+2$ for even n, but becomes $n+3$ for odd n. We have also studied a robust version of immobilization, which is insensitive to sufficiently small perturbations of the contacts. The robustness is achieved by a small increase in the number of contacts: $\lceil \frac{6}{5}(n+2) \rceil$ and $\lceil \frac{5}{4}(n+2) \rceil$ frictionless point contacts suffice for serial chains without and with parallel edges respectively.

The most appealing challenge is to get a satisfactory result for the immobilization of a chain of three polygons without parallel edges, either by showing that five contacts are sufficient or by proving that six contacts are necessary for immobilization. Although this would mean a remarkable exception, we conjecture—based on our current work—that six contacts are necessary.

Another open issue is to establish necessary numbers of contacts for immobilization and robust immobilization. Such numbers would allow us to evaluate the tightness of our bounds. We have seen that our sufficient numbers of $n+2$ and $n+3$ are tight for the immobilization of really short chains of n polygons with and without parallel edges ($n = 1, 2$), but it is unclear whether this is also true for longer chains. The bound of $n+2$ equals the number of degrees of freedom of a chain of n polygons. Even though for rigid parts without parallel edges the tight bound for immobilization (three in 2D and four in 3D) seems unrelated to the number of degrees of freedom of the system, we believe that this bound is tight for chains of any number n of polygons without parallel edges. Clearly, we also believe that the bound for chains of polygons with parallel edges of even length is tight.

The tight bounds on the number of contacts that suffice to put 2D and 3D rigid parts in form closure are in both cases equal to the dimension of the part's configuration space (and of the so-called wrench space) plus one [29]. As our notion of robust immobilization is more or less comparable to form closure, we suspect that at least $n+3$ contacts are necessary to robustly immobilize chains of n polygons. This leaves a considerable gap with our sufficient numbers $\lceil \frac{6}{5}(n+2) \rceil$ and $\lceil \frac{5}{4}(n+2) \rceil$.

The results in this paper can be extended in various directions. An extension to circular chains is straightforward. To handle a circular chain of n polygons, one temporarily removes two adjacent polygons, and (robustly) immobilizes the remaining serial chain of $n-2$ polygons. Then one puts the two adjacent polygons back in without adding contacts. Based on Lemmas 3.4 and 4.3, the resulting circular chain of n polygons without or with parallel edges is immobilized with n or at most $n+1$ contacts respectively, or robustly immobilized with $\lceil \frac{6}{5}n \rceil$ or $\lceil \frac{5}{4}n \rceil$ contacts respectively. In addition, it seems that most of our results generalize to chains of curved parts. Other possible extensions are to branching structures, and to other than rotating joints.

Finally, throughout the paper we have assumed that the placement at which the serial chain has to be immobilized is given. It seems that far less contacts suffice if we are allowed to choose the placement. For a chain of convex polygons this is evident: we simply stretch the chain to align the hinges. Now two contacts on each of the distal polygons placed along the edges incident to the hinge—in the spirit of the construction for the immobilization of two polygon—will immobilize the entire chain regardless of its length. For non-convex polygons, however, it is unclear whether a similar approach exists.

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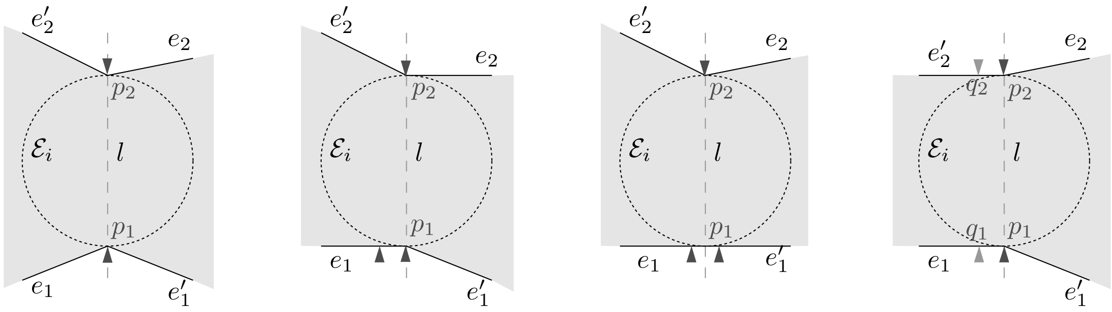
Appendix

Lemma 2.3 considers the four segments connecting these points to the adjacent vertices on the polygon. Recall that the two segments connecting an intersection point to its adjacent vertices can be two adjacent edges of the polygon (if the intersection point is the common vertex) or two parts of a single edge (if the intersection point is an interior point of that edge).

Lemma 2.3 *Let $\mathcal{B}_i$ be a polygon. Assume a largest enclosed circle $\mathcal{E}_i$ of $\mathcal{B}_i$ intersects the boundary of $\mathcal{B}_i$ at two points p_1 and p_2 that are antipodal on $\mathcal{E}_i$. Let l be the line through p_1 and p_2.*

- *Two frictionless point contacts at p_1 and p_2 suffice to immobilize $\mathcal{B}_i$ if no (part of an) edge adjacent to p_1 is parallel to (a part of) an edge adjacent to p_2 on the same side of l.*
- *Three frictionless point contacts at or close to p_1 and p_2 suffice to robustly immobilize $\mathcal{B}_i$ if no (part of an) edge adjacent to p_1 is parallel to (a part of) an edge adjacent to p_2 on the same side of l.*
- *Two frictionless point contacts at p_1 and p_2 suffice to constrain the local motions of $\mathcal{B}_i$ to translations along a single line segment if (the part of) an edge adjacent to p_1 is parallel to (the part of) the edge adjacent to p_2 on the same side of l.*

Proof: Let e_i and e'_i be the (parts of) edges adjacent to p_i ($i = 1, 2$) first when rotating in counter-clockwise and clockwise direction around p_1 from p_1p_2. Observe that the angle between each of e_1, e'_1, e_2, and e'_2 on the one hand, and the segment p_1p_2 on the other hand is at least $\pi/2$. The first claim then follows from a similar result by Czyzowicz et al. [9].
For ease of discussion we assume that the segment p_1p_2, and hence the line l, is vertical. Two (parts of) edges adjacent to p_1 and p_2 on the same side of l are parallel if and only if now either both e_1 and e'_2 are horizontal or both e'_1 and e_2 are horizontal.
If none of e_1, e'_1, e_2, and e'_2 is horizontal then an analysis of the induced half-plane constraints shows that contacts at p_1 and p_2 robustly immobilize $\mathcal{B}_i$, confirming the second claim (see Figure 9a). Assume without loss of generality that e_1 is horizontal. If neither e'_1 nor e'_2 is horizontal then points contacts at p_1, p_2, and on e_1 sufficiently close to p_1 robustly immobilize $\mathcal{B}$—even if e_2 happens to be horizontal (see Figure 9b). If e'_1 is horizontal (and hence p_1 is an interior point of an edge) and neither e_2 nor e'_2 are, then points contacts at p_2 and on e_1 and e'_1 sufficiently close to p_1 robustly immobilize $\mathcal{B}_i$ (see Figure 9c).
In the remaining cases either both e_1 and e'_2 or both e'_1 and e_2 are horizontal. Assume without loss of generality that e_1 and e'_2 are horizontal. For any point $q_1 \in e_1$ there is exactly one point $q_2 \in e_2 \cup e'_2$ at a distance at most $|p_1p_2|$; this is the point of intersection of the vertical line through q_1 with e'_2. Likewise, the only point $q_1 \in e_1 \cup e'_1$ at distance at most $|p_1p_2|$ from any point $q_2 \in e'_2$ is the point of intersection of the vertical line through q_2 with e_1. As a result, the only possible motion of $\mathcal{B}_i$ that does not violate the rigidity of $\mathcal{B}_i$ and the contacts is the one in which the point contacts slide along the vertically aligned combinations of points $q_1 \in e_1$ and $q_2 \in e'_2$ at a distance $|p_1p_2|$ (see Figure 9d). This motion is a horizontal translation to the right. If e'_1 and e_2 are also both horizontal then and polygon $\mathcal{B}_i$ is also able to translate horizontally to the left. If either e'_1 or e_2 is not horizontal, then the only two points on e'_1 and $e_2 \cup e'_2$ (and on e'_2 and $e_1 \cup e'_1$) at a distance at most $|p_1p_2|$ are p_1 and p_2. Hence, in that case there exists no motion during which one of the contacts slides along e'_1 or e_2. In summary, only horizontal translations of $\mathcal{B}$ are possible. ◻



(a) (b) (c) (d)

Figure 9: The enclosed circle $\mathcal{E}_i$ intersects the boundary of $\mathcal{B}_i$ at two points p_1 and p_2 that are antipodal on $\mathcal{E}_i$. (a) Contacts at p_1 and p_2 robustly immobilize $\mathcal{B}_i$ if none of e_1, e_1', e_2, and e_2' is horizontal. (b) Contacts at p_1 and p_2 and on e_1 robustly immobilize $\mathcal{B}_i$ if e_1 is horizontal and e_1' and e_2' are not. (c) Contacts at p_2 and on e_1 and e_1' robustly immobilize $\mathcal{B}_i$ if only e_1 and e_1' are horizontal. (d) Contacts at p_1 and p_2 allow only horizontal translations if e_1 and e_2' are horizontal.