On the creations of critical points in scale space with applications to medical image analysis

A. Kuijper and L.M.J. Florack

UU-CS-2001-25 TR
September 2001

On the creations of critical points in scale space with applications to medical image analysis

Arjan Kuijper Luc Florack

Abstract

In order to investigate the deep structure of Gaussian scale space images, one needs to understand the behaviour of spatial critical points under the influence of blurring. We show how the mathematical framework of catastrophe theory can be used to describe the various different types of annihilations and creations of pairs of spatial critical points and how this knowledge can be exploited in finding and tracing these points. We clarify the theory with an artificial image and a simulated MR image.

1 Introduction

The presence of structures of various sizes in an image demands almost automatically a collection of image analysis tools that is capable to deal with these structures. Essential is that this system is capable of handling the various, a priori unknown sizes or scales.

The concept of scale space has been introduced by (both) Witkin [20] and Koenderink [11]. They showed that the natural way to represent an image at finite resolution is by convolving it with a Gaussian of various bandwidths, thus obtaining a smoothened image at a scale determined by the bandwidth. This approach has lead to the formulation of various invariant expressions – expressions that are independent of the coordinates – that capture certain features in an image at distinct levels of scale [4].

In this paper we focus on linear, or Gaussian, scale space. This has the advantage that each scale level only requires the choice of an appropriate scale; and that the image intensity at that level follows linearly from any previous level. It is therefore possible to trace the evolution of certain image entities over scale. The exploitation of various scales simultaneously has been referred to as *deep structure* by Koenderink [11]. It pertains to information of the change of the image from highly detailed –including noise – to highly smoothened. Furthermore, it may be expected that large structures "live" longer than small structures (a reason that Gaussian blur is used to suppress noise). The image together with its blurred version was called "primal sketch" by Lindeberg [14]. Since multi-scale information can be ordered, one obtains a hierarchy representing the subsequent simplification of the image with increasing scale. In one dimensional images this has been done by several authors [8, 9, 19], but higher dimensional images are more complicated as we will discuss below.

An essentially unsolved problem in the investigation of deep structure is how to establish meaningful links across scales. A well-defined and user-independent constraint is that points are linked if they are topological equal. Thus maxima are linked to maxima, etc. This approach has been used in 2-D images by various authors [7, 13]. They linked extrema, but noticed that sometimes new extrema occurred, disrupting a good linking.

This creation of new extrema in scale space has been studied in detail by Damon, [2], proving that these creations are generic in images of dimension larger than one. That means that they are not some kind of artifact, introduced by noise or numerical errors, but that they are to be expected in any typical case. This was somewhat counterintuitive, since blurring seemed to imply that structure could only disappear, thus suggesting that only annihilations could occur. Damon, however, showed that both annihilations and creations are generic catastrophes. Whereas Damons results were stated theoretically, application of these results were reported in *e.g.* [6, 12, 14].

The main consequence is that in order to be able to use the topological approach one necessarily needs to take into account these creation events.

In this paper we describe these catastrophes in scale space. Furthermore we investigate the appearance of creations in more detail and explain why the are, albeit generic, rarely found, a reason for current applications to simply ignore them.

2 Theory

Let $L(\mathbf{x})$ denotes an arbitrary n dimensional image, the initial image. Then $L(\mathbf{x};t)$ denotes the $n+1$ dimensional *Gaussian scale space image* of $L(\mathbf{x})$, obtained by convolution of the initial image with a normalised Gaussian kernel of zero mean and standard deviation $\sqrt{2t}$. Consequently, $L(\mathbf{x};t)$ satisfies the diffusion equation:

$$\partial_t L(\mathbf{x};t) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} L(\mathbf{x};t) \stackrel{\text{def}}{=} \Delta L(\mathbf{x};t) \qquad (1)$$

Here $\Delta L(\mathbf{x};t)$ denotes the Laplacean. Differentiation is now well-defined, since derivatives of the image up to arbitrary orders at any scale are obtained by the convolution of this image with the derivatives to these orders of a Gaussian at this scale. The type of a *spatial critical point*, defined by $\nabla L(\mathbf{x};t_0) = 0$ at fixed scale t_0, is given by the eigenvalues of the Hessian H, the matrix with the second order spatial derivatives, evaluated at its location. The trace of the Hessian equals the Laplacean. For maxima (minima) all eigenvalues of the Hessian are negative (positive). At a spatial saddle point H has both negative and positive eigenvalues.

Since $L(\mathbf{x};t)$ is a continuous – even smooth – function in $(\mathbf{x};t)$-space, spatial critical points are part of a one dimensional manifold in scale space by virtue of the implicit function theorem.

Definition 1 *A* critical curve *is a one dimensional manifold in scale space on which* $\nabla L(\mathbf{x};t) = 0$.

Consequently, the intersection of all critical curves with an image at a certain fixed scale t_0 yields the spatial critical points of the image at that scale.

Catastrophe Theory The spatial critical points of a function with non-zero eigenvalues of the Hessian are called *Morse critical points*. The *Morse Lemma* states that at these points the qualitative properties of the function are determined by the quadratic part of the Taylor expansion of this function. This part can be reduced to the *Morse canonical form* by a slick choice of coordinates.

If at a spatial critical point the Hessian degenerates, so that at least one of the eigenvalues is zero, the type of the spatial critical point cannot be determined.

Definition 2 *The* catastrophe points *of* $L(\mathbf{x};t_0)$ *are defined as the points where both the spatial gradient and the determinant of the Hessian vanish:* $\nabla L(\mathbf{x};t_0) = 0 \; \wedge \; \det H(\mathbf{x};t_0) = 0$.

The term catastrophe was introduced by Thom [17, 18]. It denotes a (sudden) qualitative change in an object as the parameters on which this object depends change smoothly. This behaviour was already known by the terms perestroika, bifurcation and metamorphosis. The name catastrophe theory was suggested by Zeeman [21] to unify singularity theory, bifurcation theory and their applications and gained wide popularity. A thorough mathematical treatment on singularity theory can be found in the work of Arnol'd, see e.g. [1]. More pragmatic introductions and applications are widely published, e.g. [5].

The catastrophe points are also called *non-Morse critical points*, since a higher order Taylor expansion is essentially needed to describe the qualitative properties. Although the dimension of the variables is arbitrary, the *Thom Splitting Lemma* states that one can split up the function in a Morse and a non-Morse part. The latter consists of variables representing the k "bad" eigenvalues of the Hessian that become zero. The Morse part contains the $n-k$ remaining variables. Consequently, the Hessian contains a $(n-k)\times(n-k)$ submatrix representing a Morse function. It therefore suffices to study the part of k variables. The canonical form of the function at the non-Morse critical point thus contains two parts: a Morse canonical form of $n-k$ variables, in terms of the quadratic part of the Taylor series, and a non-Morse part. The latter can by put into canonical form called the *catastrophe germ*, which is obviously a polynomial of degree 3 or higher.

Since the Morse part does not change qualitatively under small perturbations, it is not necessary to further investigate this part. The non-Morse part, however, does change. Generally the non-Morse critical point will split into a non-Morse critical point, described by a polynomial of lower degree, and Morse critical points, or even exclusively into Morse critical points. This event is called a *morsification*. So the non-Morse part contains the catastrophe germ and a perturbation that controls the morsifications.

Then the general form of a Taylor expansion $f(\mathbf{x})$ at a non-Morse critical point of an n dimensional function can be written as (*Thom's Theorem*): $f(\mathbf{x};\lambda) = CG + PT + Q$, where $CG(x_1,\ldots,x_k)$ denotes the catastrophe germ, $PT(x_1,\ldots,x_k;\lambda_1,\ldots,\lambda_l)$ the perturbation germ with an l dimensional space of parameters, and $Q = \sum_{i=k+1}^{n} \epsilon_i x_i^2$ with $\epsilon_i = \pm 1$ the Morse part.

The infinite set of so-called simple real singularities have catastrophe germs given by the infinite series $A_k^{\pm} \stackrel{\text{def}}{=} \pm x^{k+1}, k \geq 1$ and $D_k^{\pm} \stackrel{\text{def}}{=} x^2 y \pm y^{k-1}, k \geq 4$, and the three exceptional singularities $E_6 \stackrel{\text{def}}{=} x^3 \pm y^4$, $E_7 \stackrel{\text{def}}{=} x^3 + xy^3$, and $E_8 \stackrel{\text{def}}{=} x^3 + y^5$. The germs A_k^+ and A_k^- are equivalent for $k = 1$ and k even.

Catastrophes and Scale Space The number of equations defining the catastrophe point equals $n+1$ and therefore it is over-determined with respect to the n spatial variables. In scale space, however, the number of variables equals $n+1$ and catastrophes occur as isolated points.

Although the list of catastrophes starts very simple, it is not trivial to apply it directly to scale space by assuming that scale is just one of the perturbation parameters.

For example, in one dimensional images the Fold catastrophe reduces to $x^3 + \lambda x$. It describes the change from a situation with two critical points (a maximum and a minimum) for $\lambda < 0$ to a situation without critical points for $\lambda > 0$. This event can occur in two ways. The extrema are annihilated for increasing λ, but the opposite – creation of two extrema for decreasing λ – is just as likely.

In scale space, however, there is an extra constraint: the germ has to satisfy the diffusion equation. Thus the catastrophe germ x^3 implies an extra term $6xt$. On the other hand, the perturbation term is given by $\lambda_1 x$, so by taking $\lambda = 6t$ scale plays the role of the perturbing parameter. Since we can only increase t (diffuse), the only remaining possibility for this A_2-catastrophe in one-dimensional images is an annihilation.

In higher dimensional images also the opposite – i.e. an Fold catastrophe describing creation of a pair of critical points – is possible. Then the perturbation $\lambda = -6t$ with increasing t requires a term of the form $-6xy^2$, see Definition 3.

The transfer of the catastrophe germs to scale space has been made by may authors, [2, 3, 9, 12, 14], among whom Damon's account is probably the most rigorous. He showed that the only generic morsifications in scale space are the aforementioned Fold catastrophes describing *annihilations* and *creations* of pairs of critical points. These two points have opposite sign of the determinant of the Hessian before annihilation and after creation. All other events are compounds of such events.

Definition 3 *The scale space catastrophe germs are defined by*

$$\begin{aligned} f^{A}(\mathbf{x};t) &\stackrel{\text{def}}{=} x_1^3 + 6x_1 t + Q(\mathbf{x};t)\,, \\ f^{C}(\mathbf{x};t) &\stackrel{\text{def}}{=} x_1^3 - 6x_1 t - 6x_1 x_2^2 + Q(\mathbf{x};t)\,, \end{aligned}$$

The quadratic term $Q(\mathbf{x};t)$ is defined

$$Q(\mathbf{x};t) \stackrel{\text{def}}{=} \sum_{i=2}^{n} \epsilon_i (x_i^2 + 2t),$$

where $\sum_{i=2}^{n} \epsilon_i \neq 0$ and $\epsilon_i \neq 0\ \forall i$.

Note that both the scale space catastrophe germs and the quadratic terms satisfy the diffusion equation. The germs f^{A} and f^{C} correspond to the two qualitatively different Fold catastrophes at the origin, an annihilation and a creation respectively. From Definition 3 it is obvious that annihilations occur in any dimension, but creations require at least 2 dimensions. Consequently, in 1D signals only annihilations occur. Furthermore, for images of arbitrary dimension it suffices to investigate the 2D case due to the Splitting Lemma.

3 Scale space catastrophes

A_2 Fold catastrophe The Fold catastrophe in scale space is given by

$$x^3 + 6xt + y^2 + 2t\,,$$

see Definition 3. One can verify that at the origin a saddle and an extremum meet and annihilate.

A_3 Cusp catastrophe The cusp catastrophe germ is given by x^4. Its scale space addition is $12x^2t + 12t^2$. The perturbation term contains two terms: $\lambda_1 x + \lambda_2 x^2$. Obviously, scale takes the role of λ_2. The scale space Cusp catastrophe germ with perturbation is thus defined by

$$L(x, y; t) = x^4 + 12x^2 t + 12t^2 + \alpha x + \beta(y^2 + 2t)\,.$$

Taking $\alpha = 0$ and increasing scale at the origin three critical points transform to one critical point. Morsification by the perturbation $\alpha \neq 0$ yields one Fold catastrophe and one continuing critical curve. One can verify that the $A_k, k > 3$ catastrophes represent the annihilation of critical points, albeit in more complicated appearances.

D_4^+ Hyperbolic umbilic catastrophe The hyperbolic umbilic catastrophe germ is given by $x^3 + xy^2$. Its scale space addition is $8xt$. The perturbation term contains three terms: $\lambda_1 x + \lambda_2 y + \lambda_3 y^2$. Obviously, scale takes the role of λ_1. The scale space hyperbolic umbilic catastrophe germ with perturbation is thus defined by

$$L(x, y; t) = x^3 + xy^2 + 8xt + \alpha(y^2 + 2t) + \beta y$$

where the first part describes the scale space catastrophe germ. The set (α, β) form the extra perturbation parameters. Then

$$\begin{cases} L_x &= 3x^2 + 8t + y^2 \\ L_y &= 2xy + 2\alpha y + \beta \\ \det(H) &= 12x(x + \alpha) - 4y^2. \end{cases}$$

At the combination $(\alpha, \beta) = (0, 0)$ four critical points exist for each $t < 0$. At $t = 0$ the four critical curves given by $(x, y; t) = (\pm\sqrt{-\frac{8}{3}t}, 0; t)$ and $(x, y; t) = (0, \pm\sqrt{-8t}; t)$ annihilate simultaneously at the origin (see e.g. Kalitzin [10]).

Morsification takes place in two steps. In the first step one perturbation parameter is non-zero. If $\alpha \neq 0$ and $\beta = 0$, the annihilations are separated. At the origin a Fold catastrophe occurs with critical curves $(x, y; t) = (-\sqrt{-\frac{8}{3}t}, 0; t)$. On one of these curves the other catastrophe takes place: At $(x, y; t) = (-\alpha, 0; -\frac{3}{8}\alpha^2)$ the critical curves $(-\alpha, \pm\sqrt{-3\alpha^2 - 8t}; t), t < -\frac{3}{8}\alpha^2$ annihilate in a Cusp catastrophe.

If $\alpha = 0$ and $\beta \neq 0$, the double annihilation breaks up into two Fold annihilations with symmetric non-intersecting critical curves.

Finally, if both α and β are non-zero, this second morsification results in two critical curves each of them containing an Fold annihilation.

D_4^- **Elliptic umbilic catastrophes** The elliptic umbilic catastrophe germ is given by $x^3 - 6xy^2$. Its scale space addition is $-6xt$. The perturbation term contains three terms: $\lambda_1 x + \lambda_2 y + \lambda_3 y^2$. Obviously scale takes the role of λ_1. The scale space elliptic umbilic catastrophe germ with perturbation is thus defined by

$$L(x, y; t) = x^3 - 6xy^2 - 6xt + \alpha(y^2 + 2t) + \beta y \; , \quad (2)$$

where the first part describes the scale space catastrophe germ. The set (α, β) form the extra perturbation parameters. Now

$$\begin{cases} L_x &= 3x^2 - 6t - 6y^2 \\ L_y &= -12xy + 2\alpha y + \beta \\ \det(H) &= 12x(\alpha - 6x) - 144y^2. \end{cases}$$

The combination $(\alpha, \beta) = (0, 0)$ gives two critical points for all $t \neq 0$ on the critical curves $(x, y; t) = (0, \pm\sqrt{-t}; t), t < 0$ and $(x, y; t) = (\pm\sqrt{2t}, 0; t), t > 0$. At the origin a so-called scatter event occurs: the critical curve changes from y-axis to x-axis with increasing t. Just as in the hyperbolic case, in fact two Fold catastrophes take place; in this case both an annihilation and a creation.

The morsification for $\alpha = 0, \beta \neq 0$ leads to the breaking into two critical curves without any catastrophe.

The morsification for $\alpha \neq 0$, $\beta = 0$ leads to only one catastrophe event at the origin: the Fold creation. The sign of α determines whether the critical curve contains a maximum – saddle pair or a minimum–saddle pair. Without loss of generality we may choose $\alpha = 1$.

The creation containing critical curve is given by $(x, y; t) = (\pm\sqrt{2t}, 0; t)$. The other critical curve given by $(x, y; t) = (\frac{1}{6}, \pm\sqrt{\frac{1}{72} - t}; t)$ represents two branches connected at the second catastrophe. This point is located at $(x, y; t) = (\frac{1}{6}, 0; \frac{1}{72})$, is an element of both curves. At this point two saddle points and the created extremum go through a Cusp catastrophe resulting in one saddle. Note that ignoring this catastrophe one would find the sudden change of extremum into saddle point while tracing the created critical points.

The intensity of the creation pair is given by $L(s) = 2s \pm 4s\sqrt{2s}, s \geq 0$, the intensity of the other pair by $L(s) = \frac{1}{216} + s, \; s \leq \frac{1}{72}$. The intensities of both paths are shown in Figure 1a. A close-up around the catastrophe points is given in Figure 1b.

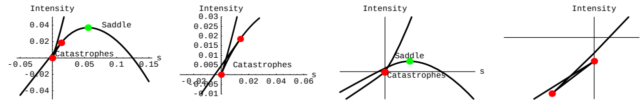

Figure 1. a) Intensities of critical paths, $\gamma = 0$. b) Close-up at both catastrophes, $\gamma = 0$. c) Intensities of critical paths, $\gamma = \frac{1}{24}\sqrt{2}$. d) Close-up at both catastrophes, $\gamma = \frac{1}{24}\sqrt{2}$.

Note that the intensity curve bottom-left contains two saddle branches. The way to read this image is that at the catastrophe in the origin two curves are created of which the extremum one (the lower curve) is annihilated at the second catastrophe with one of the saddle branches.

A complete morsification by taking $\|\beta\| \ll 1$ resolves the scatter. It can be shown that the Hessian has two real roots if and only if $\|\beta\| \leq \frac{1}{32}\sqrt{6}$. At these root points subsequently a creation and an annihilation event take place. If we take $\epsilon = \frac{1}{24}\sqrt{2}$ the creation is at $(0.013, -0.032; -0.00094)$ and the annihilation is at $(x, y; t) = (1/12, -1/24\sqrt{2}; 0)$. The intensity curves at this situation are visible in Figure 1c-d. The latter shows again a close-up around the catastrophes.

Due to this morsification the two critical curves do not intersect each other. Also in this perturbed system the minimum annihilates with one of the two saddles, while the other saddle remains unaffected.

Creations As we showed, a creation event occurs in case of a morsificated elliptic umbilic catastrophe. In most applications, however, creations are rarely found, giving raise to the (false) opinion that creations are caused by numerical errors and should be disregarded. The reason for their rare appearance lays in the specific requirements for the parameters in the (morsificated) umbilic catastrophe germ. Its general formulation is given by

$$\begin{aligned} L(x, y; t) \; = \; & \tfrac{1}{6}L_{xxx}x^3 + \tfrac{1}{2}L_{xyy}xy^2 + L_{xt}xt \\ & + \tfrac{1}{2}L_{yy}(y^2 + 2t) + L_y y \end{aligned} \quad (3)$$

where the coefficients are the derivatives evaluated at $(0, 0; 0)$. A creation requires the constraint that $L_{xxx}L_{xt} < 0$.

Theorem 1 *If the third order derivatives are uncorrelated, the possible creations in 2D form a quarter of the catastrophes. In n-D it is $\frac{1}{\pi}\arccos\frac{1}{\sqrt{n}}$.*

Proof 1 *The requirement $L_{xxx}L_{xt} < 0$ can be rewritten to $L_{xxx}(L_{xxx} + L_{xyy}) < 0$. In the (L_{xxx}, L_{xyy})-space his is the area spanned by the vectors $(1, 0)$ and $(1, -1)$,*

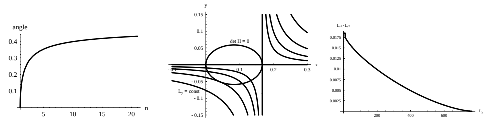

Figure 2. a) The fraction of the space of the third order derivatives in which creations can occur as a function of the dimension according Theorem 1. b) Intersections of the curves $\det(H) = 0$ and $\partial_y L = 0$ with different values for L_y. For the value given by Theorem 2 the curves touch. c) Difference in intensity between the creation and the annihilation event for L_y increasing from 0 to its critical value.

which is a quarter of the plane. For $n - D$ this extends to the $(L_{xxx}, L_{xy_iy_i})$-space, with $\dim(\mathbf{y}) = n - 1$, representing two intersecting planes. Their normal vectors are $(1, 0, \ldots, 0)$ and $(1, -1, \ldots, -1)$. They make an angle of ϕ radians, given by

$$\cos\phi = \frac{(1, 0, \ldots, 0) \cdot (1, -1, \ldots, -1)}{\mid (1, 0, \ldots, 0) \mid \cdot \mid (1, -1, \ldots, -1) \mid} = \frac{1}{\sqrt{1} \cdot \sqrt{n}}$$

Then the fraction of the space follows from twice this angle over the complete angle of 2π, i.e. $\frac{1}{\pi} \arccos \frac{1}{\sqrt{n}}$. □

Note that if $n = 1$, the fraction the space where creations can occur is zero, for $n = 2$ it is a quarter and for $n \to \infty$ the fraction converges to a half, see Figure 2a. That is: the higher the dimensions, the easier critical points can be created.

The reason that in two dimensional images the number of creations is (much) smaller than a quarter is caused by the role of the perturbation parameters. It is possible to give a tight bound to the perturbation of Equation (3) in terms of L_y:

Theorem 2 *A creation and subsequent annihilation event occur in Equation (3) if and only if*

$$\mid L_y \mid \leq \frac{3}{16} L_{yy}^2 \sqrt{\frac{-3L_{xxx}}{L_{xyy}^3}}$$

Proof 2 *The catastrophes satisfy $\partial_x L = \partial_y L = \det H = 0$. Since the solution of the system*

$$\begin{array}{rcll} \partial_y L & = & L_y + y(L_{yy} + L_{xyy}x) & = 0 \\ \det H & = & L_{xxx}x(L_{yy} + L_{xyy}x) - L_{xyy}^2 y^2 & = 0 \end{array} \quad (4)$$

only contain spatial coordinates, their intersections define the spatial coordinates of the catastrophes, e.g. if $L_y = 0$, the case considered by Damon. If they touch, there is only a point of infection in the critical curve. At this point of inflection, the spatial tangent vectors of the curves defined by Eq. (4) are equal. Solving the system Eq. (4) with respect to y results in

$$-\frac{L_y}{L_{yy} + L_{xyy}x} = \pm \frac{1}{L_{xyy}} \sqrt{L_{xxx}x(L_{yy} + L_{xyy}x)}$$

the equality of the tangent vectors yields

$$\frac{\partial}{\partial_x}\left(-\frac{L_y}{L_{yy} + L_{xyy}x}\right) = \frac{\partial}{\partial_x}\left(\pm \frac{1}{L_{xyy}} \sqrt{L_{xxx}x(L_{yy} + L_{xyy}x)}\right)$$

Solving both equalities results in

$$(x, y, L_y) = (-\frac{L_{yy}}{4L_{xyy}}, \pm\sqrt{\frac{-3L_{xxx}L_{yy}^2}{16L_{xyy}^3}}, \mp\frac{3L_{yy}^2}{16L_{xyy}} \sqrt{\frac{-3L_{xxx}}{L_{xyy}}}) \,,$$

which gives the boundary values for L_y. □

If we use $L_{xxx} = 6, L_{xyy} = -12, L_{yy} = 2$ we find $\mid L_y \mid \leq \frac{1}{32}\sqrt{6}$, as given in the previous section. In Figure 2b the ellipse $\det(H) = 0$ is plotted, together with the curves $\partial_y L = 0$ for $L_y = 0$, resulting in two straight lines, intersecting at $(x, y) = (\frac{1}{6}, 0)$, and $L_y = 2^{-i}\sqrt{6}, i = 4, \ldots, 7$. For $i > 5$ the perturbation is small enough and a creation-annihilation is observed. If $i = 5$, L_y has its the critical value and the curves touch.

Obviously the perturbation L_y can be larger if L_{yy} increases. If so, the structure becomes more elongated. It is known by various examples of creations given in literature that elongated structures play an important role.

Although creations can occur in a quarter of the 2D catastrophes, the reason that they are rarely found is that their lifetime is finite: with increasing t they annihilate.

Theorem 3 *The maximum lifetime of a creation given by Equation (3) is*

$$t_{lifetime} = \frac{L_{xxx}L_{yy}^2}{2L_{xyy}^2(L_{xxx} + L_{xyy})} \,.$$

The difference in intensity is

$$\frac{L_{xxx}(2L_{xxx} - L_{xyy})L_{yy}^3}{6L_{xyy}^3(L_{xxx} + L_{xyy})} \,.$$

Proof 3 *Observe that the lifetime is bounded by the two intersections of $\partial_y L = 0$ and $\det(H) = 0$, see Figure 2b. As L_y increases from zero, the two points move towards each other over the arch $\det(H) = 0$ until it reaches the value given by theorem 2 with lifetime equal to zero. The largest arch length is obtained for $L_y = 0$. Then the spatial coordinates are found by $\partial_y L(x, y; t) = y(L_{xyy}x + L_{yy}) = 0$ and $\det H = L_{xxx}x(L_{xyy}x + L_{yy}) - L_{xyy}^2 y^2 = 0$, i.e. $(x, y) =$*

$(0,0)$ and $(x,y) = (-\frac{L_{yy}}{L_{xyy}}, 0)$ The location in scale space is given by $\partial_x L(x,y;t) = \frac{1}{2}L_{xxx}x^2 - \frac{1}{2}L_{xyy}y^2 - L_{xt}t = 0$. Consequently, the first catastrophe takes place at the origin - since also $t = 0$ - with zero intensity. The second is located at

$$(x,y;t) = (-\frac{L_{yy}}{L_{xyy}}, 0; \frac{L_{xxx}L_{yy}^2}{2L_{xyy}^2(L_{xxx}+L_{xyy})})$$

with intensity

$$L_{cat} = \frac{L_{xxx}(2L_{xxx} - L_{xyy})L_{yy}^3}{6L_{xyy}^3(L_{xxx}+L_{xyy})} \; .$$

Then the latter is also the maximum difference in intensity. □

To show the effect of the movement along the arch $\det(H) = 0$, see Figure 2c. Without loss of generality we took $L_{xxx} = 6, L_{xyy} = -12, L_{yy} = 2$. Firstly the two solutions to $\nabla L = 0 \wedge \det(H) = 0$ were calculated as function of $L_y, \mid L_y \mid \leq \frac{1}{32}\sqrt{6}$. Secondly the difference of the intensity of the solutions was calculated for 766 subsequent values of L_y. It is clearly visible that the intensity deceases with an increase of $\mid L_y \mid$.

From the proof of theorem 3 it is again apparent that L_{yy} plays an important role in enabling a (long)lasting creation. To observe this in more detail, note that the curve $\det H = 0$ is an ellipse (see also Figure 2b). Replacing x by $x - \frac{L_{yy}}{2L_{xyy}}$, it is located at the origin. Setting $L_{xyy} = \frac{1}{b}$ and $L_{xyy}L_{xxx} = -\frac{1}{a^2}$, we find

$$\det H = 0 \Leftrightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = L_{yy}^2 \frac{b^2}{4a^2} \; .$$

Assuming that we have a creation, $a^2 > 0$. The ellipse is enlarged with an increase of L_{yy}^2.

$D_5^{\pm}$ Parabolic umbilic catastrophes Ignoring the perturbation terms, the scale space parabolic umbilic catastrophe germ is defined by

$$L(x,y;t) = \frac{1}{4!}x^4 + \frac{1}{2!}x^2t + \frac{1}{2!}t^2 + \delta(\frac{1}{2}xy^2 + xt) \quad (5)$$

where $\delta = \pm 1$ Its critical curves and catastrophes follow from

$$\left\{ \begin{array}{rcl} L_x &=& \frac{1}{6}x^3 + xt + \delta(t + \frac{1}{2}y^2) \\ L_y &=& \delta xy \\ \det(H) &=& \delta x(\frac{1}{2}x^2 + t) - y^2 \end{array} \right.$$

So the catastrophe points are located at the origin (a double point) and at $(x,y;t) = (-\frac{3}{2}\delta, 0; -\frac{9}{8}\delta^2)$. The latter is a simple annihilation (a fold catastrophe), the former is either a double annihilation if $\delta = 1$ (a hyperbolic umbilic catastrophe), or scatter if $\delta = -1$ (a elliptic umbilic catastrophe).

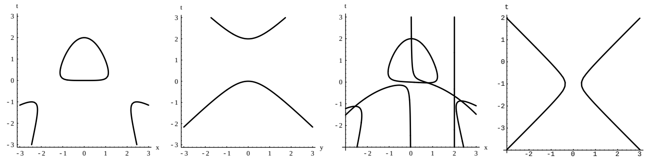

Figure 3. Critical paths of the D_6^--catastrophe. a) Unperturbed, in the $(x,0;t)$-plane. b) Unperturbed in the $(0,y;t)$-plane. c) Perturbed, with the curves $det(H) = 0$ in the $(x;t)$-plane. d) Perturbed, in the $(x,2;t)$-plane.

$D_6^{\pm}$-catastrophes From the previous sections it follows directly that D_k^+-catastrophes yield multiple annihilation events at the origin. The D_k^--catastrophes yield scatter (and thus in its morsification creation) events. In addition to the elliptic umbilic catastrophe, the morsification of the D_6^--catastrophe, called Second Elliptic Umbilic with perturbation $\lambda_1 x + \lambda_2 y + \lambda_3 x^2 + \lambda_4 y^2 + \lambda_5 x^3$, leads to a new behaviour of critical curves. Ignoring the perturbation terms for the moment, the scale space expression of the D_6^--catastrophe is given by

$$L(x,y;t) = \frac{1}{5!}x^5 + \frac{1}{3!}x^3t + \frac{1}{2!}xt^2 - (\frac{1}{2}xy^2 + xt) \quad (6)$$

Its critical curves and catastrophes follow from

$$\left\{ \begin{array}{rcl} L_x &=& \frac{1}{4!}x^4 + \frac{1}{2}x^2t + \frac{1}{2}t^2 - t - \frac{1}{2}y^2 \\ L_y &=& -xy \\ \det(H) &=& -\frac{1}{6}x^2(x^2+6t) - y^2 \end{array} \right.$$

Setting $y = 0$, four catastrophes occur: At $(x,y;t) = (\pm\sqrt{6}, 0; -1)$ two Fold annihilations, at the origin a creation and at $(x,y;t) = (0,0;2)$ again an annihilation, see Figure 3a. However, in contrast to the elliptic umbilic catastrophe, now *both* created branches annihilate with each other: the critical curve is a closed loop in scale space, see Figure 3a.

Apart from critical curves in the $(x,0;t)$-plane, also critical curves in the $(0,y;t)$-plane are present. Taking $x = 0$ yields $L_x = \frac{1}{2}t^2 - t - \frac{1}{2}y^2 = 0$, with two catastrophe points; at the origin an annihilation and at $(x,y;t) = (0,0;2)$ a creation, see Figure 3b. So the catastrophes at the t-axis without further perturbation are elliptic umbilic catastrophes.

With small perturbations, morsification into separate annihilation and creation events is obtained. For visualisation reason we added the perturbation $y^2 + 2t - \frac{1}{40}(x^2 + 2t)$ to Eq.(6). Then L_y changes to $2y - xy$, with roots at $y = 0$ and $x = 2$. The determinant of the Hessian changes to $\det(H) = (2-x)(\frac{1}{6}x^3 + xt - \frac{1}{20}) - y^2$, with four distinct roots at the x-axis, see Figure 3c. The critical curves in the $(2,y;t)$-plane do not contain catastrophe points, see Figure 3d. Therefore, the remaining catastrophes are generic Folds.

In Figure 3c the critical curves in the $(x, 0; t)$-plane together with the curves $\det(H) = 0$ are shown. The four intersections are clearly visible. The loop persists under perturbation, since λ_3, λ_4, and λ_5 are set and the perturbations λ_1 and λ_2 disturb only the symmetry and the location of a catastrophe at the origin. The lifetime of the loop in the unperturbed case is obtained by the factor $2\frac{L_{xt}}{L_{xtt}}$.

Morsification summary All non-fold catastrophes morsificate to Fold catastrophes. The morsification gives insight in the structure around the catastrophe point regarding the critical curves.

The morsification of the umbilic catastrophes (the D_k) show that the trajectories in scale space of the created critical points fall into two classes. The morsificated D_4^- and D_6^--catastrophes describe essentially different creation events. The morsificated D_4^- catastrophe describes the creation of a pair of critical points and the annihilation of one of them with another critical point. So while tracing a critical branch of a critical curve both an annihilation and a creation event are traversed. On the other hand, the morsificated D_6^--catastrophe describes an *isolated* closed critical curve, appearing *ex nihilo* with two critical branches that disappear a some larger scale. As a result, only tracing critical points starting from the initial image one finds the "D_4^-" creations, but misses the "D_6^-" loops.

4 Applications

In this section we give some examples to illustrate the theory presented in the previous sections. the effect is shown on the artificial MR image of Figure 4a. This image is taken from the web site http://www.bic.mni.mcgill.ca/brainweb. An example of creation *ex nihilo*, the D_6^--catastrophe, is shown by means of the "bridge"-image of Figure 6a.

D_4^--catastrophe The scale space image of the artificial MR image at scale 8.37 (with only the large structures remaining) is shown in Figure 4b. Now 7 extrema are found.

Figure 5a shows the critical paths in the $(x, y; t)$-space calculated in the scale range $8.37 - 33.1$. Of the branch most left a close-up of the middle part is shown in Figure 5b. It clearly shows the appearance of an annihilation-creation-pair described by the D_3^- morsification.

D_6^--catastrophe Figure 6a shows a "bridge"-image: two blobs with different intensity connected by a small bridge. First, there is only one maximum of the left blob. Then, at some scale the bridge disappears and a maximum-saddle

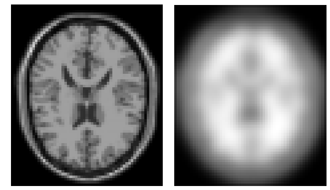

Figure 4. a: 181 x 217 artificial MR image. b) Image on scale 8.4

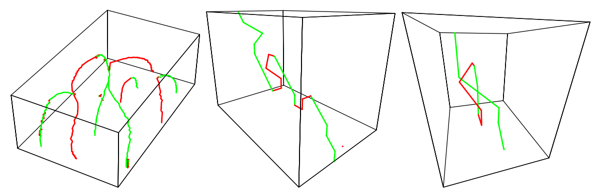

Figure 5. a) Critical paths of the MR image. b) Close-up of one of the paths. c) Idem.

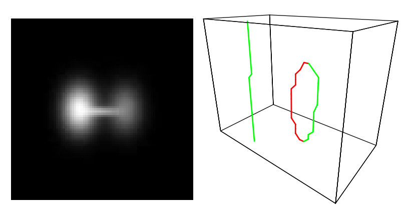

Figure 6. a: Artificial bridge image. b) Critical paths of the bridge image.

pair is created. Finally, at a large scale they annihilate again. Since the structure is elongated, the loop remains over a relatively large number of scales. Figure 6b shows the critical paths. The left string represents the largest blob, the loop the created and annihilated maximum-saddle pair. The same behaviour is derived from the MR image. Figure 5c shows a close-up of the branch most left in Figure 5a, taken at the lowest scales. Here clearly a "loop event" occurs.

5 Summary and Discussion

In this paper we investigated the (deep) structure on various catastrophe events in Gaussian scale space. Although it is known that pairs of critical points are annihilated and created (the latter if the dimension of the image is 2 or higher), it is important to describe the local structure of the image around these events. We therefore embedded catastrophes in scale space. Scale acts as one of the perturbation parameters. The morsification of catastrophes yield annihilations and creations of pairs of critical points.

Creations occur in different types. Critical paths in scale space can have bumps, a subsequent occurrence of an annihilation and a creation. In scale space images this is visible by the creation of an extremum-saddle pair, of which one critical point annihilates at some higher scale with an already present critical point, while the other remains unaffected. It is also possible that critical paths form loops: the created pair annihilates at some higher scale. The presence of both types in the MR image was shown.

Furthermore we showed that the humps in the critical paths, expressed in canonical coordinates, occur only in case of a small local perturbation. In addition, creations are less likely to happen due to the special combination of third order derivatives. We gave a dimension dependent expectation of this event.

The lifetime of a created pair is enlarged if the local structure is elongated. This is visualised by the example of the bridge image in section 4.

The theory described in this paper extends the knowledge of the deep structure of Gaussian scale space, especially with respect to the existence of creations and the scale space lifetime of the created critical points.

References

[1] V. I. Arnold. *Catastrophe Theory*. Springer, Berlin, 1984.

[2] J. Damon. Local Morse theory for solutions to the heat equation and Gaussian blurring. *Journal of Differential Equations*, 115(2):386–401, 1995.

[3] L. M. J. Florack and A. Kuijper. The topological structure of scale-space images. *Journal of Mathematical Imaging and Vision*, 12(1):65–80, February 2000.

[4] L. M. J. Florack, B. M. ter Haar Romeny, J. J. Koenderink, and M. A. Viergever. Cartesian differential invariants in scale-space. *Journal of Mathematical Imaging and Vision*, 3(4):327–348, 1993.

[5] R. Gilmore. *Catastrophe Theory for Scientists and Engineers*. Dover, 1993. Originally published by John Wiley & Sons, New York, 1981.

[6] L. D. Griffin and A. Colchester. Superficial and deep structure in linear diffusion scale space: Isophotes, critical points and separatrices. *Image and Vision Computing*, 13(7):543–557, September 1995.

[7] L. D. Griffin, A. Colchester, and G. Robinson. Scale and segmentation of grey-level images using maximum gradient paths. *Image and Vision Computing*, 10(5):389–402, 1992.

[8] P. Johansen. On the classification of toppoints in scale space. *Mathematical Imaging and Vision*, 4(1):57–67, 1994.

[9] P. Johansen, M. Nielsen, and O.F. Olsen. Branch points in one-dimensional Gaussian scale space, 2000.

[10] S. Kalitzin. Topological numbers and singularities. In *Sporring et al. [16]*, pages 181–190, 1997.

[11] J. J. Koenderink. The structure of images. *Biological Cybernetics*, 50:363–370, 1984.

[12] A. Kuijper and L.M.J. Florack. Calculations on critical points under gaussian blurring. In *Nielsen et al. [15]*, pages 318–329, 1999.

[13] L. M. Lifshitz and S. M. Pizer. A multiresolution hierarchical approach to image segmentation based on intensity extrema. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 12(6):529–540, 1990.

[14] T. Lindeberg. *Scale-Space Theory in Computer Vision*. The Kluwer International Series in Engineering and Computer Science. Kluwer Academic Publishers, 1994.

[15] M. Nielsen, P. Johansen, O. Fogh Olsen, and J. Weickert, editors. *Scale-Space Theories in Computer Vision*, volume 1682 of *Lecture Notes in Computer Science*. Springer -Verlag, Berlin Heidelberg, 1999.

[16] J. Sporring, M. Nielsen, L.M.J. Florack, and P. Johansen, editors. *Gaussian Scale-Space Theory*, volume 8 of *Computational Imaging and Vision Series*. Kluwer Academic Publishers, Dordrecht, second edition, 1997.

[17] R. Thom. *Stabilité Structurelle et Morphogénèse*. Benjamin, New york, 1972.

[18] R. Thom. *Structural Stability and Morphogenesis*. Benjamin-Addison Wesley, 1975. translated by D. H. Fowler.

[19] T. Wada and M. Sato. Scale-space tree and its hierarchy. In *ICPR90*, pages Vol–II 103–108, 1990.

[20] A.P. Witkin. Scale-space filtering. In *Proceedings of the Eighth International Joint Conference on Artificial Intelligence*, pages 1019–1022, 1983.

[21] E.C. Zeeman, editor. *Catastrophe Theory: Selected Papers, 1972-1977*. Addison-Wesley Publishing Company, 1977.