The Application of Catastrophe Theory to Medical Image Analysis

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Abstract

In order to investigate the deep structure of Gaussian scale space images, one needs to understand the behaviour of critical points under the influence of blurring. We show how the mathematical framework of catastrophe theory can be used to describe the various different types of annihilations and creations of pairs of critical points and how this knowledge can be exploited in a scale space hierarchy tree for the purpose of pre-segmentation. We clarify the theory with an artificial image and a simulated MR image.

1 Introduction

The presence of structures of various sizes in an image demands almost automatically a collection of image analysis tools that is capable to deal with these structures. Essential is that this system is capable of handling the various, a priori unknown sizes or scales. To this end various types of multi-scale systems have been developed.

The concept of scale space has been introduced by (both) Witkin [15] and Koenderink [8]. They showed that the natural way to represent an image at finite resolution is by convolving it with a Gaussian of various bandwidths, thus obtaining a smoothened image at a scale determined by the bandwidth. This approach has lead to the formulation of various invariant expressions – expressions that are independent of the coordinates – that capture certain features in an image at distinct levels of scale [4].

In this paper we focus on linear, or Gaussian, scale space. This has the advantage that each scale level only requires the choice of an appropriate scale; and that the image intensity at that level follows linearly from any previous level. It is therefore possible to trace the evolution of certain image entities over scale. The exploitation of various scales simultaneously has been referred to as *deep structure* by Koenderink [8]. It pertains to information of the change of the image from highly detailed –including noise – to highly smoothened. Furthermore, it may be expected that large structures "live" longer than small structures (a reason that Gaussian blur is used to suppress noise). The image together with its blurred version was called "primal sketch" by Lindeberg [11]. Since multi-scale information can be ordered, one obtains a hierarchy representing the subsequent simplification of the image with increasing scale. In one dimensional images this has been done by several authors [7, 14], but higher dimensional images are more complicated as we will discuss below.

An essentially unsolved problem in the investigation of deep structure is how to establish meaningful links across scales. A well-defined and user-independent constraint is that points are linked if they are topological equal. Thus maxima are linked to maxima, etc. This approach has been used in 2-D images by various authors, *e.g.* [12], noticing that sometimes new extrema occurred, disrupting a good linking. This creation of new extrema in scale space has been studied in detail by Damon [2], proving that these creations are generic in images of dimension larger than one. That means that they are not some kind of artifact, introduced by noise or numerical errors, but that they are to be expected in any typical case.

Apart from the above mentioned catastrophe points (annihilations and creations) there is a second type of topologically interesting points in scale space, *viz.* scale space critical points. These are spatial critical points with vanishing scale derivative. This implies a zero Laplacean in linear scale space. Although Laplacean zero-crossings are widely investigated, the combination with zero gradient has only been mentioned occasionally, *e.g.* by [6, 9, 10].

Since linking of topologically identical points is an intensity based approach, also the shape of iso-intensity manifolds must be taken into account. Scale space critical points, together with annihilations and creations allow us to build a hierarchical structure that can be used to obtain a so-called pre-segmentation: a partitioning of the image in which the nesting of iso-intensity manifolds becomes visible.

It is sometimes desirable to use higher order (and thus non-generic) catastrophes to describe the change of structure. In this paper we describe these catastrophes in scale space and show the implications for both the hierarchical structure and the pre-segmentation.

2 Theory

Let $L(\mathbf{x})$ denote an arbitrary n dimensional image, the *initial image*. Then $L(\mathbf{x};t)$ denotes the $n+1$ dimensional *Gaussian scale space image* of $L(\mathbf{x})$. By definition, $L(\mathbf{x};t)$ satisfies the diffusion equation: $\Delta L = \partial_t L$, where ΔL denotes the Laplacean of L. *Spatial critical points*, i.e. saddles and extrema, at a certain scale t_0 are defined as the points at fixed scale t_0 where the spatial gradient vanishes: $\nabla L(\mathbf{x};t_0) = 0$. The type of a spatial critical point is given by the eigenvalues of the Hessian H, the matrix with the second order spatial derivatives, evaluated at its location. Note that the trace of the Hessian equals the Laplacean. For maxima (minima) all eigenvalues of the Hessian are negative (positive). At a spatial saddle point H has both negative and positive eigenvalues.

Since $L(\mathbf{x};t)$ is a continuous – even smooth – function in $(\mathbf{x};t)$-space, spatial critical points are part of a one dimensional manifold in scale space, the *critical curve*.

As a result of the maximum principle, critical points in scale space, i.e. points where both the spatial gradient and the scale derivative vanish: $\nabla L(\mathbf{x};t) = 0 \wedge \partial_t L(\mathbf{x};t) = 0$, are always saddle points and called *scale space saddles*

Consequently, the *extended Hessian* $\mathcal{H}$ of $L(\mathbf{x};t)$, the matrix of second order derivatives in scale space defined by

$$\mathcal{H} = \begin{pmatrix} \nabla\nabla^T L & \Delta\nabla L \\ (\Delta\nabla L)^T & \Delta\Delta L \end{pmatrix},$$

has both positive and negative eigenvalues at scale space saddles. Note that the elements of $\mathcal{H}$ are purely spatial derivatives. This is possible by virtue of the diffusion equation.

Catastrophe Theory The spatial critical points of a function with non-zero eigenvalues of the Hessian are called *Morse critical points*. The *Morse Lemma* states that at these points the qualitative properties of the function are determined by the quadratic part of the Taylor expansion of this function. This part can be reduced to the *Morse canonical form* by a slick choice of coordinates. If at a spatial critical point the Hessian degenerates, so that at least one of the eigenvalues is zero (and consequently its determinant vanishes), the type of the spatial critical point cannot be determined. These points are called *catastrophe points*. The term catastrophe was introduced by Thom [13]. A thorough mathematical treatment can be found in the work of Arnol'd, e.g. [1]. More pragmatic introductions and applications are widely published, e.g. [5].

The catastrophe points are also called *non-Morse critical points*, since a higher order Taylor expansion is essentially needed to describe the qualitative properties. Although the dimension of the variables is arbitrary, the *Thom Splitting Lemma* states that one can split up the function in a Morse and a non-Morse part. The latter consists of variables representing the k "bad" eigenvalues of the Hessian that become zero. The Morse part contains the $n-k$ remaining variables. Consequently, the Hessian contains a $(n-k)\times(n-k)$ submatrix representing a Morse function. It therefore suffices to study the part of k variables. The canonical form of the function at the non-Morse critical point thus contains two parts: a Morse canonical form of $n-k$ variables, in terms of the quadratic part of the Taylor series, and a non-Morse part. The latter can by put into canonical form called the *catastrophe germ*, which is obviously a polynomial of degree 3 or higher.

Since the Morse part does not change qualitatively under small perturbations, it is not necessary to further investigate this part. The non-Morse part, however, does change. Generally the non-Morse critical point will split into a non-Morse critical point, described by a polynomial of lower degree, and Morse critical points, or even exclusively into Morse critical points. This event is called a *morsification*. So the non-Morse part contains the catastrophe germ and a perturbation that controls the morsifications.

Then the general form of a Taylor expansion $f(\mathbf{x})$ at a non-Morse critical point of an n dimensional function can be written as (*Thom's Theorem*): $f(\mathbf{x};\lambda) = CG + PT + Q$, where $CG = CG(x_1,\ldots,x_k)$ denotes the catastrophe germ, $PT = PT(x_1,\ldots,x_k;\lambda_1,\ldots,\lambda_l)$ the perturbation germ with an l dimensional space of parameters, and the Morse part $Q = \sum_{i=k+1}^{n} \epsilon_i x_i^2$ with $\epsilon_i = \pm 1$.

We investigate the set of so-called simple real singularities, with catastrophe germs given by the infinite series $A_k^{\pm} \stackrel{\text{def}}{=} \pm x^{k+1}, k \geq 1$ and $D_k^{\pm} \stackrel{\text{def}}{=} x^2 y \pm y^{k-1}, k \geq 4$. The germs A_k^+ and A_k^- are equivalent for $k = 1$ and k even.

Catastrophes and Scale Space The number of equations defining the catastrophe point equals $n+1$ and therefore it is over-determined with respect to the n spatial variables. In scale space, however, the number of variables equals $n+1$ and catastrophes occur as isolated points.

The transfer of the catastrophe germs to scale space has been made by may authors, [2, 3, 7, 11], among whom Damon's account is probably the most rigorous. He showed that the only generic morsifications in scale space are the aforementioned A_2 (called Fold) catastrophes, describing *annihilations* and *creations* of pairs of critical points. These two points have opposite sign of the determinant of the Hessian before annihilation and after creation. All other events are compounds of such events.

Definition 1 *The scale space fold catastrophe germs are defined by*

$$\begin{aligned} f^{A}(x_1;t) &\stackrel{\text{def}}{=} x_1^3 + 6x_1 t\,, \\ f^{C}(x_1,x_2;t) &\stackrel{\text{def}}{=} x_1^3 - 6x_1(x_2^2 + t). \end{aligned}$$

The Morse part is given by $\sum_{i=2}^{n} \epsilon_i (x_i^2 + 2t)$, *where* $\sum_{i=2}^{n} \epsilon_i \neq 0$ *and* $\epsilon_i \neq 0 \; \forall i$.

Note that both the scale space catastrophe germs and the quadratic terms satisfy the diffusion equation. The germs f^{A} and f^{C} correspond to the two qualitatively different Fold catastrophes at the origin, an annihilation and a creation respectively. From Definition 1 it is obvious that annihilations occur in any dimension, but creations require at least 2 dimensions. Consequently, in 1D signals only annihilations occur. Furthermore, for images of arbitrary dimension it suffices to investigate the 2D case due to the Splitting Lemma.

Scale Space Hierarchy From the previous section it follows that each critical curve in $(\mathbf{x}; t)$-space consists of separate branches, each of which is defined from a creation event to an annihilation event. We set $\#_C$ the number of creation events on a critical path and $\#_A$ the number of annihilation events. Since there exists a scale at which only one spatial critical points (an extremum) remains, there is one critical path with $\#_A = \#_C$, whereas all other critical paths have $\#_A = \#_C + 1$. That is, all but one critical paths are defined for a finite scale range.

One of the properties of scale space is non-enhancement of local extrema. Therefore, isophotes in the neighbourhood of an extremum at a certain scale t_0 move towards the spatial extremum at coarser scale, until at some scale t_1 the intensity of extremum equals the intensity of the isophote. The iso-intensity surface in scale formed by these isophotes form a dome, with its top at the extremum. Since the intensity of the extremum is monotonically in- or decreasing (depending on whether it is a minimum or a maximum, respectively), all such domes are nested. Retrospectively, each extremum branch carries a series of nested domes, defining increasing regions around the extremum in the input image.

These regions are uniquely related to one extremum as long as the intensity of the domes doesn't reach that of the so-called critical dome. The latter is formed by the iso-intensity manifold with its top at the extremum and containing a (nearby) scale space saddle, where both points are part of the same critical curve. That is, the scale space saddle is apparent at the saddle branch that is connected in an annihilation event with the extremum branch. The intensity at this point has a local extremum on the saddle branch.

In this way a hierarchy of regions of the input image is obtained, which can be regarded as a pre-segmentation. It also results in a partition of the scale space itself.

The crucial role is played by the scale space saddles and the catastrophe points. As long as only annihilation and creation events occur, the hierarchy is obtained straightforward. However, sometimes higher order catastrophes are needed to describe the local structure, viz. when two or

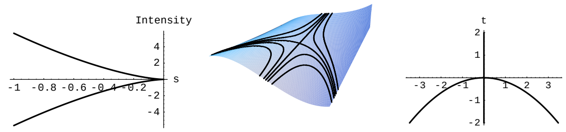

Figure 1. Fold catastrophe in 1D: a) Parametrised intensity of the critical curve. b) 1+1D intensity scale space surface. c) Segments of b), defined by the scale space saddle.

more catastrophes happen to be almost incident and cannot be segregated due to numerical imprecision or (almost) symmetries in the image.

3 Scale space catastrophes and saddles

In this section we discuss the appearance of catastrophe events in scale space and the effect on scale space saddles. Firstly, results on one dimensional images are given, because in this particular case scale space saddles coincide with catastrophe points. Secondly, multi-dimensional images are discussed. In higher dimensions the structure is more complicated, since generically scale space saddles do not coincide with catastrophe points. It suffices to investigate 2D images, since the A and D catastrophes are restricted to 2 bad variables.

A_2 catastrophe in 1D The A_2 catastrophe is called a Fold and is defined by $x^3 + \lambda x$. It scale space appearance is given by

$$L(x; t) = x^3 + 6xt \, .$$

The only perturbation parameter is given by t by the identification $\lambda_1 = 6t$. It has a scale space saddle if both derivatives are zero. So it is located at the origin with intensity equal to zero. The determinant of the extended Hessian is negative, indicating a saddle. The parametrisation of the critical curve with respect to t is $(x(s); t(s)) = (\pm\sqrt{-2s}; s), s \leq 0$ and the parametrised intensity reads $P(s) = \pm 4s\sqrt{-2s}, s \leq 0$, see Figure 1a. The critical dome is given by the isophotes $L(x; t) = 0$ through the origin, so $(x; t) = (0; t)$ and $(x; t) = (x; -\frac{1}{6}x^2)$. Figure 1b shows isophotes $L = constant$ in the $(x; t, L(x; t))$-space, where the self-intersection of the isophote $L = 0$ gives the annihilation point. This isophote gives the separatrices of the different parts of the image. The separation curves in the $(x; t)$-plane are shown in Figure 1c: for $t < 0$ four segments are present, for $t > 0$ two remain.

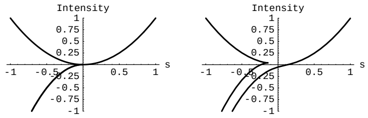

Figure 2. Parametrised intensity of the Cusp catastrophe a) $\epsilon = 0$ **b)** $0 <| \epsilon | \ll 1$

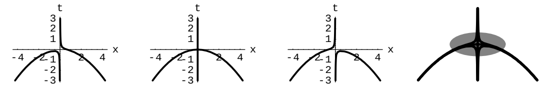

Figure 3. Critical paths in the $(x; t)$**-plane. a)** $\epsilon < 0$ **b)** $\epsilon = 0$ **c)** $\epsilon > 0$ **d) detection of the critical paths around the origin with uncertainty represented by the oval.**

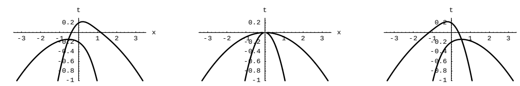

Figure 4. Critical isophotes in the $(x; t)$**-plane. a)** $\epsilon < 0$ **b)** $\epsilon = 0$ **c)** $\epsilon > 0$

1D Cusp catastrophe Although all catastrophes are generically described by fold catastrophes, one may encounter higher order catastrophes, e.g. due to numerical imprecision or symmetries in the signal when a set of two minima and one maximum change into one minimum, but one is not able to detect which minimum is annihilated. At such an event also the extended Hessian degenerates since one of the eigenvalues becomes zero.

The first higher order catastrophe describing such a situation is the A_4 (Cusp) catastrophe: $\pm x^4 + \lambda_1 x + \lambda_2 x^2$. The scale space representation of the catastrophe germ reads $\pm(x^4 + 12x^2 t + 12t^2)$. Obviously, scale fulfils the role of the perturbation by λ_2. Therefore the scale space form is given by

$$L(x; t) = x^4 + 12x^2 t + 12t^2 + \epsilon x,$$

where the two perturbation parameters are given by t for the second order term and ϵ for the first order term. If $\epsilon = 0$ the situation as sketched above occurs. The catastrophe takes place at the origin, where two minima and a maximum change into one minimum for increasing t. At the origin both L_{xx} and L_{xt} are zero, resulting in a zero eigenvalue of the extended Hessian. The parametrised intensity curves are shown in Figure 2a. Note that at the bottom left the two branches of the two minima with equal intensity coincide. The case $0 <| \epsilon | \ll 1$, where a morsification has taken place, is visualised in Figure 2b. This Figure shows the remaining Fold catastrophe of a minimum and a maximum (compare to Figure 1a), and the unaffected other minimum. Depending on the value and sign of ϵ one can find the three different types of catastrophe shown in Figure 3a-c. With an uncertainty in the measurement they may coincide, as shown in Figure 3d, where the oval represents the possible measure uncertainty.

With the degeneration of the extended Hessian at the origin if $\epsilon = 0$, also the shape of the isophotes change, as shown in Figure 4. Since one eigenvalue is zero, the only remaining eigenvector is parallel to the t-axis. So there is no critical isophote in the t-direction, but both parts pass the origin horizontally. Furthermore the annihilating minimum cannot be distinguished from the remaining minimum.

Fold catastrophe in 2D The scale space Fold catastrophe in 2D is given by:

$$L(x, y; t) = x^3 + 6xt + \alpha(y^2 + 2t), \tag{1}$$

where $\alpha = \pm 1$. Positive sign describes a saddle – minimum annihilation, negative sign a saddle – maximum one. Without loss of generality we take $\alpha = 1$. The catastrophe takes place at the origin with intensity equal to zero and the scale space saddle is located at $(x, y; t) = (-\frac{1}{3}, 0; -\frac{1}{18})$ with intensity $-\frac{1}{27}$. The surface $L(x, y; t) = -\frac{1}{27}$ through the scale space saddle is shown in Figure 5a. It has a local maximum at $(x, y; t) = (\frac{1}{6}, 0; -\frac{1}{72})$: the top of the extremum dome. The iso-intensity surface through the scale space saddle can be visualised by two surfaces touching each other at the scale space saddle. One part of the surface is related to the corresponding extremum of the saddle. The other part encircles some other segment of the image. The surface belonging to the extremum forms an dome. The critical curve intersects this surface twice. The saddle branch has a intersection at the scale space saddle, the extremum branch at the top of the dome, as shown in Figure 5a.

The parametrisation of the two branches of the critical curve with respect to t is given by $(x(s), y(s); t(s)) = (\pm\sqrt{-2s}, 0; s), s \leq 0$, see Figure 5b. The intensity of the critical curve reads $L(s) = 2s \pm 4s\sqrt{-2s}, s \leq 0$. The scale space saddle is located at $s = -\frac{1}{18}$, the catastrophe at $s = 0$. These points are visible in Figure 5b as the local minimum of the parametrisation curve and the connection point of the two curves (the upper branch representing the spatial saddle, the lower one the minimum), respectively.

Note that an alternative parametrisation of both branches of the critical curve simultaneously is given by $(x(s), y(s); t(s)) = (s, 0; -\frac{1}{2}s^2)$. Then the intensity of the critical curve is given by $L(s) = -2s^3 - s^2$, see Figure 5c. The catastrophe takes place at $s = 0$, the saddle at $s = -\frac{1}{3}$.

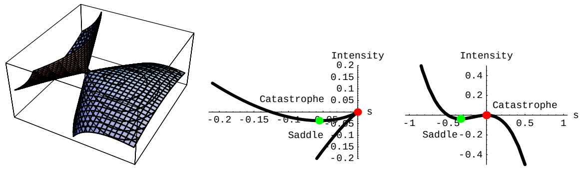

Figure 5. a) 2D Surface trough the scale space saddle. b) Intensity of the critical curve, parametrised by the x-coordinate. c) Same for the t-coordinate.

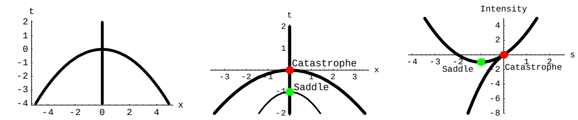

Figure 6. a) Critical paths. b) Critical paths with zero-Laplacean, catastrophe point and scale space saddle if $\alpha > 0$. c) Intensity of the critical paths. The part bottom-left represents two branches ending at the catastrophe point.

These points are visible in Figure 5c as the extrema of the parametrisation curve. The branch $s < 0$ represents the saddle point, the branch $s > 0$ the minimum.

Cusp catastrophe in 2D With the similar argumentation as in the one dimensional case it is also interesting to investigate the behaviour around the next catastrophe event. The 2D scale space extension of the Cusp catastrophe is given by

$$L(x, y; t) = \frac{1}{12}x^4 + x^2 t + t^2 + \alpha(2t + y^2) + \epsilon x$$

where, again, $\alpha = \pm 1$. If $\epsilon \neq 0$ a fold catastrophe results. The critical curves in the $(x; t)$-plane at $\epsilon = 0, y = 0$ are shown in Figure 6a. They form a so-called pitchfork bifurcation at the origin, the catastrophe point.

One can verify that the critical points lay on the curves given by $(x(s), y(s); t(s)) = (0, 0; s)$ and $(x(s), y(s); t(s)) = (\pm\sqrt{-6s}, 0; s), s \leq 0$.

The intensities are given by $L_1(s) = (0, 0; s) = s^2 + 2\alpha s$ with its extremum at $s = -\alpha$ and $L_2(s) = L(\pm\sqrt{-6s}, 0; s) = -2s^2 + 2\alpha s, s \leq 0$. The latter has an extremum at $s = \frac{1}{2}\alpha$. Since $s \leq 0$, these scale space saddles only occur if $\alpha < 0$. It is therefore essential to distinguish between the two signs of α.

Case $\alpha > 0$ For positive α, the curve $(x, y; t) = (0, 0; s)$ contains saddles if $t < 0$ and minima if $t > 0$. The other

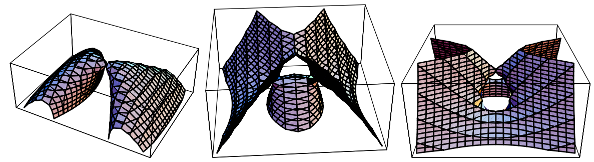

Figure 7. 2D Surfaces trough the scale space saddles at a Cusp catastrophe, a) $\alpha > 0$, b) $\alpha < 0, t = \frac{1}{2}\alpha$ and c) $\alpha < 0, t = -\alpha$

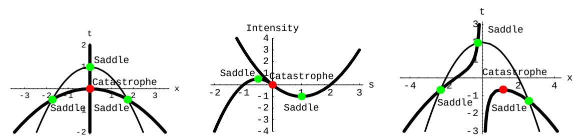

Figure 8. a) Critical paths with zero-Laplacean, catastrophe point and scale space saddle if $\alpha = -1$. b) Intensity of the critical paths. The part bottom-left represents two branches ending at the catastrophe point. c) Critical paths with $\alpha < 0$, $4\epsilon^2 < -3\alpha^3$, zero-Laplacean, catastrophe point and scale space saddle.

curve contains minima on both branches. At the origin a catastrophe occurs, at $(x, y; t) = (0, 0, -\alpha)$ a scale space saddle, see Figure 6b. The intensities of the critical curves are shown in Figure 6c; The two branches of the minima for $t < 0$ have equal intensity. The iso-intensity manifold in scale space forms a double dome since the two minima are indistinguishable, see Figure 7a.

A small perturbation $(0 <| \epsilon | \ll 1)$ leads to a generic image containing a Fold catastrophe and thus a single cone. However, as argued in section 3 this perturbation may be too small to identify the annihilating minimum. We will use this degeneration in Section 4 to identify multiple regions with one scale space saddle

Case $\alpha < 0$ If α is negative, the curve $(x, y; t) = (0, 0; s)$ contains a maximum if $t < 0$ and a saddle if $t > 0$, while the curve $(x, y; t) = (\pm\sqrt{-6s}, 0; s), s < 0$ contains saddles. Now 3 scale space saddles occur: at $(x, y; t) = (0, 0; -\alpha)$ and $(x, y; t) = (\pm\sqrt{-3\alpha}, 0; \frac{1}{2}\alpha)$, see Figure 8a. The corresponding intensities are shown in Figure 8b, where again the intensities of the two saddle branches for $t < 0$ coincide.

The iso-intensity surfaces through the scale space saddles are shown in Figure 7b-c. The scale space saddles at $t = \frac{1}{2}\alpha$ both encapsulate the maximum at the t-axis. The scale space saddle at $t = -\alpha$ is void: it is not related to an extremum. This is clear from the fact that there is only one extremum present.

If a small perturbation ($0 <| \epsilon |\ll 1$) is added, the three scale space saddles remain present in the generic image. Their trajectories in the $(x;t)$-plane are shown in Figure 8c. Now a Fold catastrophe is apparent, but also a saddle branch containing two (void) scale space saddles, caused by the neighbourhood of the annihilating saddle-extremum pair.

Degeneration of $\det(\mathcal{H})$ The extended Hessian degenerates if its determinant vanishes, yielding $4\alpha(2t - x^2) = 0$. This implies $2t = x^2$. Then $L_x = 0$ reduces to $\frac{4}{3}x^3 + \epsilon = 0$ and $L_t = 0$ implies $x^2 = -\alpha$, so the point of degeneration is located at $(x, y; t) = (\sqrt{-\alpha}, 0, -\frac{1}{2}\alpha)$, where $\alpha < 0$ and $9\epsilon^2 = -16\alpha^3$.

This special value for $\alpha, \epsilon \neq 0$ is located at the non-annihilating saddle branch where the two scale space saddle points coincide, i.e. where the saddle branch touches the zero-Laplacean. This case is non-generic, since the intersection of the critical curve and the hyperplane $\Delta L = 0$ at this value is not transverse. This value describes the transition of the case with two void scale space saddles to the case without scale space saddles: For $| \epsilon |< \frac{4}{3}\sqrt{-\alpha^3}$ two void scale space saddles occur on the non-annihilating saddle branch as shown in Figure 8c. For $| \epsilon |> \frac{4}{3}\sqrt{-\alpha^3}$ none occur since it does not intersect the zero-Laplacean. In other words: a Fold catastrophe *in scale space* occurs, regarding two scale space critical points (i.e. saddles) with different signs of $\det(\mathcal{H})$ and controlled by the perturbation parameter ϵ.

D_3^+ **catastrophe in 2D** The D_3^+ catastrophe, called hyperbolic umbilic, is given by $x^3 + xy^2$. The perturbation term contains three terms: $\lambda_1 x + \lambda_2 y + \lambda_3 y^2$. Its scale space addition is $8xt$. Obviously scale takes the role of λ_1. The scale space hyperbolic umbilic catastrophe germ with perturbation is thus defined by

$$L(x, y; t) = x^3 + xy^2 + 8xt + \alpha(y^2 + 2t) + \beta y$$

where the first part describes the scale space catastrophe germ. The set (α, β) form the extra perturbation parameters. One can verify that at the combination $(\alpha, \beta) = (0, 0)$ four critical points exist for each $t < 0$. At $t = 0$ the four critical curves annihilate simultaneously at the origin. This is non-generic, since this point is a scale space saddle with $\det(\mathcal{H}) = 0$.

Morsification takes place in two steps. In the first step one perturbation parameter is non-zero. If $\alpha \neq 0$ and $\beta = 0$, the annihilations are separated. At the origin a Fold catastrophe occurs. On the saddle branch of the critical curve both a scale space saddle and a Cusp catastrophe are found. If $\alpha = 0$ and $\beta \neq 0$, the double annihilation breaks up into two Fold annihilations with symmetric non-intersecting critical curves. A scale space saddle is not present.

Finally, if both α and β are non-zero, this second morsification results in two critical curves each of them containing an Fold annihilation. One the two critical curves contains a scale space saddle.

The extended Hessian degenerates for $x = -\alpha$. Then follows from $L_t = 0$ that $x = \alpha = 0$ and from L_y also $\beta = 0$, which is a non-generic situation.

D_3^- **catastrophe in 2D** The D_3^- catastrophe, called elliptic umbilic, is given by $x^3 - 6xy^2$. The perturbation term contains three terms: $\lambda_1 x + \lambda_2 y + \lambda_3 y^2$. Its scale space addition is $-6xt$. Obviously scale takes the role of λ_1. The scale space elliptic umbilic catastrophe germ with perturbation is thus defined by

$$L(x, y; t) = x^3 - 6xy^2 - 6xt + \alpha(y^2 + 2t) + \beta y \qquad (2)$$

where the first part describes the scale space catastrophe germ. The set (α, β) form the extra perturbation parameters. The combination $(\alpha, \beta) = (0, 0)$ gives two critical points for all $t \neq 0$. At the origin a so-called scatter event occurs: the critical curve changes from y-axis to x-axis with increasing t. Just as in the hyperbolic case, in fact two Fold catastrophes take place; in this case both an annihilation and a creation.

The morsification for $\alpha = 0, \beta \neq 0$ leads to the breaking into two critical curves without any catastrophe.

The morsification for $\alpha \neq 0$, $\beta = 0$ leads to only one catastrophe event at the origin: the Fold creation. The sign of α determines whether the critical curve contains a maximum – saddle pair or a minimum–saddle pair. Without loss of generality we may choose $\alpha = 1$. Then the generic creation germ (see Definition 1) is defined as

$$L(x, y; t) = x^3 - 6xt - 6xy^2 + y^2 + 2t \qquad (3)$$

The scale space saddle is located at $(x, y; t) = (\frac{1}{3}, 0; \frac{1}{18})$ and its intensity is $L(\frac{1}{3}, 0; \frac{1}{18}) = \frac{1}{27}$. The surface $L(x, y; t) = \frac{1}{27}$ has a local saddle at $(x, y; t) = (-\frac{1}{6}, 0; \frac{1}{72})$, see Figure 9. At creations newly created extremum domes can not be present, which is obvious from the maximum principle. Whereas annihilations of critical points leads to the annihilations of level-lines, creations of critical points are caused by the rearrangement of present level-lines. The intersection of the iso-surface through the scale space saddle and the critical curve therefore does not have a local extremum, but only local saddles.

This fact becomes clearer if we take a closer look at the structure of the critical curves. The creation containing critical curve is given by $(x, y; t) = (\pm\sqrt{2t}, 0; t)$. The other critical curve, given by $(x, y; t) = (\frac{1}{6}, \pm\sqrt{\frac{1}{72} - t}; t)$, represents two branches connected at the second catastrophe.

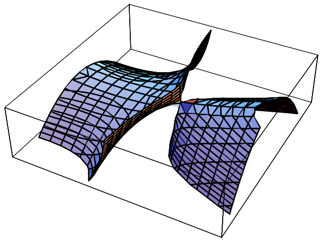

Figure 9. Iso-intensity surface of the scale space saddle of the creation germ.

This point, located at $(x, y; t) = (\frac{1}{6}, 0; \frac{1}{72})$, is an element of both curves and obviously degenerates the extended Hessian. At this point two saddle points and the created extremum go through a Cusp catastrophe resulting in one saddle. Note that ignoring this catastrophe one would find the sudden change of extremum into saddle point while tracing the created critical points. Obviously this catastrophe is located between the creation catastrophe and the scale scale space saddle. The latter therefore does not invoke a critical dome around the created extremum.

A complete morsification by taking $\beta \neq 0$ resolves the scatter. It can be shown that the Hessian has two real roots if and only if $||\beta|| < \frac{1}{32}\sqrt{6}$. At these root points subsequently a creation and an annihilation event take place on a critical curve. If $||\beta|| > \frac{1}{32}\sqrt{6}$ the critical curve doesn't contain catastrophe points.

Due to this morsification the two critical curves do not intersect each other. Also in this perturbed system the minimum annihilates with one of the two saddles, while the other saddle remains unaffected. The scale space saddle remains on the non-catastrophe-involving curve. That is, the creation-annihilation couple and the corresponding saddle curve is not relevant for the scale space saddle and thus the scale space segmentation.

The iso-intensity surface of the scale space saddle due to the creation germ does not connect a dome-shaped surface to an arbitrary other surface, but shows only two parts of the surface touching each other at a void scale space saddle, see e.g. Figure 9

4 Applications

In this section we give some examples to illustrate the theory presented in the previous sections. To show the effect of a cusp catastrophe in 2D, we firstly take a symmetric artificial image containing two Gaussian blobs and add noise to it. This image is shown in Figure 10a. Secondly, the effect is shown on the simulated MR image of Figure 10b. This image is taken from the web site http://www.bic.mni.mcgill.ca/brainweb.

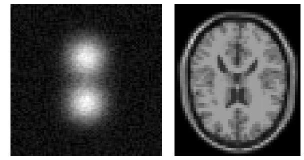

Figure 10. 2D test images a: Artificial image built by combining two maxima and additive noise. b: 181 x 217 artificial MR image.

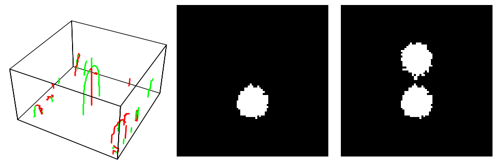

Figure 11. Example of a cusp catastrophe: a: Critical paths in scale space. b: Segment according to a fold catastrophe. c: Segment according to a cusp catastrophe.

Artificial image Of the noisy image of Figure 10a, a scale space image was built containing 41 scales ranging exponentially from $e^{\frac{10}{8}}$ to $e^{\frac{20}{8}}$. The calculated critical paths are presented in Figure 11a. Ignoring the paths on the border, caused by the extrema in the noise, the paths in the middle of the image clearly shown the pitchfork-like behaviour. Note that since the symmetric image is perturbed, instead of a cusp catastrophe a fold catastrophe occurs. The scale space saddle on the saddle branch and its intensity define a closed region around the lower maximum, see Figure 11b. However, if the noise were slightly different, one could have found the region around the upper maximum. Knowing that the image should be symmetric and observing that the critical paths indeed are pitchfork-like, it is thus desirable to identify the catastrophe as a cusp-catastrophe. Then the scale space saddle defines the two regions shown in Figure 11c, which one may want to derive given Figure 10a.

Simulated MR image Subsequently we took the 2D slice from an artificial MR image shown in Figure 10b. The scale space image at scale 8.37 with the large structures remaining is shown in Figure 12a. Now 7 extrema are found, defining a hierarchy of the regions around these extrema as shown in Figure 12b. In this case is it visually desirable to identify a region to segment S_1 with more or less similar size as region S_3. This is done by assigning a Cusp catastrophe to the annihilation of the extremum of segment S_3, in which the extremum of segment S_1 is also involved. Then the value of the scale space saddle defining segment

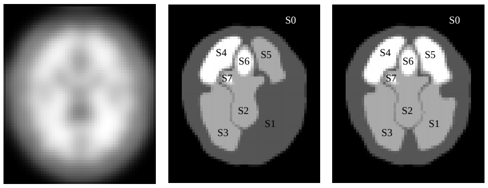

Figure 12. a) Image on scale 8.4 b) Segments of the 7 extrema of a. c) Idem, with the iso-intensity manifold of S_1 chosen equally to S_3.

S_3 also defines an extra region around the extremum in segment S_1. This is shown in Figure 12c, reflecting the symmetry present in Figure 12a. We note that in this example several creation-annihilation events occurred, as described by the morsification of the D_3^- catastrophe.

5 Summary and Discussion

In this paper we investigated the (deep) structure on various catastrophe events in Gaussian scale space. Although it is known that pairs of critical points are annihilated and created (the latter if the dimension of the image is 2 or higher), it is important to describe the local structure of the image around these events. The importance of this local description follows from the method to build the scale space hierarchy. This algorithm depends on the critical curves, their catastrophe points and the space space saddle points. We therefore embedded the mathematically known catastrophes as presented in section 2 to scale space images. Section 3 deals with the two types A and D of generic events.

Firstly, annihilations of extrema can occur in the presence of other extrema. In some cases it is not possible to identify the annihilating extremum due to numerical limitations or symmetries in the image. Then the event is described by a cusp catastrophe instead of a fold catastrophe. This description is sometimes desirable, *e.g.* if prior knowledge is present and one wishes to maintain the symmetry in the image. The scale space hierarchy can easily be adjusted to this extra information. We gave examples in section 4 on an artificial and a simulated MR image. We discussed the A_4 and the D_3^+ for this purpose, but the higher order catastrophes in the sequences $A_k, k > 4$ and $D_k^+, k > 3$ can be dealt with in a similar fashion.

Secondly, the morsification of the D_3^- catastrophe was discussed, showing he successive appearance of a creation-annihilation event on a critical curve. This doesn't influence the the hierarchical structure nor the pre-segmentation, but is only important with respect to the movement of the critical curve in scale space.

The theory described in this paper extends the knowledge of the deep structure of Gaussian scale space. It embeds higher order catastrophes within the framework of the scale space hierarchy. It explains how these events can be used, interpreted and implemented, *e.g.* if prior knowledge should be exploited.

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