Hierachical pre-segmentation without prior knowledge

A. Kuijper, L.M.J. Florack

UU-CS-2001-17
June, 2001

Hierarchical pre-segmentation without prior knowledge*

A. Kuijper

Department of Computer Science
Utrecht University
3584 CH Utrecht the Netherlands

L.M.J. Florack

Department of Mathematics
Utrecht University
3584 CH Utrecht the Netherlands

Abstract

A new method to pre-segment images by means of a hierarchical description is proposed. This description is obtained from an investigation of the deep structure of a scale space image – the input image and *the Gaussian filtered ones simultaneously. We concentrate on scale space critical points – points with vanishing gradient with respect to both spatial* and *scale direction. We show that these points are always saddle points. They turn out to be extremely useful, since the iso-intensity manifolds through these points provide a scale space hierarchy tree and induce a segmentation without a priori knowledge. Moreover, together with the so-called catastrophe points, these scale space saddles form the critical points of the parameterised critical curves – the curves along which the spatial saddle points move in scale space. Experimental results with respect to the hierarchy and segmentation are given, based on artificial images and real MRI .*

1 Introduction

One way to understand the structure of an image is to embed it in a one-parameter family. If a scale-parametrised Gaussian filter is applied, the parameter can be regarded as the "scale" or the "resolution" at which the image is observed. The resulting structure has become known as *linear*, or *Gaussian, scale space*. Main advantage is that this set of filters enables one to take derivatives of a discrete image. More detailed literature can be found in *e.g.* [2, 14, 15, 17].

In their original accounts both Koenderink [8] and Witkin [20] proposed to investigate the "deep structure" of an image, *i.e. structure at all levels of resolution simultaneously*. Encouraged by the results in specific image analysis applications, an increasing interest has recently emerged trying to establish a generic underpinning of deep structure. Results from this may serve as a basis for a diversity of multiresolution schemes. Such bottom-up approaches often rely on *catastrophe theory* [4, 18], which is now fairly well-established in the context of the scale-space paradigm. The application of catastrophe theory in Gaussian scale space has been studied *e.g.* by Damon [1]—probably the most comprehensive account on the subject—as well as by others [5, 6, 7, 9, 12, 13, 14].

The first stage in using the deep structure is to link image properties of two subsequent resolution scales. Although this may seem obvious, it is a non-trivial task in a discrete scale space. For example, if extrema at different scales correspond to an extremum at the input image, they should be linked. However, extrema may be annihilated or created. Tracking over scale therefore needs a cautious approach. Koenderink [8] mentioned a possible linking strategy using the properties of the Gaussian scale space. However, only a few heuristic attempts have been made to build such multi-scale datastructures, *e.g.* by Vincken [19]. Simmons *et al.* [16] used the idea of Koenderink's scheme for building a so-called extremum stack. However, they ignored the generic possibility of creations and only used the annihilation intensity. Their work was an extension of the results by Lifshitz and Pizer [12], who implemented Koenderink's scheme, mainly focusing on heuristics and the performance of the algorithm. At the annihilation of a minimum and a saddle point they noticed that the saddle point decreased in intensity, but passing the zero-crossing of the Laplacean, close to the annihilation, started to increase again. In response to their research Koenderink [9] showed that this happens generically for 2D saddles. Moreover, saddle points with zero-Laplacean are saddle points in scale space.

Special behaviour of critical curves at scale space saddles has been mentioned in literature by few other authors. Griffin [5] pointed out that at a catastrophe the saddle and the extremum necessarily have the same sign of the Laplacean and distinguished between ridge and trough saddles. Therefore saddles change from ridge to trough or vice versa. Lindeberg [13, 14] investigated the locations of Laplacean zero-crossings in combination with the (annihilation of) critical points and concluded that "in two and higher dimensions there is no absolute relation between lo-

* submitted to ICCV '01 and Scale Space 2001 conferences.

cations of the Laplacean zero-crossing curves and the local extrema of a signal".

The aim of this paper is to combine knowledge from catastrophe theory, properties of scale space, particularly with respect to the scale space saddles, and the multi-scale linking strategy as suggested by Koenderink. In section 2 we explain basic principles and show that scale space saddles are the key to explore the deep structure of scale space images. They give rise to the unambiguous multi-scale hierarchy describing the image presented in section 3. Images in one dimension fundamentally differ from those in higher dimensions, since only in 1D images the scale space saddles coincide with the catastrophe points. Therefore both cases are discussed separately. The results lead to a non-heuristic hierarchical multi-scale data structure and a segmentation of images without any a priori knowledge. Section 4 shows results on simple images and a 2D MRI. Main conclusions and results are given in section 5.

2 Theory

2.1 Deep Structure in Gaussian Scale Space

Given an arbitrary n-dimensional image $L(\mathbf{x})$, we denote its Gaussian scale space image by $L(\mathbf{x};t)$. Spatial critical points (extrema, saddles) of $L(\mathbf{x};t)$ at certain scale t_0 are defined as the points where $\nabla L(\mathbf{x};t_0)=0$. The behaviour of spatial critical points as the (scale) parameter changes is described by catastrophe theory. As the parameter continuously changes, the critical points move along critical curves, defined as a one dimensional manifold in scale space on which $\nabla L(\mathbf{x};t)=0$. If the determinant of the Hessian does not become zero, these critical points are called *Morse critical points*. In a typical image these points are extrema (minima and maxima) or saddles. *The Morse lemma* states that the topology of a neighbourhood of a Morse critical point can essentially be described by a second order polynomial. At isolated points on a critical curve the determinant of the Hessian may become zero. These points are called *non-Morse points*. Neighbourhoods of such points need a third or higher order polynomial, as described by *Thom's theorem* [18]. If an image is slightly perturbed, the Morse critical points may undergo a small displacement, but qualitatively nothing happens to them. A non-Morse point, however, will change. In general it will split into a number of Morse critical points. This event is called *morsification*. Thom's theorem provides a list of elementary catastrophes with canonical formulasfor the catastrophe germs and the perturbations. The Thom splitting lemma states that *canonical coordinates* exist in which these events can be described. In general, these 'curved' coordinates do not coincide with the user-defined (usually Cartesian) coordinates, but are used for notational convenience. In Gaussian scale space the only generic events are *annihilations* and *creations* of a pair of Morse points: an extremum and a saddle in the 2D case. All other events can be split into a combination of one of these events and one 'in which nothing happens'. See Damon [1] for a proof. Canonical descriptions of these events are given by the following formulae:

$$f^{\mathrm{A}}(\mathbf{x};t) \stackrel{\text{def}}{=} x_1^3 + 6x_1 t + Q(x_2,\ldots,x_n;t) \qquad (1)$$

$$f^{\mathrm{C}}(\mathbf{x};t) \stackrel{\text{def}}{=} x_1^3 - 6x_1(x_2^2+t) + Q(x_2,\ldots,x_n;t), \qquad (2)$$

where for all $a_i \neq 0$, Q is defined by

$$Q(x_2,\ldots,x_n;t) \stackrel{\text{def}}{=} \sum_{i=2}^{n} a_i\left(x_i^2+2t\right)$$

with $\sum_{i=2}^{n} a_i \neq 0$ and $a_i \neq 0,\ 2 \leq i \leq n$. Note that Eq. (1) and Eq. (2), describing annihilation and creation respectively, satisfy the diffusion equation

$$\frac{\partial L}{\partial t} = \Delta L\,. \qquad (3)$$

It can be verified that the the form $f^{\mathrm{A}}(x,y;t)$ corresponds to an annihilation at the origin via the critical path $(\sqrt{-2t}\,,\,0\,;\,t)\ ,\ t \leq 0$, and $f^{\mathrm{C}}(x,y;t)$ to a creation via the critical path $(\sqrt{2t}\,,\,0\,;\,t)\ ,\ t \geq 0$.

Note that creations are generic. They are not sometimes temporarily created, nor false extrema, nor pathological cases, nor only rarely created, although it is true that they are not as frequently encountered as annihilations.

In 1-D images only annihilations occur. Then Eq. (1) becomes $f^{\mathrm{A}}(x;t) \stackrel{\text{def}}{=} x^3 + 6xt$. See *e.g.* Lindeberg [14] for a proof.

A consequence of the Gaussian scale space representation is the strong smoothing property, usually mentioned for its *non-enhancement of local extrema*. It corresponds to the *extremum principle* for parabolic differential equations: If at a certain scale $t_0 > 0$ a point $\mathbf{x}_0$ is a local maximum (minimum) of the function $L(\mathbf{x};t_0)$, then the Laplacean $\Delta L(\mathbf{x}_0;t_0)$ at this point is negative (positive). This means that $\partial_t L(\mathbf{x}_0;t_0)$ is strictly negative (positive). In other words, small local variations will be suppressed. See *e.g.* Lindeberg [14] for more details.

As a result, the structure of iso-intensity manifolds in scale space close to an extremum is umbrella-shaped: At some scale an iso-intensity manifold (an isophote in 2D) encapsulates an extremum, *e.g.* a maximum. The intensity of this iso-intensity manifold is smaller than that of the maximum. Due to the extremum principle the intensity of the maximum decreases and at a certain scale it equals the intensity of the manifold when it reaches the top of the umbrella. At coarser scales the iso-intensity manifold around this extremum has disappeared.

Thus the evolution of extrema induce a family of iso-intensity umbrellas, nested like union peels.

2.2 Scale Space Critical Points

Scale space critical points of $L(\mathbf{x};t)$ are defined as the points with zero gradient and zero Laplacean: $\nabla L(\mathbf{x};t) = 0 \wedge \Delta L(\mathbf{x};t) = 0$, since $\partial_t L(\mathbf{x};t) \stackrel{\text{def}}{=} \Delta L(\mathbf{x};t)$ by definition. The type of these critical points is determined by the eigenvalues of the matrix of second order derivatives, $\mathcal{H}$. We call this matrix the *extended Hessian*:

$$\mathcal{H} = \begin{pmatrix} H & \Delta\nabla L \\ (\Delta\nabla L)^T & \Delta\Delta L \end{pmatrix}. \tag{4}$$

Here H is the spatial Hessian defined by $H_{i,j} = L_{i,j}$, all evaluated at the location of the critical point of interest. Points are maxima (minima) if all eigenvalues are all negative (positive). If at least two eigenvalues have a different sign, the point is a saddle. Since $\mathcal{H}$ is symmetric, all eigenvalues are real.

Theorem 1 *The matrix $\mathcal{H}$ has both positive and negative eigenvalues if $\Delta L = 0$.*

Proof 1 *Let the point $(\mathbf{x}_0;t_0)$ be a critical point of the function $L(\mathbf{x};t)$. Then $(\mathbf{x}_0;t_0)$ is also a critical point of the function $L(\mathbf{x};t_0)$ at scale t_0. If $(\mathbf{x}_0;t_0)$ is an extremum of $L(\mathbf{x};t)$, it is also an extremum of $L(\mathbf{x};t_0)$. But then the extremum principle states that the Laplacean is non-zero. So $(\mathbf{x}_0;t_0)$ is a saddle point.* □

As a consequence, critical points in scale space are *always* saddle points. These scale space saddle points form a subset of the spatial saddles, since critical points with vanishing Laplacean in spatial sense are always saddle points.

This notion extends the idea of non-creation of local (spatial) extrema, valid only in the one dimensional case, but sometimes erroneously extended to higher dimensions. It is known that in spatial coordinates, while increasing scale, new extrema can occur, except for 1D. However, in the full coordinate system, *viz.* including scale, this intuitive notion of non-creation is true, albeit not for spatial, but for scale space extrema, since the latter do not even exist on the interior of the scale space. Moreover, it is even requested by the notion of causality, that states that isophotes in scale space only disappear and never appear (no spurious detail).

The only spatial critical point traversing the scale space saddle is the spatial saddle. Since the manifold $\nabla L = 0$ intersects the manifold $\Delta L = 0$ transversally, the intensity of this spatial saddle has an extremum at the scale space saddle. Therefore, its intensity first increases and then decreases, or vice versa.

2.3 Critical Curves in Scale Space

In scale space each critical curve contains branches representing critical points. Branches are connected at catastrophe points, where two critical points are annihilated or created. These two critical points differ with respect to the sign of one eigenvalue of the Hessian, that becomes zero at the catastrophe. Of all other eigenvalues the number of positive and negative signs is equal. Note that a critical curve can contain several catastrophe points.

In two dimensional images these two points necessarily are a saddle and an extremum, in one dimensional ones they are a maximum and a minimum. In higher dimensions interactions become more complicated, since also catastrophes of saddles of different type are also possible. For writing convenience we will use the terminology saddle and extremum (minimum, maximum) to distinguish between the two types of critical points.

It is known from catastrophe theory that each branch of the critical curve is bounded with respect to scale: at some scale the critical points annihilate. Critical points are present from the initial scale or they are created at a certain (catastrophe) point in scale space. If the scale is taken coarse enough only one extremum remains. Then there exists one critical curve bounded by the coarsest scale. Apart from catastrophe points a second type of points comprises special behaviour, viz. scale space saddles.

On critical curves the intensities of the critical points is well-defined. The intensity of extrema is damped continuously in scale space. Each minimum (maximum) therefore increases (decreases) monotonically towards its annihilation point. At certain spatial and scale distance from the annihilation, the intensity of corresponding saddle will generically tend to move towards the intensity of extremum, i.e. it decreases (increases) to the intensity of minimum (maximum). So the signs of the Laplacean of both critical points at that scale will be opposite. At the catastrophe point, however, they necessarily have the same sign and both points approach the intensity of the annihilation decreasing (increasing).

Therefore, at saddle-branch of the critical curves, the saddle will generically pass a point at which the Laplacean equals zero: a scale space saddle. Since the sign of the Laplacean changes passing the scale space saddle, the intensity will have a switch in intensity.

The intensity of the critical curve of the both the annihilation as the creation of critical points can be generically shown as a curve with two extrema, where the minimum corresponds to the scale space saddle and the maximum to the catastrophe, in case of a saddle-minimum annihilation; and vice versa.

A parametrisation of a critical curve leads to a 1D-function of the intensity of the critical points. The extrema of this function have the following properties:

Theorem 2 *Let $(\mathbf{x}(s);t(s))$ be a parametrisation of $(\mathbf{x};t)$, such that $\nabla L(\mathbf{x}(s);t(s)) = 0$, i.e. $(\mathbf{x}(s);t(s))$ defines a critical curve. Then $L(\mathbf{x}(s);t(s))$ has its extrema at the scale space saddle(s) and the catastrophe point(s). For 1D* *images (signals) the parametrisation has a point of inflection.*

Proof 2 *The total differentiation of $L(\mathbf{x}(s);t(s))$ with respect to s is defined by*

$$\frac{dL(\mathbf{x}(s);t(s))}{ds} = \nabla L \cdot \mathbf{x}_s + \Delta L \cdot t_s. \quad (5)$$

Here

$$\mathbf{x}_s \stackrel{\text{def}}{=} \frac{d\mathbf{x}}{ds}, \quad t_s \stackrel{\text{def}}{=} \frac{dt}{ds}$$

Since $\nabla L = 0$, the critical points of Eq. (5) are given by $\Delta L \cdot t_s = 0$. The scale space saddles are defined as the points where $\Delta L = 0$, whereas the catastrophes take place at the location where the saddle and the extremum 'meet' in scale space, i.e. where the parametrisation of scale has its local extremum. The critical points of $L(\mathbf{x}(s);t(s))$ are extrema, since both the Laplacean and the catastrophe point are non-degenerate and do not coincide for n-D images, $n > 1$. For 1D images they do, so the point is a point of inflection. $\Box$

Although this results holds for any parametrisation of the critical curves, in practice the intensities of critical points are obtained at the calculated scales of the scale space. In other words, they are measured as a function of scale. Then $t = s$, so $t_s = 1$ and $L(\mathbf{x}(s);t(s))$ is obtained as the union of its different branches. Each branch is defined on a closed interval $s_1 \leq s \leq s_2$, where s_1 is either the initial or the creation scale, and s_2 is the annihilation scale of the spatial critical point.

2.4 The Structure of Iso-intensity Manifolds

Each extremum is encircled by iso-intensity manifolds. The shape of these manifolds is determined by the presence of scale space saddles and the annihilation point, the intensity at these points and the intensity of the manifold at other points.

At the annihilation intensity the manifold has a horseshoe shape, as known from catastrophe theory, see Figure 1d.

At scale space saddles the manifold is umbrella-shaped around the extremum belonging to the saddle point. At the saddle point it touches another manifold with the same intensity, see Figure 1b.

At intensities between the scale space saddle and the annihilation point the manifold around the extremum transforms from umbrella-shaped to horseshoe, see Figure 1c.

At other intensities 'before' the annihilation the manifolds around the extremum are umbrella-shaped encapsulating a bounded region , see Figure 1a.

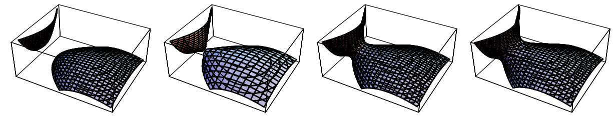

Figure 1: 2D Iso-intensity surfaces a) Before the scale space saddle b) At the scale space saddle c) Between the scale space saddle and the catastrophe point d) At the catastrophe point

As a dual expression it follows that each extremum forms the top of an iso-intensity umbrella in scale space, until its intensity equals that of the related scale space saddle. Then the umbrella transforms to a horseshoe shape at the annihilation. In case of a minimum (maximum) there are only pure umbrellas at intensities smaller (larger) than the intensity of the scale space saddle.

3 Scale Space Hierarchy and Pre-segmentation

Since each extremum encapsulates a series of umbrellas from the initial scale to a scale space saddle, the intensities of the collection of these umbrellas define a segment. The boundaries of various segments follow directly from the intensities of the scale space saddles. A natural hierarchy is obtained as scale space segments are defined by the regions encapsulated by the iso-intensity manifolds through the scale space saddles. This hierarchy avoids the problems of a straight-forward segmentation of an image based on the intensities of the saddle points. Although saddles have different intensities in the initial image (since they are Morse-saddles), at some scales intensities of saddles are equal, see e.g. Lindeberg [14]. Then, for example, a saddle isophote contains another saddle and encircles three extrema. In scale space, however, the scale space saddles generically have different intensities.

The hierarchy tree contains as nodes the scale space saddles and their intensities. The branches are formed by the segments, defined by the collection of internal umbrellas, bounded by the iso-intensity manifold through the scale space saddle. So one branch represents the set of umbrellas of the corresponding extremum. The scale space saddles are ordered by scale. Segments in the tree can be joined if they have a scale space saddle in common. Subtrees contain parts of the image and can be selected or deselected. To obtain a simple segmentation, only the part of the tree with large scales can be regarded. See section 4 for applications.

The scale space hierarchy is uniquely found by the following algorithm:

1 Build a scale space.

2 Find the extrema and the saddle points at each level.

3 Construct critical paths.

4 Connect the critical paths.

5 Find the scale space saddles.

6 Build the hierarchical tree.

In the following these items will be explained and illustrated.

1 Scale Space
Input is an image of arbitrary size and dimension. Only for the sake of illustration we consider the one and two dimensional cases. Images of higher dimension are comparable to the two dimensional ones, albeit that they allow saddle-saddle pairs. A scale space image is obtained by convolving the input image with a normalised Gaussian filter of variable size. The intermediate levels are logarithmically sampled, see e.g. [2, 8, 10, 14].

2 Extremum and Saddle Stacks
Each level in scale space is a blurred image. Its critical points can be calculated by various methods, e.g. zero-crossings of the derivatives, winding-numbers, or neighbourhood-relations.

3 Extremum and Saddle Paths
Since critical points can be annihilated and created, they inherit, apart form movement in scale direction, also spatial drift. This movement can be calculated accurately by means of derivatives up to third order, see e.g. [3, 11]. To link critical points at two subsequent scales, for each critical point at scale i both its current and expected location, predicted by its spatial drift, are compared with the critical points at scale $i + 1$. The outcome of this procedure are two stacks each containing doubly linked lists. The head of each list corresponds with the creation of the critical point (or the initial scale), its tail with the annihilation.

4 Connected Critical Paths
Since the annihilation of an extremum involves a saddle, each tail of an extremum list at a certain scale i corresponds to a tail of some saddle list at the same scale i. At catastrophes the spatial drift becomes undetermined since $\det(H) = 0$. Then the movement of a critical point can still be accurately predicted, see [3, 11]. This results in chains of extremum-saddle pairs, viz. critical curves.

5 Scale Space Saddles
Scale space saddles have the property that they are the local extrema of the parametrised intensity-curve, obtained by taking the intensity along the saddle branches as function of scale, as argued in section 2. Saddle lists can have zero or multiple extrema with respect to intensity. If no extrema are found then the Laplaceans of the extremum and the saddle have either the same or the opposite sign at all scales. The former signals that the scale space saddle is absent. To identify a segment with the extremum, the intensity of the saddle in the first image of the scale space stack can be taken. The latter case represents a scale space saddle located closer to the catastrophe point than is measured. The saddle at the coarsest scale is assigned as scale space saddle. If multiple scale space saddles are found within one saddle list, the one at coarsest scale is chosen. Since each extremum list is linked to a saddle list, each extremum is linked to a scale space saddle. Equivalently, the iso-intensity manifold through the scale space saddle encapsulates the corresponding extremum.

6 Hierarchical Tree
The scale space saddles are sorted from coarse to fine according to scale at which the extremum saddle pair annihilates. Each scale space saddle defines an iso-intensity manifold around an extremum: the part of the image encapsulated by this manifold is a segment of the image at that scale. Segments may have sub-segments, defined by scale space saddles within the segment. At the coarsest scale only one extremum remains. Since it has no corresponding saddle branch containing a scale space saddle, it does not have an a priori critical umbrella. These umbrellas, however, are defined as the iso-intensity manifolds through an extremum at a pre-defined scale, viz. at which a scale space saddle occurs. Therefore the iso-intensity manifold of the last extremum can be chosen having the intensity of the extremum at the coarsest scale. Since the heat equation is energy preserving, it is known that the input image converges to an image of constant value equalling the average value of the input image. Consequently the value of the iso-intensity manifold of the remaining extremum can be set to this value.

Segmentation
A natural segmentation, or rather "pre-segmentation", of scale space is thus obtained by the iso-intensity manifolds of the scale space saddles with their corresponding extrema. Consequently a spatial segmentation of the image at any scale t is found by the intersection of the scale space segmentation and this fixed scale t; A full (partial) segmentation of the initial image is found by taking into account the intensities of all (a proper subset of all) scale space saddles. At a partial segmentation each selection of scale space saddles define segments with a certain grey-level histogram. If knowledge of the grey-level distribution of the image is present, this is a semantical choice.

4 Applications

1D images
As an example of segmentation at various scales and a hier-

Figure 2: a) Initial image b) After one saddle, c) after the second one d) and after the last one.

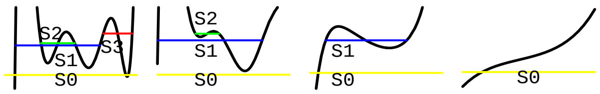

Figure 3: a) Initial image with 4 segments. S2 is a subset of S1. S1 and S3 are subsets of S0 b) After the first saddle S3 vanishes. c) After the second saddle the segment S1 and S2 remain. d) After the second saddle the only segment is S0.

archy tree in 1D we use Figure 2a. The scale space image contains three scale space saddles (equivalent: catastrophes, annihilations). The four essential different appearances are shown in Figure 2.

At each scale a minimum-maximum pair defines a segment based on the intensity of their scale space saddle. These segments vary in scale space. For the images of Figure 2a-c the corresponding segments labelled S1, S2, and S3 are shown in Figure 3a-c.

The scale space hierarchy tree is shown in Figure 4. At high scales there is only one segment: the whole image from boundary to boundary. Decreasing scale one reaches scale space saddle 3, from which point the image contains two segments: S1 and the complement of S1: the parts that range from the boundaries to S1. Continuing the descent one reaches scale space saddle 2, from which point segment S1 contains a subsegment, viz. S2. Decreasing scale even more one ends up with scale space saddle 1, from which point a new segment S3 is obtained from the boundary part.

Interpreting Figure 4 the other way round one concludes that at increasing scale firstly segment S3 vanishes at the boundary, secondly S2 is "gulped down" by S1, and finally S1 disappears.

This notion of disappearing of structure at special points gave rise to the gist that the essence of segmenting images should be based on catastrophe points instead of scale space points. This misinterpretation is caused by the coincidence of scale space saddles and catastrophe points in 1D.

2D images

To show the results of the scale space saddle hierarchy and pre-segmentation in 2D we took an 81x81 artificial image, made by the combination of four maxima and one (induced) minimum, see Figure 5a. The simplicity of this image enables a quantitative check of the outcome. Subsequently we took the 2D slice from an MR image shown in Figure 5b to illustrate the use of the hierarchy tree.

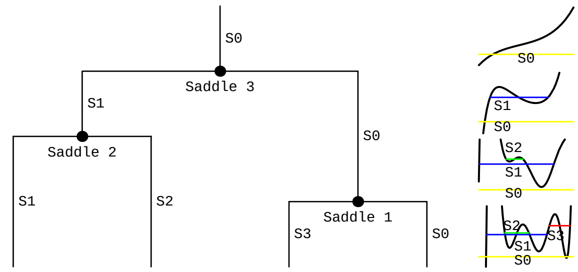

Figure 4: Hierarchy tree of Figure 2a. The three 'Saddles' denote the topological changes of the image in scale space. The branches denote the segments present at the distinguished scales. The stack of images at the right show the corresponding image between the scale space saddle scales. Note that segment S2 is a subsegment of segment S1.

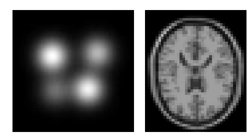

Figure 5: 2D test images. a) Artificial 81 x 81 image build by combining four maxima and one minimum. b) 181 x 217 MR image.

The 81 x 81 artificial image contains 5 extrema. Since the image at (very) large scale contains only one blob, 4 extrema must be annihilated. To obtain the scale space hierarchy firstly a scale space consisting of 113 levels was built. Levels were calculated at scales $e^{i/32}, i = 2, \ldots, 114$. Secondly at each level the spatial critical points were calculated.

Thirdly the spatial critical points of subsequent scales were linked resulting in the critical paths. Figure 6a shows the locations of spatial critical points in scale space. For visualisation purposes this 81x81x113 space was reduced to a 41x41x113 volume of interest space. Dark points correspond to extrema, light points to saddle points. At three scales a pair of created and directly annihilated critical points is visible. The algorithm is able to detect these points and finds the right linking.

Fourthly extrema and saddle points were pairwise grouped by means of the catastrophe points. The parametrised critical paths, viz. the intensities of the critical curves containing the branches of saddle and extremum branches, are shown in Figure 6b. The four catastrophes are visible as the end of two branches of critical points.

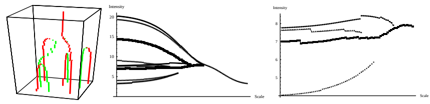

Figure 6: a) Spatial critical points of Figure 5a in $(x, y; t)$ scale space. Dark points correspond to extrema, light points to saddle points. b) Intensities of the critical points as function of scale. c) Intensities of the saddles as function of scale.

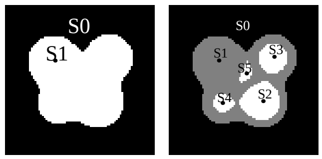

Figure 7: a) Segments S_1 and S_0 of Figure 5a projected at the initial image. b) Pre-segmentation of the initial image.

Fifthly the scale space saddles are derived from the saddle branches. These are shown in Figure 6c. It can be seen that the upper three saddle branches, although containing multiple local extrema with respect to the intensity, have a global maximum, viz. the scale space saddle of interest. The fourth saddle branch is monotonically increasing, just as its corresponding minimum. Therefore the intensity of the spatial saddle at the first level is chosen as value for the minimum encapsulating manifold.

Finally an unambiguous hierarchy based on the catastrophe points and the scale space saddles, just as in the 1D case, can be made. The presence of 5 extrema results in 5 inner regions $S_i, i = 1, \ldots, 5$ and a boundary region S_0. The first region is defined by the remaining extremum. The scale space umbrella defined by this maximum is the iso-intensity manifold valued by the intensity of the extremum at coarsest scale. Since the diffusion equation is energy preserving, this value converges to the average intensity of the initial image. This convergence can also be seen in Figure 6b. This segment S_1 and its dual S_0 projected to the initial image are shown in Figure 7a.

To find the next segment, scale is decreased until the second extremum appears. The segment S_2 corresponding to it is located at the bottom right part of the image. The value of the iso-intensity manifold is obtained from the scale space saddle of the spatial saddle corresponding to this extremum. The other segments are found in the same way, resulting in the pre-segmentation of the image as shown in Figure 7b. Furthermore the hierarchy associated with this pre-segmentation is given by the successive annihilations in scale space, shown in Figure 8.

Having a hierarchical description tree of the image, one can disregard parts of the tree. Combined with knowledge of the image one can thus obtain a pre-segmentation useful for e.g. further segmentation. Figure 10a shows a 2D slide from an simulated MRI of brain tissue. This image is taken from the web-site http://www.bic.mni.mcgill.ca/brainweb. In order to investigate the large structures of this 2D image, we focused on the part of the scale space with scales varying from 8.4 to 32.9. Within these scales, of 7 extrema 6 annihilate. The parametrised critical paths are shown in Figure 9a, the saddle branches in Figure 9b.

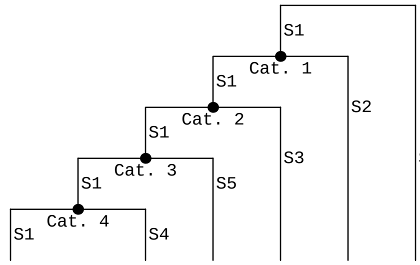

Figure 8: Hierarchy tree of Figure 5a. Segments are labelled corresponding to Figure 7. Segments $S_2, \ldots, S_5$ are subsegments of segment S_1, but annihilate in the sequence S_4, S_5, S_3, S_2 at increasing scale.

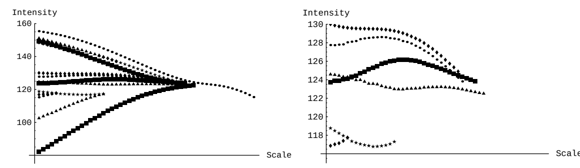

Figure 9: a) Intensities of the critical points of Figure 5b as function of scale. b) Intensities of the saddles as function of scale.

Figure 10d shows a direct intersection of the original image Figure 10a with the manifolds of these 7 extrema. The range of values is reduced from $0, \ldots, 255$ to $0, \ldots, 8$. With the simulated MRI, also the distributions of the white and gray matter are given as ground-truth. These images are shown in Figure 10b-c, respectively. Selecting the manifold obtained by the scale space saddle of the last catastrophe, Figure 10e is obtained. Selecting the region between two manifolds, that is: deselecting a part of the hierarchy tree, Figure 10f is obtained.

5 Summary and Conclusions

We developed a method to calculate the hierarchical structure of an arbitrary input image. Consequently, this structure can be represented as a pre-segmentation. The method is based on the scale space image of the input image and the critical paths within it. The latter exist of branches of extrema and saddle points. The range of scales at with these

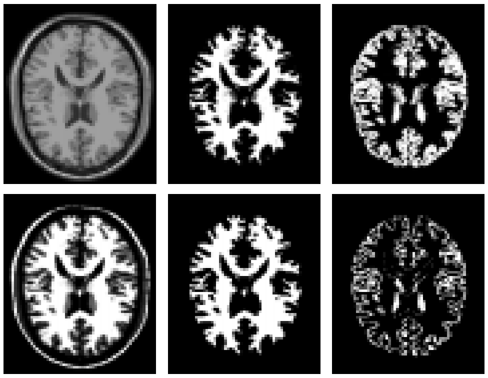

Figure 10: Top row: a) Simulated MR brain image. b) Ground-truth white matter. c) Ground-truth grey matter. Bottom row: d) Segmentation by 7 extrema. e) Segment of the extremum of the last catastrophe. f) Region of the segment of a selected extremum and its successive subsegment.

branches exist follow from their catastrophe points in scale space. To each extremum that annihilates an iso-intensity manifold is assigned. The value of this manifold equals that of the scale space saddle located at the saddle branch annihilating with the extremum branch. This point is a critical point in scale space. The iso-intensity manifold encapsulates the extremum in scale space. The manifolds through the extrema are nested and non-intersecting and thus form a hierarchy. Consequently, a pre-segmentation of the image without any a priori knowledge is obtained by the intersection of the image and the manifolds. The proposed algorithm has two main advantages. Firstly it has a severe mathematical underpinning which encourages and facilitates future improvements, and admits reproducible, predictable, and provable segmentation results. Secondly it has the potential to include semantics enabling an intelligent choice of the nodes, either by deterministic, statistic or probabilistic means. Experimental results based on artificial images and simulated MRI with respect to the hierarchy and segmentation were given and showed intuitive results.

References

[1] J. Damon. Local Morse theory for solutions to the heat equation and Gaussian blurring. *Journal of Differential Equations*, 115(2):386–401, 1995.

[2] L. M. J. Florack. *Image Structure*, volume 10 of *Computational Imaging and Vision Series*. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.

[3] L. M. J. Florack and A. Kuijper. The topological structure of scale-space images. *Journal of Mathematical Imaging and Vision*, 12(1):65–80, February 2000.

[4] R. Gilmore. *Catastrophe Theory for Scientists and Engineers*. Dover, 1981. Originally published by John Wiley & Sons, New York, 1981.

[5] L. D. Griffin and A. Colchester. Superficial and deep structure in linear diffusion scale space: Isophotes, critical points and separatrices. *Image and Vision Computing*, 13(7):543–557, September 1995.

[6] P. Johansen. On the classification of toppoints in scale space. *Mathematical Imaging and Vision*, 4(1):57–67, 1994.

[7] P. Johansen. Local analysis of image scale space. In *Sporring et al. [17]*, pages 139–146, 1997.

[8] J. J. Koenderink. The structure of images. *Biological Cybernetics*, 50:363–370, 1984.

[9] J. J. Koenderink. A hitherto unnoticed singularity of scale-space. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 11(11):1222–1224, 1989.

[10] Jan J. Koenderink. *Solid Shape*. MIT Press, Cambridge, Massachusetts, 1990.

[11] A. Kuijper and L.M.J. Florack. Calculations on critical points under gaussian blurring. In *Nielsen et al. [15]*, pages 318–329, 1999.

[12] L. M. Lifshitz and S. M. Pizer. A multiresolution hierarchical approach to image segmentation based on intensity extrema. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 12(6):529–540, 1990.

[13] T. Lindeberg. Scale–space behaviour of local extrema and blobs. *Journal of Mathematical Imaging and Vision*, 1(1):65–99, 1992.

[14] T. Lindeberg. *Scale-Space Theory in Computer Vision*. The Kluwer International Series in Engineering and Computer Science. Kluwer Academic Publishers, 1994.

[15] O. Fogh Olsen M. Nielsen, P. Johansen and J. Weickert, editors. *Scale-Space Theories in Computer Vision*, volume volume 1682 of *Lecture Notes in Computer Science*. Springer -Verlag, Berlin Heidelberg, 1999.

[16] A. Simmons, S.R. Arridge, P.S. Tofts, and G.J. Barker. Application of the extremum stack to neurological MRI. *IEEE Transactions on Medical Imaging*, 17(3):371–382, June 1998.

[17] J. Sporring, M. Nielsen, L.M.J. Florack, and P. Johansen, editors. *Gaussian Scale-Space Theory*, volume volume 8 of *Computational Imaging and Vision Series*. Kluwer Academic Publishers, Dordrecht, second edition, 1997.

[18] R. Thom. *Structural Stability and Morphogenesis*. Benjamin-Addison Wesley, 1975. translated by D. H. Fowler.

[19] K.L. Vincken, A.S.E. Koster, and M.A. Viergever. Probabilistic multiscale image segmentation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 19(2):109–120, 1997.

[20] A.P. Witkin. Scale-space filtering. In *Proceedings of the Eighth International Joint Conference on Artificial Intelligence*, pages 1019–1022, 1983.