# Approximations for $\lambda$-Coloring of Graphs

Hans L. Bodlaender [a], Ton Kloks [b], Richard B. Tan [a,c] and Jan van Leeuwen [a]

[a] *Department of Computer Science*
*Utrecht University*
*Padualaan 14, 3584 CH Utrecht*
*The Netherlands*
{hansb,rbtan,jan}@cs.uu.nl

[b] *Department of Mathematics and Computer Science*
*Vrije Universiteit, 1081 HV Amsterdam*
*The Netherlands*
kloks@cs.vu.nl

[c] *Department of Computer Science*
*University of Sciences & Arts of Oklahoma*
*Chickasha, OK 73018*
*U.S.A.*

**Abstract**

A $\lambda$-coloring of a graph $G$ is an assignment of colors from the integer set $\{0, \ldots, \lambda\}$ to the vertices of the graph $G$ such that vertices at distance at most two get different colors and adjacent vertices get colors which are at least two apart. The problem of finding $\lambda$-coloring with small or optimal $\lambda$ arises in the context of radio frequency assignment. We show that the problem of finding the minimum $\lambda$ for planar graphs, bipartite graphs, chordal graphs and split graphs are NP-Complete. We then give approximation algorithms for $\lambda$-coloring and compute upper bounds of the best possible $\lambda$ for outerplanar graphs, graphs of treewidth $k$, permutation and split graphs. With the exception of the split graphs, all the above bounds for $\lambda$ are linear in $\Delta$, the maximum degree of the graph. For split graphs, we give a bound of $\lambda \leq \Delta^{1.5} + 2\Delta + 2$ and show that there are split graphs with $\lambda = \Omega(\Delta^{1.5})$. We also give a bound of $\lambda = \Omega(\Delta^2)$ for bipartite graphs. Similar results are also given for variations of the $\lambda$-coloring problem.

## 1 Introduction

Radio frequency assignment is a widely studied area of research. The task is to assign radio frequencies to transmitters at different locations without causing interference. The problem is closely related to graph coloring where the vertices of a graph represent the transmitters and adjacencies indicate possible interferences.

In [21], Griggs and Yeh introduced a problem proposed by Roberts which they call the $L(2,1)$-labeling problem. It is the problem of assigning radio frequencies (integers) to transmitters such that transmitters that are close (distance 2 apart) to each other receive different frequencies and transmitters that are *very* close together (distance 1 apart) receive frequencies that are at least two apart. To keep the frequency bandwidth small, they are interested in computing the difference between the highest and lowest frequencies that have been assigned to the radio network. They call the *minimum* range of frequencies, $\lambda$. The problem is then equivalent to assigning an integer from $\{0,\ldots,\lambda\}$ to the nodes of the networks satisfying the $L(2,1)$-labeling constraint.

Subsequently, different bounds of $\lambda$ were obtained for various graphs. A common parameter used is $\Delta$, the maximum *degree* of a graph. The obvious lower bound for $\lambda$ is $\Delta+1$, achieved for the tree $K_{1,\Delta}$. In [21] it was shown that for every graph $G$, $\lambda \leq \Delta^2+2\Delta$. This upper bound was later improved to $\lambda \leq \Delta^2+\Delta$ in [10]. An interesting conjecture of Griggs and Yeh is that for every graph $G$, $\lambda \leq \Delta^2$. This remains an open problem and has been the motivation of some research since.

For some special classes of graphs, tight bounds are known and can be computed efficiently. These include paths, cycles, wheels and complete $k$-partite graphs [21], trees [10,21], cographs [10], $k$-almost trees [15], cacti, unicycles and bicycles [24], and grids, hexagonal grids and cellular grids [4]. Other types of graphs have also been studied, but only approximate bounds are known for them. These are chordal graphs and unit interval graphs [29], interval graphs [10], hypercubes [18,19,24], bipartite graphs [31], and outerplanar and planar graphs [24].

In this paper, we extend the upper bounds of $\lambda$ to other graphs and also improve some existing bounds for some classes of graphs. Precisely, new upper bounds are provided for graphs of treewidth $k$, permutation graphs and split graphs. We also improve the bound in [24] for outerplanar graphs. Efficient algorithms for labeling the graphs achieving these bounds are also given. With the exception of split graphs, all the above bounds are linear in $\Delta$. For split graphs, we give a bound of $\lambda \leq \Delta^{1.5}+2\Delta+2$ and show that there are split graphs with $\lambda=\Omega(\Delta^{1.5})$. This is the first bound for $\lambda$ that we know of that is neither linear in $\Delta$ nor $\Delta^2$. For bipartite graphs, we give a lower bound of $\Omega(\Delta^2)$ thus showing that there are graphs with $\lambda = \Theta(\Delta^2)$.

In [21], it was shown that determining $\lambda$ of a graph is an $\mathsf{NP}$-Complete problem, even for graphs with diameter two. It was further shown in [15] that it is also $\mathsf{NP}$-Complete to determine if $\lambda \leq k$ for every fixed integer $k \geq 4$ (the case when $\lambda \leq 3$ occurs only when $G$ is a disjoint union of paths of length at most 3). In this paper, we show that the problem remains $\mathsf{NP}$-Complete when restricted to planar graphs, bipartite graphs, chordal graphs and split graphs.

The $L(2,1)$-labeling problem proposed by Roberts is basically a problem of avoiding *adjacent-band interferences* – adjacent bands must have frequencies sufficiently far apart. There are several variations of the $\lambda$-coloring problems in the context of frequency assignment in multihop radio networks. Two other common type of collisions (frequencies interference) that have been studied are: direct and hidden collisions. In *direct collisions*, a radio station and its neighbors must have different frequencies, so their signals will not collide (overlap). This is just the normal vertex-coloring problem with its associated chromatic number $\mathcal{X}(G)$. In *hidden collisions*, a radio station must not receive signals of the same frequency from any of its adjacent neighbors. Thus, the only requirement here is that for each station, all its neighbors must have distinct frequencies (colors), but there is no requirement on what the color of the station itself should be.

In [3,28], the special case of avoiding *hidden collisions* in multihop radio networks were studied. We call this the $L(0,1)$-labeling problem (this notation was not used in [3,28]). In [2,11,12], the problem is to avoid both *direct* and *hidden collisions* in the radio network. Thus, a station and all of its neighbors must all have distinct colors. This is called $L(1,1)$-labeling in [31]. It is also known as distance-2 coloring problem and is equivalent to the normal coloring of the square of a graph, $G^2$, and has also been well-studied. These variations of $\lambda$-coloring are $\mathsf{NP}$-Complete even for planar graphs [3].

Perhaps a more applicative term for all these $\lambda$-variations is the one given by Harary [22]: *Radio-Coloring*. We apply our algorithms to these variations as well and obtain similar bounds.

We include a table (Figure 1) summarizing most of the known results.

The paper is organized as follows. We give some definitions of special graphs and generalizations of the $\lambda$-coloring problem in the next section. Then different upper bounds and algorithms for graphs of treewidth $k$, outerplanar graphs and permutation graphs are presented in Section 3. Section 4 contains the approximation for split graphs along with the complexity result for split graphs and chordal graphs. The complexity results for planar graphs and bipartite graphs are given in Section 5. Finally, in the last section we mention

| Type of Graphs | Bounds | Complexity |
|---|---|---|
| Paths | $\lambda = 2, 3$ or $4$ [21] | P [21] |
| Cycles | $\lambda = 4$ [21] | P [21] |
| Hexagonal Grids | $\lambda = 5$ [4] | P [4] |
| Bidimensional Grids | $\lambda = 6$ [4] | P [4] |
| Cellular Grids | $\lambda = 8$ [4] | P [4] |
| Trees | $\lambda = \Delta + 1$ or $\Delta + 2$ [21] | P [10] |
| Outerplanar | $\lambda \leq \Delta + 8$ [∗] | |
| HyperCube | $\Delta + 3 \leq \lambda \leq 2\Delta + 1$ [30,24]<br>$\lim_{\Delta \to \infty} \lambda/\Delta = 1$ [30] | |
| Strongly Chordal | $\lambda \leq 2\Delta$ [10] | |
| Permutation | $\lambda \leq 5\Delta - 2$ [∗] | |
| Planar | $\lambda \leq 2\Delta + 25$ [23] | NP-Complete [∗],[16] |
| Treewidth $\leq k$ | $\lambda \leq k\Delta + 2k$ [∗] | |
| Split | $\lambda = \Theta(\Delta^{1.5})$ [∗] | NP-Complete [∗] |
| Chordal | $\lambda \leq \frac{1}{4}(\Delta + 3)^2$ [29] | NP-Complete [∗] |
| Bipartite | $\lambda = \Theta(\Delta^2)$ [24],[∗] | NP-Complete [∗] |
| Graphs of Diameter 2 | $\lambda \leq \Delta^2$ [21] | NP-Complete [21] |
| General Graphs | $\lambda \leq \Delta^2 + \Delta$ [10] | NP-Complete [21] |
| Cographs | | P [10] |
| Graphs of Degree $\leq 3$ | | NP-Complete [15] |
| Degree $\leq 7$ & Planar | | NP-Complete [∗] |
| $\Delta$-regular | $\lambda \leq \Delta^2$ ?? | |

Fig. 1. Summary of Results ([∗] denotes *this paper*)

some open problems.

A shorter preliminary version of this paper was presented at the $17^{th}$ Annual Symposium on Theoretical Aspects of Computer Science *STACS 2000* [7]. That report contains an erroneous theorem (on planar graphs) which has been deleted from this paper.

## 2 Preliminaries

### *2.1 $\lambda$-coloring*

Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. The number of vertices in $G$ is denoted by $n$ and the maximum degree of $G$ by $\Delta$.

**Definition 1** *Let $G$ be a graph and $d_1, d_2$ be two non-negative integers. A* $\lambda$-coloring *is an assignment of colors from the integer set $\{0, \ldots, \lambda\}$ to the vertices of the graph. The $\lambda$-coloring satisfies the $L(d_1, d_2)$-constraint if each pair of vertices at distance $i, 1 \leq i \leq 2$, in the graph gets colors that differ by at least $d_i$. If a $\lambda$-coloring of $G$ satisfies the $L(d_1, d_2)$-constraint, then we say that $G$ has an $L(d_1, d_2)$-labeling. The minimum value $\lambda$ for which $G$ admits a $\lambda$-coloring satisfying the $L(d_1, d_2)$-constraint is denoted by $\lambda_{d_1,d_2}(G)$, or, when $G$ is clear from the context, by $\lambda_{d_1,d_2}$.*

In this paper, we shall focus mainly on particular $L(d_1, d_2)$-labelings which have been studied in the literature: $L(2, 1)$-labeling ([21]), $L(1, 1)$-labeling ([2,31]) and $L(0, 1)$-labeling ([3,28]).

**Fact 2** *For any graph $G$, the following lower bounds hold:*

*(1)* $\lambda_{0,1} \geq \Delta - 1$ [3],
*(2)* $\lambda_{1,1} \geq \Delta$ [31],
*(3)* $\lambda_{2,1} \geq \Delta + 1$ [21].

All these bounds are easily obtained by considering the tree $K_{1,\Delta}$, which is contained in any graph of maximum degree $\Delta$.

### *2.2 Special Graphs*

**Definition 3** *Let $k$ be a positive integer. A $k$-*tree *is a graph of $n \geq k + 1$ vertices defined recursively as follows. A clique of $k + 1$ vertices is a $k$-tree. A $k$-tree with $n + 1$ vertices can be formed from a $k$-tree with $n$ vertices by making a new vertex adjacent to exactly all vertices of a $k$-clique in the $k$-tree with $n$ vertices.*

**Definition 4** *A graph is a* partial *$k$-tree if it is a subgraph of a $k$-tree.*

**Definition 5** *The* treewidth *of a graph is the minimum value $k$ for which the graph is a subgraph of a $k$-tree.*

A useful way of dealing with the treewidth of a graph is via its tree-decomposition.

**Definition 6** *A* tree decomposition *of a graph* $G = (V, E)$ *is a pair* $(\{X_i \mid i \in I\}, T = (I, F))$ *with* $\{X_i \mid i \in I\}$ *a collection of subsets of* $V$*, and* $T = (I, F)$ *a tree, such that*

- $\bigcup_{i \in I} X_i = V$
- *for all edges* $(v, w) \in E$ *there is an* $i \in I$ *with* $v, w \in X_i$
- *for all* $i, j, k \in I$*: if* $j$ *is on the path from* $i$ *to* $k$ *in* $T$*, then* $X_i \cap X_k \subseteq X_j$.

*The* width *of a tree decomposition* $(\{X_i \mid i \in I\}, T = (I, F))$ *is* $\max_{i \in I} |X_i| - 1$. *The* treewidth *of a graph* $G = (V, E)$ *is the minimum width over all tree decompositions of* $G$.

It can be shown that the above definitions of treewidth are equivalent and that every graph with treewidth $\leq k$ is a partial $k$-tree and conversely, that every partial $k$-tree has treewidth $\leq k$. For more details on treewidth, $k$-trees and other equivalent definitions, consult, for example, [6,26].

We now define a few more special graphs. Other definitions and results concerning these special graphs can be found in [9,20].

**Definition 7** *A graph is* chordal *or* triangulated *iff every cycle of length* $\geq 4$ *has a chord (i.e., there is no induced cycle of length* $\geq 4$*).*

**Definition 8** *A vertex of a graph* $G$ *is* simplicial *if its neighbors induce a clique.*

**Definition 9** *A* perfect elimination scheme *for a graph* $G$ *is an ordering of vertices* $(v_1, \ldots, v_n)$*, such that for each* $i$*,* $1 \leq i \leq n$*, the vertex* $v_i$ *is a simplicial vertex in the subgraph induced by* $(v_{i+1}, \ldots, v_n)$.

**Fact 10** [14] *The following statements are equivalent:*

*(1)* $G$ *is chordal.*
*(2)* $G$ *has a perfect elimination scheme.*
*(3)* $G$ *is an* intersection model of subtrees of a tree*, i.e. there is a tree* $T$*, such that for every vertex* $v \in V$ *one can associate a subtree* $T_v$ *of* $T$ *with* $\{v, w\} \in E \iff T_v \cap T_w \neq \emptyset$*, for all* $v, w \in V, v \neq w$.

**Definition 11** *A* split graph *is a graph* $G$ *of which the vertex set can be split into two sets* $K$ *and* $S$*, such that* $K$ *induces a clique and* $S$ *induces an independent set in* $G$.

A *permutation graph* can be obtained from a permutation $\pi = \{\pi_1, \ldots, \pi_n\}$ of integers from 1 to $n$ in the following visual manner. Line up the numbers 1 to $n$ horizontally on a line. On the line below it, line up the corresponding permutation so that $\pi_i$ is right below $i$. Now connect each $i$ and $\pi_j$ such that

$\pi_j = i$ with an edge. Such edges are called *matching edges* and the resulting diagram is referred to as a *matching diagram*. The *inversion graph* is the graph $G_\pi = (V, E)$ with $V = \{1, \ldots, n\}$ and $\{i, j\} \in E$ iff the matching lines of $i$ and $j$ in the matching diagram intersect. Formally, one can define a permutation graph as follows.

**Definition 12** *Let $\pi = \{\pi_1, \ldots, \pi_n\}$ be a permutation of integers from 1 to $n$. Then the permutation graph determined by $\pi$ is the graph $G_\pi = (V, E)$ with $V = \{1, \ldots, n\}$ and $\{i, j\} \in E$ iff $(i - j)(\pi_i^{-1} - \pi_j^{-1}) < 0$, where $\pi_i^{-1}$ is the* inverse *of $\pi_i$ (i.e., the position of the number $i$ in the sequence $\pi$). A graph $G$ is a* permutation graph *if there exists a permutation $\pi$ such that $G$ is isomorphic to the inversion graph $G_\pi$.*

## 3 Bounds and Algorithms

Following [21,24,29,31], we use the following heuristic, or small modifications of it, to $\lambda$-color a graph $G$. First we find an *elimination sequence*, an ordering of the vertices, $(v_1, \ldots, v_n)$, satisfying certain conditions for $G$. In order to do this, we rely on the fact that all the graphs considered have the *hereditary* property, i.e., when a special vertex is eliminated from a graph considered, the induced subgraph remains the same type of graph. Then we simply apply the *greedy* algorithm to color each vertex in the sequence by using the smallest available color in $\{0, \ldots, \lambda\}$, satisfying the $L(d_1, d_2)$-constraint. For each graph $G$, we next estimate the total number of vertices at distance two that a vertex can have among the vertices that have been colored so far. Finally we compute the upper bound for $\lambda$.

### *3.1 Graphs of treewidth k*

For a graph $G = (V, E)$ of treewidth $k$, we first take a tree-decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ with $\{X_i \mid i \in I\}$ a collection of subsets of $V$, and $T = (I, F)$ a tree. Let $d(v, w)$ be the distance between vertices $v$ and $w$.

**Algorithm 1** *Algorithm for Graphs of Treewidth k*

1: Add *a set of virtual edges $E'$:*
   *Here $\{v, w\} \in E'$ iff $\{v, w\} \notin E$ and $\exists i : v, w \in X_i$.*

2: Find *a Perfect Elimination Sequence:* $\{v_1, \ldots, v_n\}$ in $G' = (V, E \cup E')$.

3: **For** $i := n$ **to** 1

Label $v_i$ *with the smallest available color, such that for all already colored vertices* $w$*:*

If $(v, w) \in E$, then *the color of* $v$ *and* $w$ *differ by at least 2,*
If $(v, w) \in E'$, then *the color of* $v$ *and* $w$ *differ by at least 1,*
If $d(v, w) = 2$, then *the color of* $v$ *and* $w$ *differ by at least 1.*

**Theorem 13** *There is an algorithm for labeling a graph* $G$ *of treewidth* $k$ *with* $\lambda_{2,1} \leq k\Delta + 2k$ *.*

**PROOF.** After all the virtual edges are added in the above algorithm, the new graph $G' = (V, E \cup E')$ is chordal, hence has a perfect elimination sequence (from Fact 10). Also, $G'$ has the same tree decomposition as $G$ so has treewidth at most $k$.

We proceed by induction. The first vertex $v_n$ can simply be colored with color 0. Suppose we have colored the vertices in the elimination sequence $(v_{i+1}, \ldots, v_n)$, $i < n$. When we are ready to color the vertex $v_i$, it has at most $k$ colored neighbors because $v_i$ with its neighbors forms a clique in $G'$. Now by the perfect elimination sequence property, an already colored vertex at distance two of $v_i$ must be adjacent to an already colored neighbor of $v_i$ in $G'$ also. Hence, at most $3k$ colors are unavailable due to the neighbors of $v_i$ since each of these neighbors can account for at most 3 more colors: if we color one of these vertices by color $c$, then colors $c - 1$ and $c + 1$ are forbidden for $v_i$. Now $v_i$ has at most $(\Delta - 1)k$ colors unavailable due to the vertices at distance two. If we have $k\Delta + 2k + 1$ colors, then a color for $v_i$ is always available. The bound now follows. $\square$

**Corollary 14** *There is an algorithm for labeling a* $k$*-tree with* $\lambda_{2,1} \leq k\Delta - k^2 + 3k$*.*

**PROOF.** As a $k$-tree is always triangulated, there are at most $k(\Delta - 1 - (k - 1)) = k\Delta - k^2$ distance-2 neighbors of $v_i$. The total number of colors needed is then $k\Delta - k^2 + 3k + 1$ or $\lambda_{2,1} \leq k\Delta - k^2 + 3k$. $\square$

**Theorem 15** *For graphs of treewidth* $k$*,* $\lambda_{0,1} \leq k\Delta - k$ *and* $\lambda_{1,1} \leq k\Delta$ *.*

**PROOF.** We apply the same algorithm as in Algorithm 1 to find an eliminating sequence $(v_n, \ldots, v_1)$ and then greedily color each $v_i$ in the sequence with the smallest available color satisfying the $L(0, 1)$-constraint. As in the proof of Theorem 13, each $v_i$ in the eliminating sequence has at most $k\Delta - k$ neighbors at distance two, which must have different colors from $v_i$. Now $v_i$ can have at most $k$ colored neighbors, but they can have the same color as $v_i$. So we can color $v_i$ and its neighbors with only one extra color. The bound for $\lambda_{0,1}$ now follows.

Similar argument applies for $\lambda_{1,1}$. $\square$

The labeling algorithms given in this section can be implemented in time $O(kn\Delta)$, $n$ the number of vertices, assuming that we are given the tree-decomposition, which can be found in linear time for treewidth of constant size $k$ [5]. Step 1 of the algorithm can be done as follows: first, just list all pairs $\{v, w\}$ such that $\exists i : v, w \in X_i$. Then, radix sort the pairs $\{v, w\}$ of this type together with the edges $e \in E$. This can be done in linear time (see e.g. [13], chapter 9.) Given this sorted sequence, pairs $\{v, w\}$ such that $\exists i : v, w \in X_i$ and $\{v, w\} \in E$ can easily be filtered out. Step 2 can be done in linear time with standard methods ($G'$ is chordal; see e.g. [20].) Step 3 is not hard to implement, such that coloring a vertex can be done in $O(k\Delta)$ time. Each vertex maintains the set of colors, given to its neighbors; when a vertex is colored, these sets of its neighbors are updated. When a vertex must be colored, these color sets of its neighbors and the colors given to the neighbors are inspected. One thus gets a multiset of $O(k\Delta)$ elements in $\{0, \cdots, k\Delta - k^2 + 3k\}$ and finding the smallest non-negative integer not in this multiset can easily be done in $O(k\Delta)$ time.

Recently, in [32] it has been shown that $\lambda_{1,1}$ can be computed in polynomial time for graphs with constant treewidth $k$. A similar argument would yield the result for $\lambda_{0,1}$ as well.

### *3.2 Outerplanar Graphs*

We now specialize to outerplanar graphs. As outerplanar graphs are graphs of treewidth two, we have $\lambda_{2,1} \leq 2\Delta + 4$, from Theorem 13. Jonas in [24] already has a slightly better bound of $\lambda_{2,1} \leq 2\Delta + 2$. It turns out we can improve this bound even further. First we need a technical lemma concerning outerplanar graphs in general.

**Lemma 16** *In an outerplanar graph, there exists a vertex of degree at most two which has a neighbor of degree at most four.*

**PROOF.** If $G$ has at most two vertices, the Lemma trivially holds. We now assume $G$ has at least three vertices. First, we claim that a biconnected outerplanar graph $G$ (with at least three vertices) has a vertex of degree two with a neighbor of degree at most four.

Suppose not. Then let $G$ be a biconnected outerplanar graph with at least three vertices such that all neighbors of a vertex of degree at most two have degree five or more. We consider the *inner dual* $G^*$ of graph $G$, formed by taking the dual of $G$ and then removing the vertex that represents the outer region. It is easy to see that $G^*$ is a tree [24]. Note that all leaves in $G^*$ correspond to a face with at least one vertex of degree two in $G$, as at most one edge of the face is shared by another inner-face, hence two adjacent edges belong to the face and the outer region, so that the vertex shared by the edges has degree two. Consider now the two leaves, $u^*$ and $v^*$, of the inner dual with maximum distance in $G^*$. Suppose to the contrary that all neighbors of a vertex of degree $\leq 2$ have more than four neighbors in $G$. The face represented by $u^*$ has a vertex $x$ of degree two in $G$, and both neighbors $y$ and $z$ have degree more than four. See Figure 2. Thus, there are at least four faces adjacent to $y$, and these form a path of length at least three starting from $u^*$ in $G^*$. Similarly, there is a different path of length three starting from $u^*$ in $G^*$ for the faces that contain $z$. One can now observe that there is a path in $G^*$ of greater length than the path from $u^*$ to $v^*$, say from $w^*$ or $w^{'*}$, which is a contradiction. Hence $G$ has a vertex of degree two that has a neighbor of degree at most four.

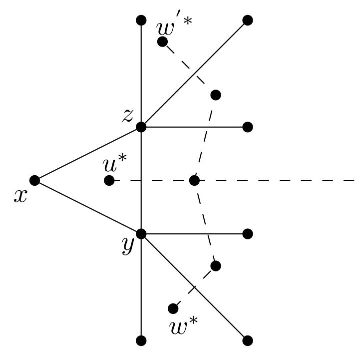


Fig. 2. $u^*$ cannot be endpoint of a path of maximum distance in $G^*$.

Finally, every outerplanar graph $G = (V, E)$ is a subgraph of a biconnected outerplanar graph $H = (V, F)$ [25]. A vertex $v$, that has degree at most two and has a neighbor of degree at most four in $H$, has degree at most one in $G$ or has degree two and a neighbor of degree at most four in $G$. □

**Algorithm 2** *Algorithm for Outerplanar Graphs*

**For** $i := 1$ **to** $n$
  Find *a vertex* $v_i$ *of degree* $\leq 2$ *with a neighbor of degree at most 4 or of degree* $\leq 1$*.*
  **If** *neighbors of* $v_i$ *are not adjacent*
    **Then** add *a* virtual *edge between them*
  Temporarily *remove* $v_i$ *from* $G$*.*

**For** $i := n$ **to** 1
  Label $v_i$ *with the* smallest *available color in* $\{0, \ldots, \lambda\}$
    *satisfying the* $L(2, 1)$-constraint
  **If** *neighbors of* $v_i$ *have a virtual edge*
    **Then** remove *the edge*

In Algorithm 2, we first find an elimination sequence, $(v_1, \dots, v_n)$, using the condition in Lemma 16. Note that this can always be done due to the hereditary property of outerplanarity. Then, we use this order to color the vertices in a greedy manner, making sure that the color of vertex $v_i$ differs by at least two from the colors of its already colored neighbors, and differs by at least one from the colors already colored vertices at distance two. In this way, we make sure that the $L(2, 1)$-constraint is fulfilled. The virtual edge (if there is one) is there to guarantee that the colors of $v_i$'s two neighbors are different when they are colored. Note that an operation that removes $v_i$ and adds a virtual edge between its two neighbors does not increase the maximum degree $\Delta$.

We can now compute the value of the maximum color used in Algorithm 2.

**Theorem 17** *There is an algorithm for finding an $L(2, 1)$-labeling of an outerplanar graph with $\lambda_{2,1} \leq \Delta + 8$.*

**PROOF.** Again we use induction. We color the first vertex $v_n$ in the elimination sequence $(v_n, \dots, v_1)$ with color 0. and suppose we have colored the vertices in the elimination sequence $(v_{i+1}, \dots, v_n)$, $i < n$. When we want to color the vertex $v_i$, it can have at most two colored neighbors. First, suppose $v_i$ has two colored neighbors. As before, each of these two neighbors can account for at most 3 more colors. Now $v_i$ has at most $\Delta - 1 + 3$ vertices at distance two, which means another $\Delta + 2$ colors that possibly cannot be used for $v_i$. If there are at least $\Delta + 9$ colors, then there is always at least one color available for $v$, i.e., $\lambda_{2,1} \leq \Delta + 8$. A similar analysis can be used if $v_i$ has one colored neighbor. □

**Corollary 18** *There is an algorithm for finding an $L(2, 1)$-labeling of a triangulated outerplanar graph with $\lambda_{2,1} \leq \Delta + 6$.*

**PROOF.** In a triangulated graph, there are at most $\Delta - 2 + 2 = \Delta$ distance-2 neighbors of $v_i$. The total number of colors needed is then $\Delta + 7$ or $\lambda_{2,1} \leq \Delta + 6$. □

This improves the bound of $\lambda_{2,1} \leq 2\Delta + 2$ in [24] for outerplanar graphs, for sufficiently large $\Delta$.

We can apply our algorithm of outerplanar graphs to the other $\lambda$-variants. By doing a simple count, we easily obtain the following results.

**Theorem 19** *For outerplanar graphs, there are polynomial time algorithms for labeling the graphs such that $\lambda_{0,1} \leq \Delta + 2$ and $\lambda_{1,1} \leq \Delta + 4$.*

It is not hard to see that the algorithms mentioned above can be implemented to run in time $O(n\Delta)$, $n$ the number of vertices.

**Theorem 20** *There is a polynomial time algorithm for labeling a permutation graph with* $\lambda_{2,1} \leq 5\Delta - 2$ *.*

**PROOF.** Suppose the vertices are numbered $1, 2, \ldots, n$, and we have a permutation $\pi$, with $(i, j) \in E$ iff $(i-j)(\pi_i^{-1} - \pi_j^{-1}) < 0$ (i.e., the lines cross). We color the vertices from 1 to $n$ in order, using the smallest color available satisfying the usual $L(2,1)$-constraint. To show that the stated bound is sufficient, we make use of the following two claims.

Suppose we are in the midst of this algorithm, ready to color a vertex $v$. Let $w$ be a vertex at distance two from $v$. Note that $v$ and $w$ can have distance two via a path across a colored vertex or via a path across an uncolored vertex.

**Claim 21** *Suppose there is a path* $(v, y, w)$ *with* $y$ *a vertex that is not yet colored. Let* $x$ *be the neighbor of* $v$ *such that* $\pi_x^{-1}$ *is minimal. Then either* $w$ *is a neighbor of* $v$ *or* $x$*, or* $w$ *is not colored.*

**PROOF.** Suppose $w$ is not a neighbor of $v$ (i.e. $\{v, w\} \notin E$) and $w$ is already colored. then we have $w < v$ and $\pi_w^{-1} < \pi_v^{-1}$. Informally, the matching edges in the matching diagram of $w$ and $v$ should not cross, while the matching edges of $v$ and $y$, and also the matching edges of $w$ and $y$ both cross. Given the order of the top endpoints, $w < v < y$, the situation shown in Figure 3 is the only one possible. Now, by assumption, $\pi_x^{-1} < \pi_y^{-1}$, i.e. the lower endpoint of the matching edge of $x$ is to the left of $\pi_y^{-1}$, hence if this matching edge crosses the matching edge of $v$, it also crosses the corresponding matching edge of $w$; thus $\{x, w\} \in E$.

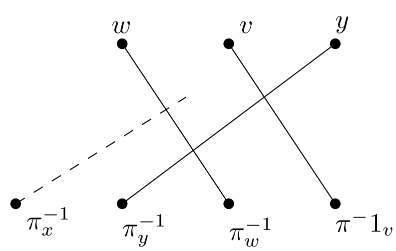


Fig. 3. Matching diagram for vertices in permutation graph in the proof of Claim 21.

Formally, $\{v, w\} \in E \Rightarrow \pi_y^{-1} < \pi_v^{-1}$. Also $y > v > w$, so $\{y, w\}$ is an edge and $\pi_y^{-1} < \pi_w^{-1}$. By assumption, $\pi_x^{-1} \leq \pi_y^{-1}$, $\pi_x^{-1} < \pi_v^{-1}$. Now $x > v > w$, so $\pi_w^{-1} > \pi_y^{-1} \geq \pi_x^{-1}$. Therefore $\{w, x\}$ is an edge. $\square$

The proof of the next claim is similar to the proof above.

**Claim 22** *Suppose there is a path* $(v, y, w)$ *with* $y$ *a vertex that is already* *colored. Let $x$ be the neighbor of $v$ having a minimal $\pi_x^{-1}$ among the neighbors. Then either $w$ is a neighbor of $v$ or $x$, or $w$ is not colored.*

We now count the total number of colors needed. The vertices $x$ in the above two claims are adjacent to $v$, so they both have at most $\Delta - 1$ neighbors that are at distance two to $v$. We thus get a total of $3\Delta + 2(\Delta - 1) + 1 = 5\Delta - 1$ colors. □

**Theorem 23** *For permutation graphs, there are polynomial time algorithms for labeling the graphs such that $\lambda_{0,1} \leq 2\Delta - 2$ and $\lambda_{1,1} \leq 3\Delta - 2$.*

All the above algorithms can also be implemented in $O(n\Delta)$ time.

### *3.4 Bipartite graphs*

Bipartite graphs may require $\lambda = \Omega(\Delta^2)$, as is shown in the following lemma using a well known technique. Of course, $\Delta^2 + \Delta$ is also for bipartite graphs an upper bound on $\lambda$.

**Theorem 24** *For every $\Delta \geq 2$, there is a bipartite graph with maximum degree $\Delta$, with $\lambda_{0,1} \geq (\lfloor \Delta/2 \rfloor + 1) \cdot \lceil \Delta/2 \rceil \geq \Delta^2/4$.*

**PROOF.** Let $\Delta_1 = \lfloor \Delta/2 \rfloor + 1$, and $\Delta_2 = \lceil \Delta/2 \rceil$. Take a graph $G$ with vertices $\{v_{i,j} \mid 1 \leq i \leq \Delta_1, 1 \leq j \leq \Delta_2\} \cup \{w_{i,j} \mid 1 \leq i \leq \Delta_1, 1 \leq j \leq \Delta_2\}$, and take edges $\{v_{i,j}, w_{i',j'}\}$ for all pairs with $i = i'$ or $j = j'$. Note that the maximum degree of $G$ is $\Delta$. Also, for every pair of distinct vertices $v_{i,j}$, $v_{i',j'}$, their distance is two, hence all $\Delta_1 \cdot \Delta_2$ vertices of the form $v_{i,j}$ must have a different color in any $\lambda$-coloring of $G$. □

Note that the proof also holds when we consider $\lambda_{1,1}$-colorings and $\lambda_{2,1}$ colorings.

## 4 Split Graphs

So far all the bounds for $\lambda$ that we have obtained are linear or quadratic in $\Delta$. For split graphs we give a non-linear and non-quadratic bound for $\lambda$ and show that there are split graphs that require this bound.

**Theorem 25** *There is a polynomial time algorithm for labeling a split graph with $\lambda_{2,1} \leq \Delta^{1.5} + 2\Delta + 2$ .*

**PROOF.** Let $S$ be the independent set and $K$ the clique that split $G$. Note that $|K| \leq \Delta + 1$. We use colors $0, 2, \ldots, 2\Delta$ to color the vertices in $K$. For $S$, we will use colors from the set $\{2\Delta + 2, 2\Delta + 3, \ldots, \Delta^{1.5} + 2\Delta + 2\}$. If $|S| \leq \Delta^{1.5}$, we just give every vertex in $S$ a distinct color and we are done. Suppose $|S| > \Delta^{1.5}$. We claim that there is always a vertex $v$ in $S$ with degree $\leq \Delta^{0.5}$. Suppose we have $r, 1 \leq r \leq \Delta + 1$ vertices in $K$. Each can have at most $\Delta - (r-1)$ neighbors in $S$. Hence, the total number of edges emanating from $S$ is at most $r(\Delta - r + 1)$. The minimum degree vertex in $S$ is thus at most

$$\frac{r(\Delta - r + 1)}{\Delta^{1.5} + 1} \leq \frac{(\frac{\Delta+1}{2})^2}{\Delta^{1.5} + 1} \leq \Delta^{0.5}.$$

Now, let $v$ be a vertex in $S$ with degree $\leq \Delta^{0.5}$. Recursively color the graph obtained by removing $v$ from $G$. As $v$ has at most $\Delta^{1.5}$ vertices in $S$ at distance two, and neighbors of $v$ of distance one are already colored, with the available $\Delta^{1.5} + 1$ colors to color $S$, we always have a color left for $v$. Finally, note that adding $v$ back cannot decrease the distances between other vertices in $G$ as the neighborhood of $v$ is a clique. □

**Theorem 26** *For split graphs, there are polynomial time algorithms for labeling the graphs such that* $\lambda_{0,1} \leq \Delta^{1.5}$ *and* $\lambda_{1,1} \leq \Delta^{1.5} + \Delta + 1$.

We now show that the above bounds for $\lambda$ is actually tight (within constant factor).

**Theorem 27** *For every* $\Delta$*, there is a split graph with* $\lambda_{2,1} \geq \lambda_{1,1} \geq \lambda_{0,1} \geq \frac{1}{3}\sqrt{\frac{2}{3}}\Delta^{1.5}$.

**PROOF.** Consider the following split graph. We take an independent set of $\frac{1}{3}\sqrt{\frac{2}{3}}\Delta^{1.5}$ vertices. This set is partitioned into $\sqrt{\frac{2}{3}\Delta}$ groups, each consisting of $\Delta/3$ vertices. The clique consists of $\Delta/3 + 1$ vertices. Note that we have less than $(\sqrt{\frac{2}{3}\Delta})^2/2 = \Delta/3$ distinct pairs of groups. For each such pair of groups, we take one unique vertex in the clique, and make that vertex adjacent to each vertex in these two groups. In this way, the maximum degree is exactly $\Delta$: each vertex in the clique is adjacent to $\Delta/3$ vertices in the clique and at most $2\Delta/3$ vertices in the independent set.

Now, the resulting graph has diameter two, and any pair of vertices in the independent set have distance exactly two. So, in any $L(0,1)$-labeling of the graph (or $L(1,1)$- or $L(2,1)$-labeling), all vertices in the independent set must receive different colors. □

As split graphs are also chordal graphs [20], the above theorem provides a non-linear lower bound for the upper bound of $\frac{3}{4}\Delta^2$ in [29] for chordal graphs.

We now show the NP-completeness result for split graphs. The proof below is a modification of that by Griggs and Yeh [21].

**Theorem 28** *Let $\mathcal{G}$ be a class of graphs, such that*

*(1) If $G = (V, E) \in \mathcal{G}$, then the graph obtained by adding a new vertex $v$ and making it adjacent to every vertex in $V$ also belongs to $\mathcal{G}$.*
*(2)* HAMILTONIAN PATH *is* NP*-complete for graphs whose complement belongs to $\mathcal{G}$.*

*Then, the problem to decide for a given graph $G = (V, E) \in \mathcal{G}$ whether $\lambda_{2,1}(G) \leq |V|$ is* NP*-complete.*

**PROOF.** Clearly the problem belongs to NP.

In order to show NP-hardness, we transform from HAMILTONIAN PATH for graphs in the complement of $\mathcal{G}$. Let a graph $G = (V, E)$ in the complement of $\mathcal{G}$ be given. Assume $G'$ is the graph obtained by taking the complement of $G$ and then adding a new vertex $v_0$, and making $v_0$ adjacent to every vertex in $V$, i.e., $G' = (V \cup \{v_0\}, E \cup \{\{v_0, x\} \mid x \in V\})$. Write $n = |V| + 1$.

By the first assumption, $G' \in \mathcal{G}$.

**Claim 29** *$\lambda_{2,1}(G') \leq n$, if and only if $G$ has a Hamiltonian path.*

**PROOF.** Suppose $x_1, x_2, \ldots, x_{n-1}$ forms a Hamiltonian path in $G$. Then if we color $v_0$ with 0 and $x_i$ with $i+1$, $1 \leq i \leq n-1$, we obtain a lambda-coloring of $G'$.

Assume we have a lambda-coloring $f : V \cup \{v_0\} \to \{0, 1, \ldots, n\}$ of $G'$. As every vertex is adjacent to $v_0$, we must have that all vertices have a different color. As colors $f(v_0) - 1$ and $f(v_0) + 1$ cannot be used, we must have that $f(v_0) = 0$ or $f(v_0) = n$, otherwise we have $n - 2$ colors left for $n - 1$ vertices.

Without loss of generality, suppose $f(v_0) = 0$. So, every vertex in $V$ receives a distinct color from the set $\{2, 3, \ldots, n\}$. If $f(w) = i$ and $f(x) = i + 1$, then $w$ and $x$ are not adjacent in $G'$, hence $\{w, x\} \in E$. So $f^{-1}(2), f^{-1}(3), \ldots, f^{-1}(n-1), f^{-1}(n)$ forms a Hamiltonian path in $G$. □

NP-hardness of the lambda-coloring problem for $\mathcal{G}$ now follows. □

**Corollary 30** *The following problem is* NP*-Complete:*

Instance*: A split graph $G = (V, E)$.*
Question*: Is $\lambda_{2,1} \leq |V|$?*

**PROOF.** HAMILTONIAN PATH is NP-complete for split graphs, and hence, as the complement of a split graph is also a split graph, for complements of split graphs. Also, when one adds a vertex that is adjacent to all vertices to a split graph, one obtains again a split graph. So Theorem 28 can be applied. □

As split graphs are chordal, we also have the following NP-Complete problem.

*Instance*: A chordal graph $G = (V, E)$.
*Question*: Is $\lambda_{2,1} \leq |V|$?

**Theorem 31** *The following problems are* NP*-Complete:*

Instance*: A split graph* $G = (V, E)$*.*
Question*: Is* $\lambda_{0,1} \leq 3$*?*

Instance*: A split graph* $G = (V, E)$ *and an integer* $r$*.*
Question*: Is* $\lambda_{1,1} \leq r$*?*

**PROOF.** (i) $\lambda_{0,1}$: We transform from 3-coloring. Let $G = (V_G, E_G)$ be an undirected graph. We take a split graph $H$ as follows. $H$ has three types of vertices: for each vertex $v \in V_G$, we take also a vertex $v$ in $H$; for each edge $e \in E_G$, we take a vertex $e$ in $H$, and then we add one additional new vertex $x_0$ to $H$. We turn $E_G \cup \{x_0\}$ into a clique, and add edges $\{v, e\}$ to $H$ whenever $e$ is an edge with $v$ as endpoint in $G$. Now we claim that $\lambda_{0,1}(H) \leq 4$, if and only if the chromatic number of $G$ is at most three. Suppose $c : V_G \rightarrow \{1, 2, 3\}$ is a 3-coloring of $G$. Then, the function that colors every vertex in $E_G \cup \{x_0\}$ with color 0, and every vertex in $V_G$ with $c(v)$ is a $\lambda_{0,1}$-coloring of $H$. Conversely, suppose we have a $\lambda_{0,1}$-coloring $c$ of $G$ that uses the colors $\{0, 1, 2, 3\}$. Without loss of generality, suppose $c(x) = 0$. Then all vertices in $V_G$ have color 1, 2, or 3. If $\{v, w\} \in E_G$, then $v$ and $w$ have distance two in $H$, hence $c(v) \neq c(w)$, so $G$ is 3-colorable. This shows the result.

(ii) $\lambda_{1,1}$: We can use the same construction as in part (i), but without the vertex $x_0$, and with $r = |E_G| + 2$. If $G$ is 3-colorable, then we can construct a $\lambda_{1,1}$-coloring of $H$, by using three colors for the vertices in $V$, and giving each vertex in $E$ a different color. Conversely, note that in any $\lambda_{1,1}$-coloring of $H$, all vertices in $E_G$ must have a different color, and as every vertex in $V_G$ has distance one or two to each vertex in $E_G$, all vertices in $G$ are differently colored from the vertices in $E_G$. So, if we have a $\lambda_{1,1}$-coloring $c$ of $H$ with $|E_G| + 3$ colors, then $c$, restricted to $V$ gives a 3-coloring of $G$. This proves part (ii). □

Again, this also implies NP-completeness of the problems to decide $\lambda_{0,1}$ and $\lambda_{1,1}$ for chordal graphs.

## 5 Complexity of $\lambda$-Coloring Bipartite Planar Graphs

In this section we will show that it is $\mathsf{NP}$-complete to decide whether $\lambda_{2,1} \leq 8$ for a given bipartite planar graph $G = (V, E)$. Independently, an NP-completeness proof for $\lambda$-coloring of planar graphs has been given in [16].

We say a graph $G = (V, E)$ is 3-colorable, if and only if there is a function $c : V \rightarrow \{1, 2, 3\}$ such that for all edges $\{v, w\} \in E$: $c(v) \neq c(w)$. An edge 4-coloring of a graph $G = (V, E)$ is a function $f : E \rightarrow \{1, 2, 3, 4\}$ such that for all edges $e, e' \in E, e \neq e'$, if $e$ and $e'$ share a common endpoint, then $f(e) \neq f(e')$.

**Theorem 32 (Garey, Johnson, Stockmeyer [17])** *The following problem is* $\mathsf{NP}$*-Complete.*

Instance*: A planar graph $G = (V, E)$ of maximal degree four.*
Question*: Is $G$ 3-colorable?*

We first show that the following problem (though somewhat contrived is needed in a later reduction) is $\mathsf{NP}$-Complete.

[3-Coloring for planar graphs with given 4-edge coloring]
*Instance*: A planar graph $G = (V, E)$ and a 4-edge coloring $f$ of $G$.
*Question*: Is $G$ 3-colorable?

**Lemma 33** 3-Coloring for planar graphs with given 4-edge coloring *is* $\mathsf{NP}$*-complete.*

**PROOF.** Clearly, the problem is in $\mathsf{NP}$. To show $\mathsf{NP}$-hardness, we use a technique from Garey, Johnson and Stockmeyer [17].

We use a transformation from the 3-coloring problem for planar graphs with maximum degree four. Suppose we have a planar graph $G = (V, E)$ with maximum degree four. Now, replace every vertex $v \in V$ by a copy of the subgraph $S$, shown in Figure 4.

The subgraph $S$ is given together with a 4-edge coloring, which will be used below. There are four marked vertices (with an *) in this subgraph $S$. Each edge to $v$ in $G$ now goes to one of the marked vertices, such that each marked vertex gets at most one of these edges, and the graph stays planar. This reduction is almost identical to the reduction used in [17] to show $\mathsf{NP}$-hardness of 3-coloring of planar graphs with maximum degree four. Let $G' = (V', E')$ be the resulting graph. One can construct a 4-edge coloring of $G'$ by coloring the edges in the replacement subgraph $S$ as in Figure 4, and coloring the original edges of $G$ (the dotted lines in Figure 4) as follows: if $e$ is an edge between two

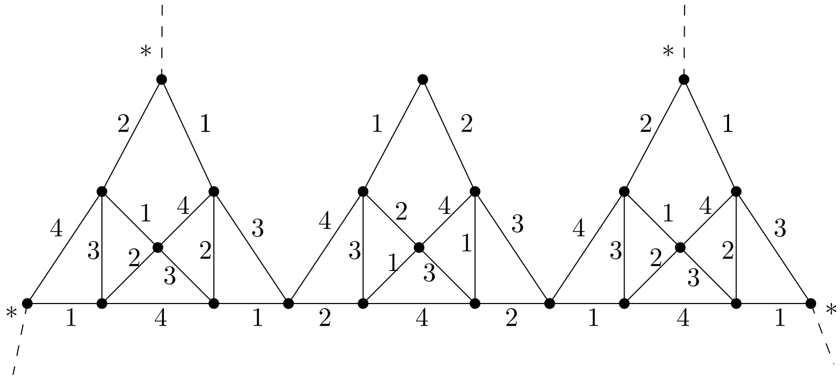


Fig. 4. Subgraph $S$ that replaces a vertex.

subgraphs, then note that both endpoints have degree three and one edge is already colored with 1, hence there is at least one color from $\{2, 3, 4\}$ left for the edge. So, $G'$ with a 4-edge coloring of $G'$ can be constructed in polynomial time.

Finally $G$ is 3-colorable, if and only if $G'$ is 3-colorable. This follows directly from the proof in [17]. $\square$

**Theorem 34** *The following problem in* NP*-Complete.*

Instance*: A planar bipartite graph* $G = (V, E)$ *of maximal degree seven.*
Question*: Is* $\lambda_{2,1}(G) \leq 8$*?*

**PROOF.** Again, clearly, the problem is in NP. To show NP-hardness, we use a transformation from 3-COLORING OF PLANAR GRAPHS WITH A GIVEN 4-EDGE COLORING.

Suppose we are given a planar graph $G = (V, E)$ with a 4-edge coloring $f : E \to \{1, 2, 3, 4\}$.

We now transform $G$ in a number of steps.

First, we subdivide every edge of $G$. We distinguish five types of vertices:

- original vertices: vertices belonging to $V$.
- 0-vertices: vertices that resulted from subdividing an edge $e$ with $f(e) = 1$.
- 1-vertices: vertices that resulted from subdividing an edge $e$ with $f(e) = 2$.
- 7-vertices: vertices that resulted from subdividing an edge $e$ with $f(e) = 3$.
- 8-vertices: vertices that resulted from subdividing an edge $e$ with $f(e) = 4$.

Now, if an original vertex has degree less than four, it receives new neighbors: if original vertex $v$ has no neighbor that is a 0-vertex, we add one new vertex that is only adjacent to $v$. The vertex is called an *extra 0-vertex*. Similarly, original vertices that have no neighbor that is a 1-vertex, 7-vertex, or 8-vertex get a new neighbor of degree one that is an extra 1-vertex, extra 7-vertex, or extra 8-vertex.

In the next few steps, additional edges and vertices are added to the graph.

To every 0-vertex and 8-vertex, we add five new neighbors of degree one each, as in Figure 5.

To every extra 0-vertex and to 8-vertex, we add six new neighbors of degree one each, as in Figure 6.

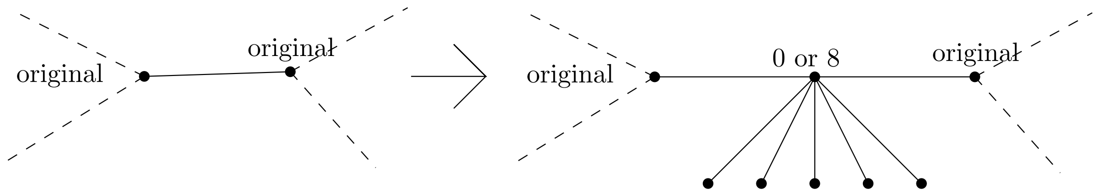


Fig. 5. Replacing an edge by a subdivision with a 0- or 8-vertex

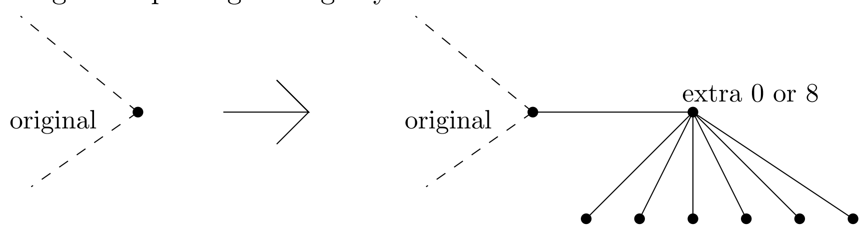


Fig. 6. Adding an extra 0-vertex or extra 8-vertex

To every 1-vertex and 7-vertex, we add two subtrees of a specific form, as shown in Figure 7. To every extra 1-vertex and extra 7-vertex we add three of these subtrees, as in Figure 8.

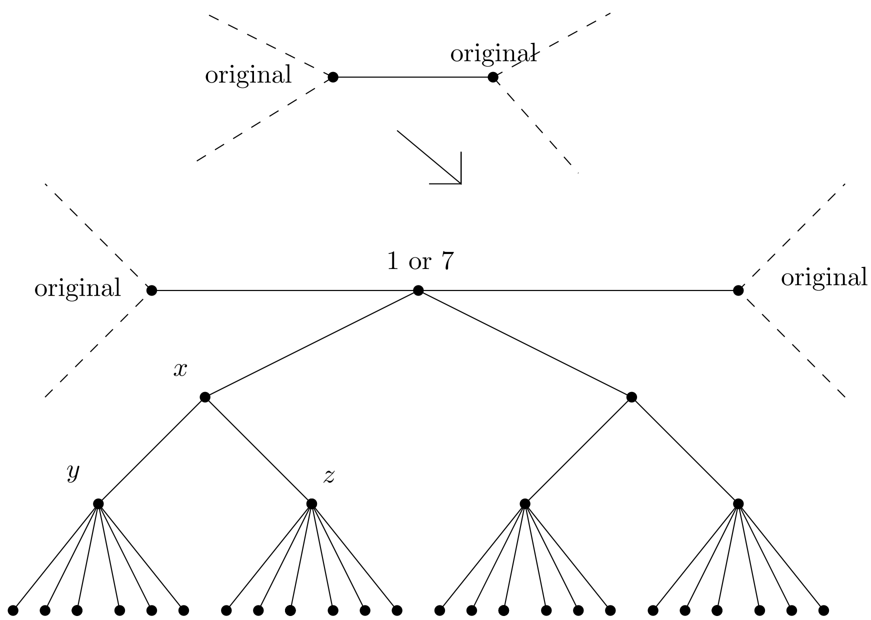


Fig. 7. Adding a subgraph to 1-vertices and 7-vertices

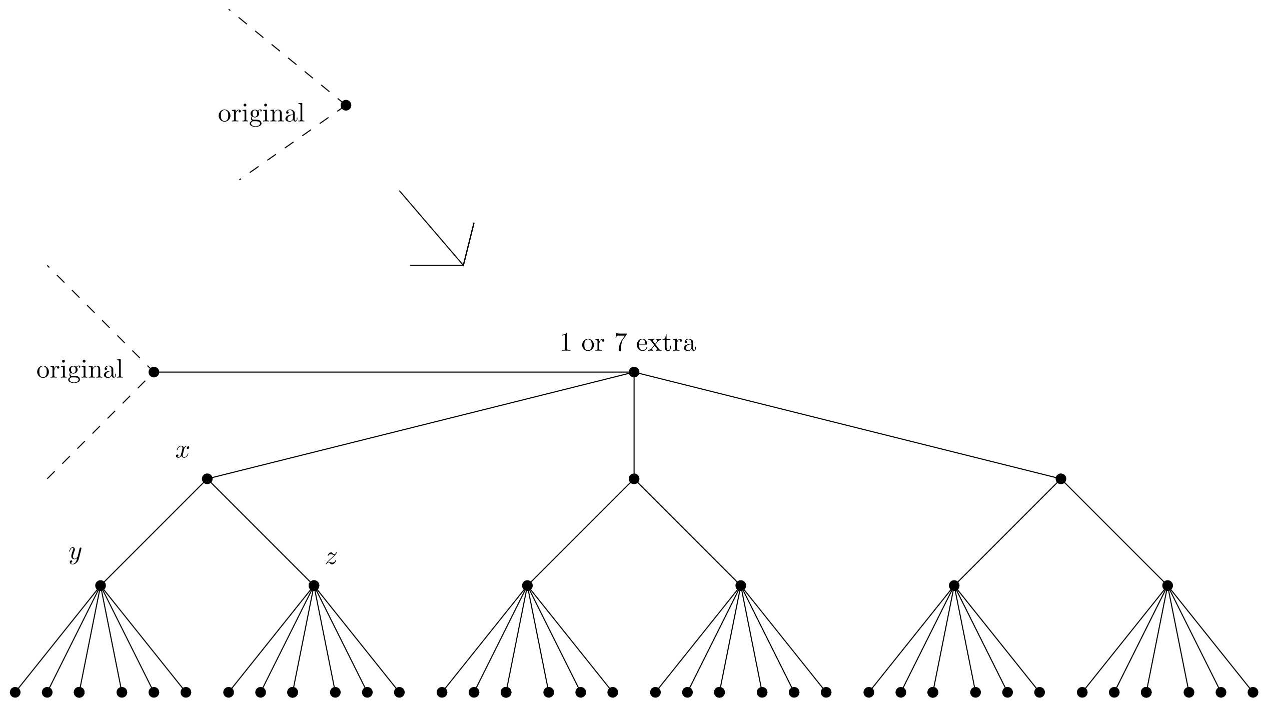


Fig. 8. Extra 1-vertices and extra 7-vertices

Let $G' = (V, E)$ be the resulting graph. As $G'$ can be constructed in polynomial time from $G$ and $f$, and $G$ is a planar bipartite graph of maximum degree 7, the NP-completeness result follows from the next lemma.

**Lemma 35** *$G$ is 3-colorable, if and only if $\lambda_{2,1}(G') \leq 8$.*

**PROOF.** First, suppose that $G$ is 3-colorable. Let $c : V \to \{1, 2, 3\}$ be a 3-coloring of $G$. We will give a lambda-coloring of $G'$ with colors $0, 1, \ldots, 8$.

Every original vertex $v$ of $G'$ is colored with color $c(v) + 2$. For $i \in \{0, 1, 7, 8\}$, every $i$-vertex and every extra $i$-vertex is colored with color $i$.

The neighbors of 0-vertices, extra 0-vertices, 8-vertices, and extra 8-vertices can be colored greedily. E.g., a neighbor of a 0-vertex has one neighbor with color 0, and six vertices at distance two, so there is always a color available from $\{2, 3, 4, 5, 6, 7, 8\}$.

We now show how to color the subtrees attached to 1-vertices (the other cases of 7-vertices, extra 1-vertices, or extra 7-vertices are similar.) See Figure 7. The vertex labeled $x$ is adjacent to one already colored vertex, which is colored with 1, and has at most three already colored vertices at distance two. There is at least one color in the set $\{3, 4, 5, 6\}$ not already assigned to a vertex at distance two of $x$. Give $x$ this color. Then, color the vertex marked $y$ with 0, and the vertex marked $z$ with 8. The leaves adjacent to $x$ and $y$ can now be colored greedily, similar as the vertices, adjacent to 0-vertices and 8-vertices. In this way, all subtrees attached to 1-vertices (and 7-vertices, extra 1-vertices, and extra 7-vertices) can be colored.

A tedious case analysis shows that the resulting coloring indeed is a $\lambda_{2,1}$-coloring.

Now, suppose we have a $\lambda_{2,1}$-coloring $c$ of $G'$. With help of a series of claims, we show that $G$ is 3-colorable.

**Claim 36** *If a vertex $v$ has degree 7, then $v$ must be colored with 0 or 8.*

**PROOF.** All neighbors of $v$ must receive a distinct color. If $v$ has a color, different from 0 and 8, then at most 6 colors are left for its neighbors. □

**Claim 37** *0-vertices, 8-vertices, extra 0-vertices, and extra 8-vertices have color either 0 or 8.*

**PROOF.** These vertices have degree 7. □

**Claim 38** *Original vertices have a color from $\{2, 3, 4, 5, 6\}$.*

**PROOF.** They are adjacent to two vertices that must contain color 0 or 8. As these vertices receive a different color, an original vertex is adjacent to a vertex with color 0, and hence cannot get color 0 or 1, and an original vertex is adjacent to a vertex with color 8, and hence cannot get color 7 or 8. □

**Claim 39** *1-vertices, 7-vertices, extra 1-vertices, and extra 7-vertices are colored with 1 or 7.*

**PROOF.** Suppose $v$ is a 1-vertex, 7-vertex, extra 1-vertex or extra 7-vertex. As $v$ has an original vertex as a neighbor, and an original vertex has a neighbor colored 0 and a neighbor colored 8, $v$ has vertices with color 0 at distance two, and vertices with color 8 at distance two, so $v$ has a color from $\{1, 2, 3, 4, 5, 6, 7\}$. Consider the subtrees attached to $v$. See Figure 7. The vertices $y$ and $z$ have degree 7, hence must be colored with 0 or 8, and the color of $y$ is different from the color of $z$. So, $x$ has a neighbor with color 0 and a neighbor with color 8, so must have a color from the set $\{2, 3, 4, 5, 6\}$. Now, $v$ is adjacent to four vertices that have a color from $\{2, 3, 4, 5, 6\}$, and, as these vertices have mutually distance two, they must be colored differently. It follows that $v$ cannot be colored with a color from $\{2, 3, 4, 5, 6\}$, as this would leave too few colors available for its neighbors. Hence, the color of $v$ is 1 or 7. □

**Claim 40** *An original vertex has color 3, 4, or 5.*

**PROOF.** An original vertex has two neighbors with color 1 or 7, and as these have different colors, it has a neighbor with color 1 and a neighbor with color

7. Hence, it must have color 3, 4, or 5. □

Now, we can make a 3-coloring of $G$ as follows. For every $v \in V$, if the color given to $v$ in the lambda-coloring is $i$, then write $c(v) = i - 2$, where $c : V \to \{1, 2, 3\}$ is a function, and for all $\{v, w\} \in E$, we have that the colors given by the lambda-coloring to $v$ and $w$ are different, as $v$ and $w$ have distance two in $G'$, so $c(v) \neq c(w)$, thus $c$ indeed is a 3-coloring of $G$. This ends the proof of Lemma 35. □

By the results above, we now have shown the NP-completeness of $\lambda$-coloring of planar bipartite graphs of maximum degree seven, with nine colors. □

It is possible to generalize the result somewhat:

**Theorem 41** *The following problem is* NP*-Complete. Let $r \geq 8$ be an* even *integer.*

Instance*: A planar bipartite graph $G = (V, E)$ of maximal degree $r - 1$.*
Question*: Is $\lambda_{2,1}(G) \leq r$?*

Basically, this proof goes as follows: every original vertex is adjacent to an $i$-vertex or extra $i$-vertex for all $i \in \{0, 1, \dots, r/2-3, r/2+3, r/2+4, \dots, r-1, r\}$. To these $i$-vertices, trees are attached in a way, similar to the 1-nodes, forcing them to receive color $i$ or $r - i$. Then, a structure, similar as in the proof above, (using in addition inductive arguments) can be used.

It seems that the proof technique used above cannot help to show NP-completeness for the problem to decide whether a given planar graph $G$ has $\lambda_{2,1}(G) \leq r$ for any odd values of $r$. We leave this as an open problem.

## 6 Concluding remarks

We have given upper bounds of $\lambda$ for some of the well-known graphs. However, we still lack examples of graphs where these bounds are matched, for example those for planar graphs and permutation graphs. It should be possible to tighten the constant factors in the bounds somewhat. For example, in outerplanar graphs, the conjecture is that $\lambda_{2,1} \leq \Delta + 2$ and we have $\lambda_{2,1} \leq \Delta + 8$. Similar comments apply to the other graphs studied in this paper as well.

For chordal graphs, in [29] it has been shown that $\lambda_{2,1} < \Delta^2$. We have shown for split graphs (a special case of chordal graphs) that $\lambda_{2,1} = \Theta(\Delta^{1.5})$. What is the best bound for chordal graphs?

For graphs of treewidth $k$, the $L(0, 1)$-labeling and $L(1, 1)$-labeling problems are polynomial for constant $k$ [32]. The corresponding problem for $L(2,1)$-labelings appears to be an interesting (but apparently not easy) open problem. The corresponding problem for interval graphs and outerplanar graphs also remain open.

It is conjectured in [21] that $\lambda_{2,1} \leq \Delta^2$ for any graph. This is true for all the special graphs that have been studied, but the general problem remains open. As any graph of degree $\Delta$ can be turned into a $\Delta$-regular graph in polynomial time by just adding at most $\Delta + 2$ vertices [8], $\Delta$-regular graphs constitute the *hardest* type of graphs for $\lambda$-coloring. We already have a lower bound of $\frac{\Delta^2}{4}$ for bipartite graphs hence for $\Delta$-regular graphs as well. The question thus remains if $\lambda_{2,1} \leq \Delta^2$ for $\Delta$-regular graphs.

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