

Defining a non-concrete recursive type in HOL which includes sets.

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1 Introduction

In HOL [GM93], a set of tools is provided which – for a certain class of commonly used *concrete recursive types* – automatically carries out all the formal proofs necessary to define these types. The class of recursive types supported by these tools are:

$$\begin{array}{l} op = C_1 t_1^1 \dots t_1^{k_1} \\ | \dots \\ | \dots \\ | C_m t_1^m \dots t_1^{k_m} \end{array} \tag{1.1}$$

where each t_i^j is either a type expression already defined as a type (this type must *not* include op), or is the name op itself. For recursive types where the t_i^j 's do include op , these tools do not work, and the user has to define such a type manually. In this paper, we shall describe how we manually defined the following recursive data type to HOL:

```
Val = NUM num
      | SET (Val)set
      | LIST (Val)list
      | TREE (Val)ltree
```

Section 2 outlines the general approach one has to follow when manually defining a recursive data type in HOL as it is described in [Mel89]. Sections 2 up to 4 describe the application of this approach to the specific data type mentioned above. Section 5, subsequently, shows how functions can be defined on this data type. Section 6, finally, concludes. All the results in this technical report have been mechanically verified using the HOL theorem proving environment [GM93].

2 The general approach for defining a new type in HOL

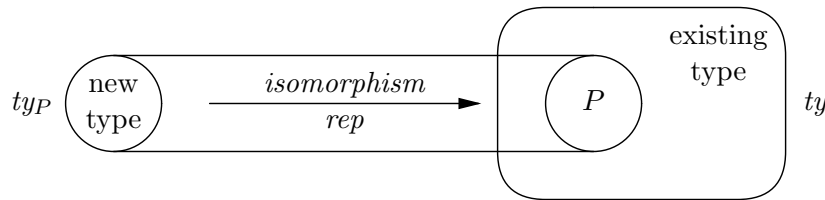
The approach to defining a new logical type as described in [Mel89], involves the following three steps:

1. find an appropriate non-empty subset of an existing type to represent the new type
2. extend the syntax of logical types to include a new type symbol, and use a type definition axiom to relate this new type to its representation
3. derive from the type definition axiom and the properties of the representing type a set of theorems that serves as an “axiomatisation” of the new type.

In the first step, a model for the new type is given by specifying a set of values that will be used to represent it. This is done by defining a predicate P on an existing type such that the set of values satisfying P is non-empty and has exactly the properties that the new type is expected to have.

In the second step, the syntax of types is extended to include a new type symbol which denotes the set of values of the new type. In HOL, this can be done by means of *type definition axioms*, a mechanism formalised by Mike Gordon in [Gor85] which defines a new type by adding a definitional axiom to the logic asserting that the new type is isomorphic to an appropriate non-empty subset of an existing type. The SML function for doing this in HOL is:

```
new_type_definition : {name:string, pred:term, inhab_thm:thm} → thm
```



If ty is an existing type of the HOL logic, and P is a term of type $ty \rightarrow \text{bool}$ denoting a non-empty¹ (i.e. we can prove $\vdash \exists x :: P x$) subset of ty , then evaluating:

```
new_type_definition : {name = " tyP", pred = P, inhab_thm =  $\vdash \exists x :: P x$  }
```

results in ty_P being declared as a new type symbol characterised by the following definitional axiom:

$$\frac{\vdash \exists rep :: (\forall x y :: (rep.x = rep.y) \Rightarrow x = y) \wedge (\forall r :: P.r = (\exists x :: r = rep.x))}{(ty_P_TY_DEF)}$$

¹Due to the formalisation of Hilbert’s ε -operator, HOL types must be non-empty (see Appendix A).

which states that the set of values denoted by the new type is isomorphic to, and consequently has the same properties as, the subset of ty specified by P . By adding this definition to the logic, the new type ty_P is defined in terms of existing type ty , and the isomorphism rep can be thought of as a representation function that maps a value of the new type ty_P to the value of type ty that represents it. The type definition axiom (ty_P -TY_DEF) above, asserts only the *existence* of a bijection from ty_P to the corresponding subset of ty . To introduce constants that in fact denote this isomorphism and its inverse, the following SML function is provided:

```
define_new_type_bijections :
      {ABS:string, REP:string, name:string, tyax:thm} → thm
```

If REP_ty_P and ABS_ty_P are the desired names under which to store the representation function and its inverse, then evaluating:

```
define_new_type_bijections
  {ABS = "ABS_ty_P", REP = "REP_ty_P",
   name = "ty_P_ISO_DEF", tyax = ty_P_TY_DEF}
```

defines two constants $REP_ty_P:ty \rightarrow ty_P$ and $ABS_ty_P:ty_P \rightarrow ty$, and creates the following theorem which is stored under the name $ty_P_ISO_DEF$:

$$\begin{aligned} \vdash (\forall a :: ABS_ty_P.(REP_ty_P.a) = a) \\ \wedge (\forall r :: P.r = (REP_ty_P.(ABS_ty_P.r) = r)) \end{aligned} \quad ty_P_ISO_DEF$$

It is straightforward to prove that the representation and abstraction functions are injective (one-to-one) and surjective (onto), using provided SML functions:

$$\begin{aligned} \vdash (\forall a a' :: (REP_ty_P.a = REP_ty_P.a') = (a = a')) & \quad ty_P_REP_ONE_ONE \\ \vdash \forall r :: P.r = (\exists a :: r = REP_ty_P.a) & \quad ty_P_REP_ONTO \\ \vdash \forall r r' :: P.r \Rightarrow P.r' \Rightarrow ((ABS_ty_P.r = ABS_ty_P.r') = (r = r')) & \quad ty_P_ABS_ONE_ONE \\ \vdash \forall a :: \exists r :: (a = ABS_ty_P.r) \wedge P.r & \quad ty_P_ABS_ONTO \end{aligned}$$

So, after the second step we actually have the new type ty_P , we know that the values of this type have exactly the same properties as the values in the subset P of ty , and we have a representation and abstraction function to go back and forth between ty_P and ty . Consequently, stating that some property H is true for all elements of the new type ty_P is equal to stating that for all elements in P , H is true of their image under ABS_ty_P :

$$\vdash (\forall x :: (H.x)) = (\forall r :: (P.r) \Rightarrow (H.(ABS_ty_P.r))) \quad ty_P_PROP$$

In the third step, a collection of theorems is proved that state abstract characterisations of the new type. These characterisations capture the essential properties of the new type without reference to the way its values are represented and therefore acts as an abstract “axiomatisation” of it. For an inductive type σ , the assertion of the unique existence of a function g satisfying a recursion equation whose form coincides with the primitive recursion scheme of this type σ – that is, g is a paramorphism [Mee90] – provides an adequate and complete abstract characterisation for σ . From this characterisation it follows that every value of σ is constructed by one or more applications of σ ’s constructors, and consequently completely determines the values of σ up to isomorphism without reference to the way these are represented. Moreover, in [Mee90] it is proved that all functions with source type σ are expressible in the form of paramorphism g .

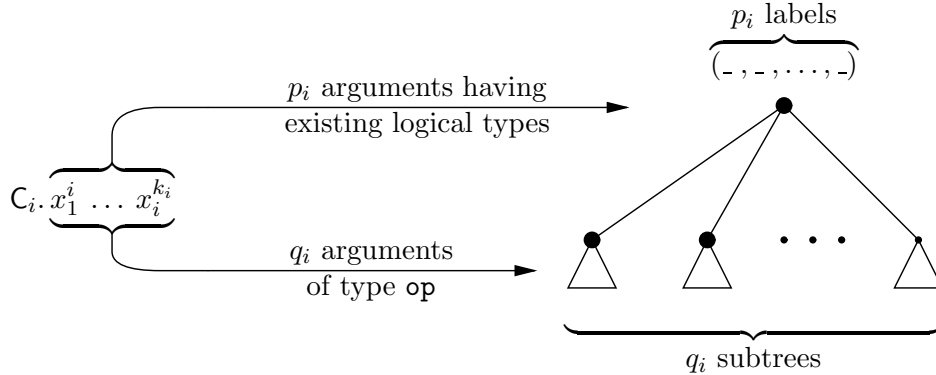
3 The representation and type definition

In [Mel89] a concrete recursive type of the form:

$$\begin{array}{l} \text{op} = C_1 t_1^1 \dots t_1^{k_1} \\ | \quad \dots \\ | \quad C_m t_1^m \dots t_1^{k_m} \end{array} \quad (3.1)$$

is represented by a non-empty subset of labelled trees (see Appendix E for labelled trees (`ltree`)). Each of the m constructors C_i , $1 \leq i \leq m$, of the concrete recursive data type is represented by a labelled tree using the scheme outlined below:

Let us consider the following instantiation of the i th constructor: $C_i.x_1^1 \dots x_i^{k_i}$, where each x_i^j is of type t_i^j which can be an existing logical type, or is the type `op` itself. Let p_i denote the number of arguments which have existing types and let q_i be the number of arguments which have type `op`, where $p_i + q_i = k_i$, then the abstract value of `op` denoted by $C_i.x_1^1 \dots x_i^{k_i}$ can be represented by a labelled tree which has p_i values associated with its root node, and q_i subtrees (for the recursive occurrences of `op`). In a diagram:

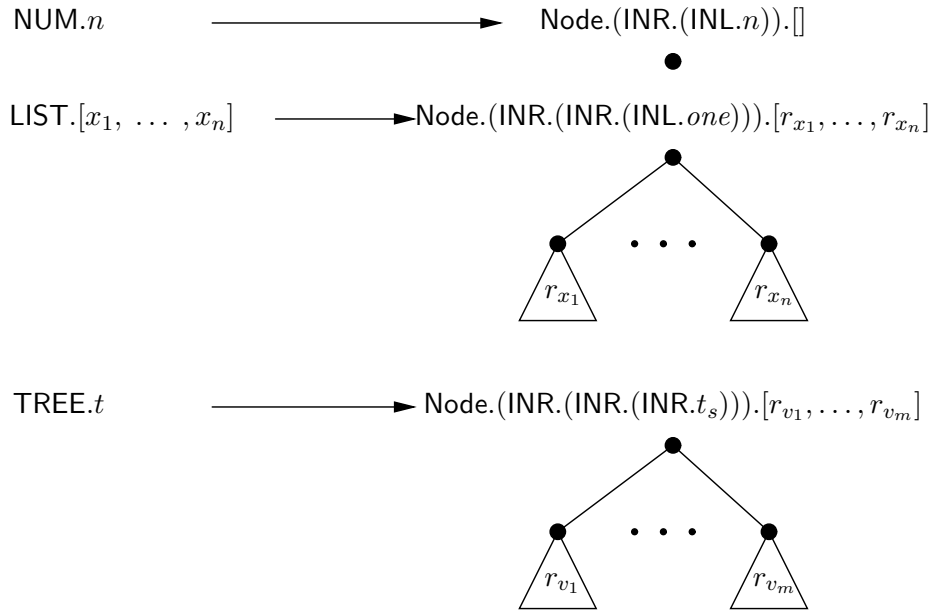


the root of the representing tree is labelled by a p_i -tuple of values. Each of these values is one of the p_i arguments to C_i which are not of type `op`. When $p_i = 0$, the representing tree is labelled with *one*, the one and only element of type `one`. The q_i subtrees shown in the diagram are the representations of the arguments of C_i that have type `op`. When $q_i = 0$, the tree will have no subtrees. Each of the m constructors can be represented by a labelled tree in this way, and consequently the representing type for `op` will be:

$$\overbrace{\left(\underbrace{(- \# \dots \# -)}_{\text{product of } p_1 \text{ types}} + \dots + \underbrace{(- \# \dots \# -)}_{\text{product of } p_m \text{ types}} \right)}^{\text{sum of } m \text{ products}} \text{ltree}$$

The subset predicate P can now be defined to specify a subset of labelled subtrees of the above type.

This method from [Mel89] only has to be adjusted a bit, in order to represent a subset of the new data type `Val`. Let us be ignorant of the sets for a while, and start using the ideas outlined above. That is, we use $((\text{one} + \text{num} + \text{one} + \text{tree}))\text{ltree}$ as the representing type, and make representations for the constructors `NUM`, `LIST` and `TREE` as follows:

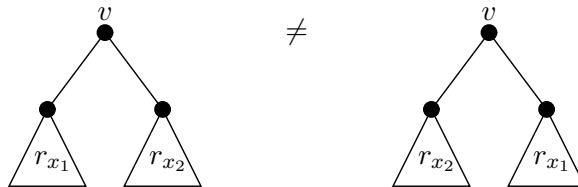


where, r_x denotes the representation as a $((\text{one} + \text{num} + \text{one} + \text{tree}))\text{ltree}$ of value x of type Val . The t_s in the root of the representation tree of $\text{TREE } t$ denotes the shape of the tree t , and $[v_1, \dots, v_m]$ is the list of Val typed values stored at the nodes of tree t . Although t_s is not an argument to the constructor TREE we can use this position at the root of the representation tree to store the shape of the Val tree which we obviously need in order to be able to go from the representation as a $((\text{one} + \text{num} + \text{one} + \text{tree}))\text{ltree}$ – where all the values of the Val tree are put in a list not containing any information about the shape of the original tree – to an abstract value of type Val .

As already indicated, the sets in the new type Val constitute a problem when proceeding with the method outlined above. When representing $\text{SET}. \{x_1, \dots, x_n\}$ as a labelled ltree of which the subtrees are the representations of the values x_1, \dots, x_n the resulting representation function is *not* an injection, since:

$$\text{SET}. \{x_1, x_2\} = \text{SET}. \{x_2, x_1\}$$

but,



The solution is to represent $\text{SET}. \{x_1, \dots, x_n\}$ by an equivalence class of ltrees in which ltrees like the two above are considered to be equivalent. Consequently, the existing type to represent our new type Val by shall consist of equivalence classes of ltrees, that is:

$$((\text{one} + \text{num} + \text{one} + \text{tree}))\text{ltree} \rightarrow \text{bool}$$

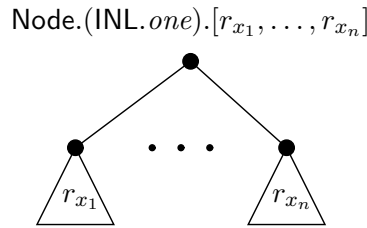
Before the subset predicate can be defined, we first need to formalise the equivalence relation `equiv`, that given an ltree of type `((one + num + one + tree))ltree`, returns the equivalence class of that ltree:

```
equiv : ((one + num + one + tree))ltree →
        ((one + num + one + tree))ltree → bool
```

The representation of a:

`NUM.n` value obviously has to consist of the equivalence class containing only the ltree `(Node.(INR.(INL.n)).[])`. Consequently, `equiv.(Node.(INR.(INL.n)).[])` must return a function that only delivers `true` for argument `(Node.(INR.(INL.n)).[])`.

`SET.{x1, ..., xn}` value, has to consist of the class containing all ltrees that are equivalent to:



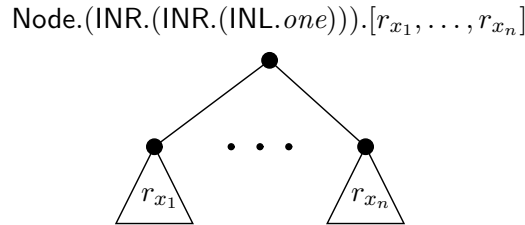
For the `SET` case this must be the class of ltrees:

- that have `(INL.one)` at their root
- of which the **sets** of images of their subtrees under equivalence are identical. Note that, because of the absence of ordering in sets, the requirement that these particular sets are identical ensures that two ltrees as displayed on page 5 are equivalent.

Consequently, `equiv.(Node.(INL.one).tl1)` must return a function that only delivers `true` when given an argument `(Node.(INL.one).tl2)` such that:

$$\text{image.equiv.}(l2s.tl_1) = \text{image.equiv.}(l2s.tl_2)$$

`LIST.xs` value, has to consist of the class containing all ltrees that are equivalent to:



For the `LIST` case this must be the class of ltrees:

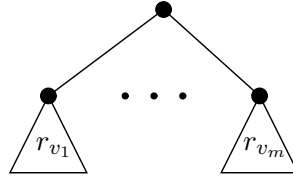
- have `(INR.(INR.(INL.one)))` at their root
- of which the **list** of images of their subtrees under equivalence are identical.

Consequently, `equiv.(Node.(INR.(INR.(INL.one))).tl1)` must return a function that only delivers `true` for an argument `(Node.(INR.(INR.(INL.one))).tl2)` such that:

$$\text{map.equiv.}tl_1 = \text{map.equiv.}tl_2$$

TREE. t value, has to consist of the class containing all ltrees that are equivalent to:

$$\text{Node}.\text{(INR}.\text{(INR}.\text{(INR}.\text{t}_s\text{)}))\text{)}.[r_{v_1}, \dots, r_{v_m}]$$



Where t_s denotes the shape of tree t , and $[v_1, \dots, v_m]$ is the list of **Val** typed values stored at the nodes of tree t . For the TREE case this must be the class of ltrees:

- have $(\text{INR}.\text{(INR}.\text{(INR}.\text{t}_s\text{)}))$ at their root
- of which the **list** of images of their subtrees under equivalence are identical.

Consequently, $\text{equiv}.\text{(Node}.\text{(INR}.\text{(INR}.\text{(INR}.\text{t}_s\text{)}))\text{)}.tl_1)$, must return a function that only delivers *true* for an argument $(\text{Node}.\text{(INR}.\text{(INR}.\text{(INR}.\text{t}_s\text{)}))\text{)}.tl_2)$ such that:

$$\text{map.equiv}.\text{tl}_1 = \text{map.equiv}.\text{tl}_2$$

Below the formal definition of **equiv** is given:

Definition 3.1 EQUIVALENCE RELATION

equiv_DEF ◀

$$\begin{aligned} &\text{equiv}.\text{(Node}.\text{v}_1.\text{tl}_1)\text{.}(\text{Node}.\text{v}_2.\text{tl}_2) = \\ & (v_1 = v_2) \\ & \wedge \\ & ((tl_1 = tl_2 \wedge (\exists n :: v_1 = \text{INR}.\text{(INL}.\text{n}))) \\ & \vee \\ & (\text{image.equiv}.\text{(l2s}.\text{tl}_1) = \text{image.equiv}.\text{(l2s}.\text{tl}_2) \wedge \text{ISL}.\text{v}_1) \\ & \vee \\ & (\text{map.equiv}.\text{tl}_1 = \text{map.equiv}.\text{tl}_2) \\ & \wedge \\ & (v_1 = \text{INR}.\text{(INR}.\text{(INL}.\text{one}))} \vee \exists t :: v_1 = \text{INR}.\text{(INR}.\text{(INR}.\text{t}))) \\ &) \\ &) \end{aligned}$$

Proving that the relation **equiv** is an equivalence relation is tedious but straightforward. Using the very nice way to represent equivalence relations from [Har93], we have:

Theorem 3.2 **equiv** IS AN EQUIVALENCE RELATION

equiv_EQUIV_REL ◀

$$\text{equiv}.\text{t}_1.\text{t}_2 = (\text{equiv}.\text{t}_1 = \text{equiv}.\text{t}_2)$$

The subset predicate P that has to specify a non-empty subset of equivalence classes of ltrees can now be defined as the quotient set of an appropriate subset Q of ltrees by the equivalence relation **equiv**. Looking at the representations of the different **Val** values, we can see that this Q must contain ltrees $(\text{Node}.\text{v}.\text{tl})$ for which hold that:

- (1) if $(v = \text{INR}.\text{INL}.n)$ for some number n at their root , then $tl = []$.
- (2) if $(v = \text{INR}.\text{INR}.\text{INR}.t)$ for some tree t , then t and tl form an ltree
- (3) for all ltrees in tl , (1) and (2) from above also hold.

in a formula:

Definition 3.3

Q_DEF

$$\begin{aligned}
Q.(\text{Node}.v.tl) = & \\
& (\exists n :: (v = \text{INR}.\text{INL}.n)) \Rightarrow tl = [] \\
& \wedge (\exists t :: v = \text{INR}.\text{INR}.\text{INR}.t) \Rightarrow \text{Is_ltree}.\text{OUTR}.\text{OUTR}.\text{OUTR}.v, tl \\
& \wedge (\forall t :: t \in tl \Rightarrow Q.t)
\end{aligned}$$

Finally, the subset predicate P is defined as the quotient set of Q by equiv:

Definition 3.4

Is_pvt_REP

$$P = Q/\text{equiv}$$

Theorem 3.5

Is_pvt_REP_THM

$$P = (\lambda s. \exists t :: (s = \text{equiv}.t) \wedge (Q.t))$$

It is not hard to prove that P is not empty, and consequently we can use the SML functions `new_type_definition` and `define_new_type_bijections` to extend the syntax of logical types to include our new type `Val`, define the type bijections `ABS_Val` and `REP_Val` between `Val` and P , and prove that these are injective and surjective:

Definition 3.6

Val_ISO_DEF

$$(\forall a :: \text{ABS_Val}.\text{REP_Val}.a = a) \wedge (\forall r :: P.r = (\text{REP_Val}.\text{ABS_Val}.r) = r)$$

Theorem 3.7

Val_REP_ONE_ONE

$$(\forall a a' :: (\text{REP_Val}.a = \text{REP_Val}.a') = (a = a'))$$

Theorem 3.8

Val_REP_ONTO

$$\forall r :: P.r = (\exists a :: r = \text{REP_Val}.a)$$

Theorem 3.9

Val_ABS_ONE_ONE

$$\forall r r' :: P.r \Rightarrow P.r' \Rightarrow ((\text{ABS_Val}.r = \text{ABS_Val}.r') = (r = r'))$$

Theorem 3.10

Val_ABS_ONTO

$$\forall a :: \exists r :: (a = \text{ABS_Val}.r) \wedge P.r$$

Theorem 3.11

Val_PROP

$$(\forall x :: (H.x) = (\forall r :: (P.r) \Rightarrow (H.\text{ABS_Val}.r)))$$

4 The axiomatisation

The abstract axiomatisation of `Val` will be based upon four constructors:

```

NUM   : num → Val
SET   : (Val)set → Val
LIST  : (Val)list → Val
TREE  : (Val)ltree → Val

```

To define the constructors, we need a function that given an equivalence class of `ltrees` returns an element of that equivalence class. We will call this function `pick`, and define it using Hilbert's ε -operator (see Appendix A). It satisfies the following properties:

Definition 4.1 `pick` *pick*

`pick.c = $\varepsilon t. c.t$`

Theorem 4.2 *equiv_pick_REP_pvt*

`equiv ∘ pick ∘ REP_Val = REP_Val`

Theorem 4.3 *Q_pick_REP_pvt*

`∀x :: Q.((pick ∘ REP_Val).x)`

Now the constructors can be defined as follows (see Appendix D for the definition of `s2l`).

Definition 4.4 *NUM_DEF*

`NUM.n = ABS_Val.(equiv.(Node.(INR.(INL.n)).[]))`

Definition 4.5 *SET_DEF*

`SET.s = ABS_Val.(equiv.(Node.(INL.one)
.(map.(pick ∘ REP_Val).(s2l.s))))`

Definition 4.6 *LIST_DEF*

`LIST.l = ABS_Val.(equiv.(Node.(INR.(INR.(INL.one))
.(map.(pick ∘ REP_Val).l))))`

Definition 4.7 *TREE_DEF*

`TREE.t = ABS_Val.(equiv.(Node.(INR.(INR.(INR.(shape.t))))
.(map.(pick ∘ REP_Val).(values.t))))`

Having defined the constructors, we can state the theorem which abstractly characterises the new type `Val`. Following [Mee90] we characterise our new type by stating the unique existence of a paramorphism `para` as follows:

Theorem 4.8 ABSTRACT CHARACTERISATION OF Val*pvf_Axiom*

$$\begin{aligned}
& \forall f_n f_s f_l f_t :: \\
& \quad \exists! \text{para} :: \\
& \quad (\forall n :: \text{para}.\text{NUM}.n = f_n.n) \\
& \quad \wedge \\
& \quad (\forall s :: (\text{FINITE}.s) \Rightarrow (\text{para}.\text{SET}.s = f_s.\text{image}.\text{split}.\text{para}.s))) \\
& \quad \wedge \\
& \quad (\forall l :: \text{para}.\text{LIST}.l = f_l.\text{map}.\text{split}.\text{para}.l) \\
& \quad \wedge \\
& \quad (\forall t :: \text{para}.\text{TREE}.t = f_t.\text{map_tree}.\text{split}.\text{para}.t)
\end{aligned}$$

where:

Definition 4.9 SPLIT*split*

$$\forall f x :: \text{split}.f.x = (f.x, x)$$

The proof of Theorem 4.8 consists of two parts, the proof of the existence of a paramorphism *para*, and the proof that such a paramorphism is unique.

The existence proof is based upon the following theorem about quotient sets. Informally, this theorem states that: for all equivalence relations E on α , and subsets Q on α , if ABS and REP are mutually inverse bijections between some set β and Q/equiv , and h is a function of type $\alpha \rightarrow \gamma$ that does not distinguish between different elements in the same equivalence class, then there exists a unique function g of type $\beta \rightarrow \gamma$ such that the following diagram commutes:

$$\begin{array}{ccc}
Q & \xrightarrow{E} & Q/E \\
h \downarrow & & \text{ABS} \downarrow \uparrow \text{REP} \\
\gamma & \xleftarrow{g} & \beta
\end{array}$$

Theorem 4.10 QUOTIENT SETS*QUOTIENT_THM*

For all equivalence relations E on α ; for all Q defining a subset of α ; for all $\text{ABS} : (\alpha \rightarrow \text{bool}) \rightarrow \beta$ and $\text{REP} : \beta \rightarrow (\alpha \rightarrow \text{bool})$, abstraction and representation functions respectively, the following theorem holds for all functions h of type $\alpha \rightarrow \gamma$:

$$\begin{array}{c}
(\forall a :: \text{ABS}.\text{REP}.a = a) \wedge (\forall r :: ((Q/E).r) = (\text{REP}.\text{ABS}.r) = r) \\
(\forall t_1 t_2 :: (E.t_1.t_2) \Rightarrow (h.t_1 = h.t_2)) \\
\hline
\exists! g :: \forall t :: (Q.t) \Rightarrow (g.\text{ABS}.(E.t)) = (h.t)
\end{array}$$

Instantiating Theorem 4.10 with $((\text{one} + \text{num} + \text{one} + \text{tree}))\text{tree}$ for α , equiv for E , ABS_Val for ABS , REP_Val for REP , and Q (Definition 3.3) for Q , obviously makes g a good candidate for para . Applying modus ponens to this instantiation and Val_ISO_DEF gives us a unique function g of type $\text{Val} \rightarrow \gamma$ for which it holds that:

$$\frac{\forall t_1 t_2 :: (\text{equiv}.t_1.t_2) \Rightarrow (h.t_1 = h.t_2)}{\forall t :: (Q.t) \Rightarrow (g.(\text{ABS_Val}.\text{equiv}.t)) = (h.t)} \quad (4.1)$$

Using theorem 4.3 (and Theorem E.3), it is easy to prove that:

Theorem 4.11 *Q_NUM_REP*

$\forall n :: Q (\text{Node}.\text{INR}.\text{INL}.n).\text{[]}$

Theorem 4.12 *Q_SET_REP*

$\forall s :: Q.\text{Node}.\text{INL}.\text{one}.\text{map}.\text{pick} \circ \text{REP_Val}.\text{s2l}.s$

Theorem 4.13 *Q_LIST_REP*

$\forall l :: Q.\text{Node}.\text{INR}.\text{INR}.\text{INL}.\text{one}.\text{map}.\text{pick} \circ \text{REP_Val}.l$

Theorem 4.14 *Q_TREE_REP*

$\forall t :: Q.\text{Node}.\text{INR}.\text{INR}.\text{INR}.\text{shape}.t.\text{map}.\text{pick} \circ \text{REP_Val}.\text{values}.t$

These theorems together with (4.1) and the definitions of the constructors give us:

$$\frac{\forall t_1 t_2 :: (\text{equiv}.t_1.t_2) \Rightarrow (h.t_1 = h.t_2)}{\begin{aligned} \forall n :: g.\text{NUM}.n &= h.\text{Node}.\text{INR}.\text{INL}.n.\text{[]} \\ \forall s :: g.\text{SET}.s &= h.\text{Node}.\text{INL}.\text{one}.\text{map}.\text{pick} \circ \text{REP_Val}.\text{s2l}.s \\ \forall l :: g.\text{LIST}.l &= h.\text{Node}.\text{INR}.\text{INR}.\text{INL}.\text{one}.\text{map}.\text{pick} \circ \text{REP_Val}.l \\ \forall t :: g.\text{TREE}.t &= h.\text{Node}.\text{INR}.\text{INR}.\text{INR}.\text{shape}.t.\text{map}.\text{pick} \circ \text{REP_Val}.\text{values}.t \end{aligned}}$$

Consequently, in order to finish the existence part of the proof of Theorem 4.8, we have to find a function h , that satisfies the following properties:

- (i). $\forall t_1 t_2 :: (\text{equiv}.t_1.t_2) \Rightarrow (h.t_1 = h.t_2)$
- (ii). $h.\text{Node}.\text{INR}.\text{INL}.n.\text{[]}$
 $= f_n.n$
- (iii). $h.\text{Node}.\text{INL}.\text{one}.\text{map}.\text{pick} \circ \text{REP_Val}.\text{s2l}.s$
 $= f_s.\text{image}.\text{split}.g.s$, for all finite sets s
- (iv). $h.\text{Node}.\text{INR}.\text{INR}.\text{INL}.\text{one}.\text{map}.\text{pick} \circ \text{REP_Val}.l$
 $= f_l.\text{map}.\text{split}.\text{para}.l$
- (v). $h.\text{Node}.\text{INR}.\text{INR}.\text{INR}.\text{shape}.t.\text{map}.\text{pick} \circ \text{REP_Val}.\text{values}.t$
 $= f_t.\text{map_tree}.\text{split}.\text{para}.t$

We claim that the following function, defined by “primitive recursion” on `ltrees`, satisfies these conditions, and to finish the proof of the existence part of theorem 4.8, it only remains for us to validate this claim:

$\forall v \, tl :: h.(\text{Node}.v.tl) = k.(\text{map}.h.tl).v.tl$, where:

$$\begin{aligned}
k &= \lambda xs \, v \, tl. \\
&\text{ISL}.v \rightarrow f_s.(\text{l2s}.\text{zip}.(xs, (\text{map}.\text{ABS_Val} \circ \text{equiv}).tl))) \\
&\quad | \text{ISL}.\text{OUTR}.v \Rightarrow f_n.(\text{OUTL}.\text{OUTR}.v) \\
&\quad | \text{ISL}.\text{OUTR}.\text{OUTR}.v \Rightarrow f_l.(\text{zip}.(xs, (\text{map}.\text{ABS_Val} \circ \text{equiv}).tl))) \\
&\quad | f_t.(\text{zip}.tree \\
&\quad \quad ((\text{ABS_ltree}.\text{OUTR}.\text{OUTR}.\text{OUTR}.v), xs), \\
&\quad \quad (\text{ABS_ltree}.\text{OUTR}.\text{OUTR}.\text{OUTR}.v), \text{map}.\text{ABS_Val} \circ \text{equiv}).tl)))
\end{aligned}$$

In order to prove requirement (i), we have the following theorem, the proof of which is straightforward and tedious and hence will not be given here.

Theorem 4.15

ltree_Axiom_PRESERVES_equiv

For all functions h of type $((\text{one} + \text{num} + \text{one} + \text{tree}))\text{ltree} \rightarrow \gamma$ defined by “primitive recursion” on $((\text{one} + \text{num} + \text{one} + \text{tree}))\text{ltrees}$ (i.e. having the form $(\forall v \, tl :: h.(\text{Node}.v.tl) = k.(\text{map}.h.tl).v.tl)$ for an arbitrary function k of type $(\gamma)\text{list} \rightarrow ((\text{one} + \text{num} + \text{one} + \text{tree}))\text{list} \rightarrow \gamma$):

$$\begin{aligned}
&\forall xs_1 \, xs_2 \, tl_1 \, tl_2 \, v :: \\
&\quad (\text{equiv}.\text{Node}.v.tl_1).\text{Node}.v.tl_2 \wedge \\
&\quad (\text{ISL}.v \Rightarrow (\text{length}.xs_1 = \text{length}.tl_1) \wedge (\text{length}.xs_2 = \text{length}.tl_2) \wedge \\
&\quad \quad ((\text{l2s}.\text{zip}.(xs_1, \text{map}.equiv.tl_1))) = (\text{l2s}.\text{zip}.(xs_2, \text{map}.equiv.tl_2))) \\
&\quad (\text{ISR}.v \Rightarrow (xs_1 = xs_2)) \\
&\quad \Rightarrow \\
&\quad ((k.xs_1.v.tl_1) = (k.xs_2.v.tl_2)) \\
\hline
&\quad \forall t_1 \, t_2 :: (\text{equiv}.t_1.t_2) \Rightarrow (h.t_1 = h.t_2)
\end{aligned}$$

Proving that our function k satisfies the premise of theorem 4.15 is again straightforward and tedious since, as the proof of theorem 4.15 itself, it involves lots of lemmas involving `zip`. We shall not give this proof here either, and consider (i) to be proven.

It is easy to prove that h satisfies property (ii):

$$\begin{aligned}
&h.(\text{Node}.\text{INR}.\text{INL}.n).\text{[]}) \\
&= (\text{definition } h \text{ and } k) \\
&\quad f_n.(\text{OUTL}.\text{OUTR}.\text{INR}.\text{INL}.n) \\
&= (\text{OUTL}, \text{OUTR}, \text{INR}, \text{INL}) \\
&\quad f_n.n
\end{aligned}$$

In order to show that h satisfies property (iii), we first need to prove:

$$\forall s :: \text{map}.(h \circ \text{pick} \circ \text{REP_Val}).(\text{s2l}.s) = \text{map}.g.(\text{s2l}.s) \tag{4.2}$$

proof of (4.2)

Let us consider an arbitrary set s of `Val`-typed values. Since $h \circ \text{pick} \circ \text{REP_Val}$ is mapped only on elements in $(\text{s2l } s)$, it will be sufficient to prove:

$$\forall t : t \in (\text{s2l } s) : g = (h \circ \text{pick} \circ \text{REP_Val})$$

From the definition of Q (3.3) and theorem 4.12, we know that:

$$\forall x : x \in (\text{map}.\text{(pick} \circ \text{REP_Val)}.\text{(s2l.s)}) : (Q.x) \text{ holds,}$$

and thus (Theorem B.10)

$$\forall t : t \in (\text{s2l.s}) : (Q.\text{(pick} \circ \text{REP_Val)}.t))$$

From (i) and (4.1) we can now deduce that

$$\forall t : t \in (\text{s2l.s}) : (g \circ \text{ABS_Val} \circ \text{equiv} \circ \text{pick} \circ \text{REP_Val}) = (h \circ \text{pick} \circ \text{REP_Val})$$

which with theorem 4.2 and `Val_ISO_DEF`, rewrites to:

$$\forall t : t \in (\text{s2l.s}) : g = (h \circ \text{pick} \circ \text{REP_Val})$$

end proof of (4.2)

Now we can proceed with the proof of property (iii) as follows:

$$\begin{aligned} & h.\text{(Node.(INL.one).(\text{map}.\text{(pick} \circ \text{REP_Val)}.\text{(s2l.s))})} \\ = & \text{(definition } h) \\ & k.\text{(map.h.(\text{map}.\text{(pick} \circ \text{REP_Val)}.\text{(s2l.s))}.(\text{INL.one}).(\text{map}.\text{(pick} \circ \text{REP_Val)}.\text{(s2l.s))})} \\ = & \text{(map composition (Theorem B.8) and 4.2)} \\ & k.\text{(map.g.(s2l.s)).(INL.one).(\text{map}.\text{(pick} \circ \text{REP_Val)}.\text{(s2l.s))})} \\ = & \text{(definition } k) \\ & f_s.\text{(l2s.(zip.(\text{map}.\text{g.(s2l.s)}, \text{map}.\text{(ABS_Val} \circ \text{equiv}).(\text{map}.\text{(pick} \circ \text{REP_Val)}.\text{(s2l.s))})))} \\ = & \text{(map composition (Theorem B.8), theorem 4.2 and Val_ISO_DEF)} \\ & f_s.\text{(l2s.(zip.(\text{map}.\text{g.(s2l.s)}, (\text{s2l.s})))} \\ = & \text{(zip and split (Theorem B.9))} \\ & f_s.\text{(l2s.(map.(split.g).(s2l.s))} \\ = & \text{(l2s, map and image (Theorem C.9))} \\ & f_s.\text{(image.(split.g).(l2s.(s2l.s))} \\ = & \text{(s is a finite set (Theorem D.6))} \\ & f_s.\text{(image.(split.g).s)} \end{aligned}$$

The proofs of properties (iv) and (v) are similar to the proof of (iii) and will not be given. We hereby finish the proof of the existence part of theorem 4.8, and continue with the proof that the existing paramorphism is unique. That is we shall prove that: for all function x and y of type `Val` $\rightarrow \gamma$:

$$\begin{aligned} \forall n :: x.\text{(NUM.n)} &= f_n.n \\ \forall s :: \text{finite.s} \Rightarrow x.\text{(SET.s)} &= f_s.\text{(image.(split.x).s)} \\ \forall l :: x.\text{(LIST.l)} &= f_l.\text{(map.(split.x).l)} \\ \forall t :: x.\text{(TREE.t)} &= f_t.\text{(map_tree.(split.x).t)} \\ \forall n :: y.\text{(NUM.n)} &= f_n.n \\ \forall s :: \text{finite.s} \Rightarrow y.\text{(SET.s)} &= f_s.\text{(image.(split.y).s)} \\ \forall l :: y.\text{(LIST.l)} &= f_l.\text{(map.(split.y).l)} \\ \forall t :: y.\text{(TREE.t)} &= f_t.\text{(map_tree.(split.y).t)} \end{aligned} \quad (4.3)$$

$$(x = y)$$

In order to be able to prove this, we first need an induction theorem for type `Val`.

Theorem 4.16 INDUCTION ON `Val`
pvt_Induct

For all properties H ,

$$\begin{array}{l}
 \forall n :: H.(\text{NUM}.n) \\
 \forall s :: ((\text{finite}.s) \wedge (\forall p :: (p \in s) \Rightarrow (H.p))) \Rightarrow (H.(\text{SET}.s)) \\
 \forall l :: (\text{every}.H.l) \Rightarrow (H.(\text{LIST}.l)) \\
 \forall t :: (\text{every_tree}.H.t) \Rightarrow (H.(\text{TREE}.t)) \\
 \hline
 \forall p :: H.p
 \end{array}$$

The proof of this induction theorem is not too hard. Here we shall only give a sketchy proof to give the reader an idea. We start with the following lemma, that is easy to prove using `Val_PROP`.

Lemma 4.17
induct_lemma4

$$(\forall p :: H.p) = (\forall t r :: ((r = \text{equiv}.t) \wedge (Q.t)) \Rightarrow (H \circ \text{ABS_Val} \circ \text{equiv}).t)$$

To prove theorem 4.16 we assume:

$$\begin{array}{l}
 \mathbf{A}_1) \forall n :: H.(\text{NUM}.n) \\
 \mathbf{A}_2) \forall s :: ((\text{finite}.s) \wedge (\forall p :: (p \in s) \Rightarrow (H.p))) \Rightarrow (H.(\text{SET}.s)) \\
 \mathbf{A}_3) \forall l :: (\text{every}.H.l) \Rightarrow (H.(\text{LIST}.l)) \\
 \mathbf{A}_4) \forall t :: (\text{every_tree}.H.t) \Rightarrow (H.(\text{TREE}.t))
 \end{array}$$

we have to prove that:

$$\begin{array}{l}
 (\forall p :: H.p) \\
 (= \text{lemma 4.17}) \\
 (\forall t r :: ((r = \text{equiv}.t) \wedge (Q.t)) \Rightarrow (H \circ \text{ABS_Val} \circ \text{equiv}).t) \\
 (\Leftarrow \text{ltree induction (Theorem E.4) and definition of every (Definition B.6)}) \\
 \text{for arbitrary } h \text{ and } tl, \text{ we have to prove:}
 \end{array}$$

$$\frac{((r = \text{equiv}.(\text{Node}.h.tl)) \wedge (Q.(\text{Node}.h.tl))) \quad \forall t :: (t \in tl) \Rightarrow \forall r :: ((r = \text{equiv}.t) \wedge (Q.t)) \Rightarrow ((H \circ \text{ABS_Val} \circ \text{equiv}).t)}{((H \circ \text{ABS_Val} \circ \text{equiv}).(\text{Node}.h.tl))}$$

Moving the antecedents of this proof obligation into the assumptions, we get for an arbitrary h and tl that:

$$\begin{array}{l}
 \mathbf{A}_5) \forall t :: (t \in tl) \Rightarrow (\forall r :: ((r = \text{equiv}.t) \wedge (Q.t)) \Rightarrow ((H \circ \text{ABS_Val} \circ \text{equiv}).t)) \\
 \mathbf{A}_6) r = \text{equiv}.(\text{Node}.h.tl) \\
 \mathbf{A}_7) Q.(\text{Node}.h.tl)
 \end{array}$$

The proof that $(H \circ \text{ABS_Val} \circ \text{equiv}).(\text{Node}.h.tl)$, now proceeds by case distinction on h .

We shall prove the SET case (i.e. ISL.h), the other cases are similar. So suppose:

A₈) ISL.h

From the definition of equiv, and the properties of Q, pick, ABS_Val and REP_Val it follows that:

Lemma 4.18

SET_L2S_EQ_ABS

For all lists tl of $((one + num + one + tree))l$ trees:

$$\forall t :: (t \in tl) \Rightarrow (Q.t)$$

$$\overline{(\text{SET}.\text{l2s}.\text{map}.\text{(ABS_Val} \circ \text{equiv).tl}))} = \text{(ABS_Val}.\text{(equiv}.\text{(Node}.\text{(INL.one).tl))})$$

Continuing with the proof of 4.16:

$$\begin{aligned} & (H \circ \text{ABS_Val} \circ \text{equiv}).\text{(Node.h.tl)} \\ = & (\mathbf{A}_8, \text{ the type of } h, \text{ one, and } \circ) \\ & H.\text{(ABS_Val}.\text{(equiv}.\text{(Node}.\text{(INL.one).tl))}) \\ = & (\text{rewriting assumption } \mathbf{A}_7 \text{ with } Q, \text{ and lemma 4.18}) \\ & H.\text{(SET}.\text{l2s}.\text{map}.\text{(ABS_Val} \circ \text{equiv).tl))} \\ \Leftarrow & (\text{assumption } \mathbf{A}_2) \\ & \text{finite}.\text{l2s}.\text{map}.\text{(ABS_Val} \circ \text{equiv).tl)} \\ & \wedge \\ & \forall p :: (p \in \text{l2s}.\text{map}.\text{(ABS_Val} \circ \text{equiv).tl))} \Rightarrow (H.p) \\ = & (\text{lists are finite (Theorem C.10)}) \\ & \forall p :: (p \in \text{l2s}.\text{map}.\text{(ABS_Val} \circ \text{equiv).tl))} \Rightarrow (H.p) \\ = & (\text{element of l2s and map (Theorems C.9, C.5 and C.11)}) \\ & \forall p :: (\exists t :: (t \in tl) \wedge ((\text{ABS_Val} \circ \text{equiv}).t) = p) \Rightarrow (H.p) \end{aligned}$$

Making the antecedents of this proof obligation into assumptions, gives us an t , such that for arbitrary p :

A₉) $t \in tl$

A₁₀) $p = ((\text{ABS_Val} \circ \text{equiv}).t)$

leaving us with proof obligation:

$$\begin{aligned} & H.p \\ = & (\text{assumption } \mathbf{A}_{10}) \\ & H.\text{((ABS_Val} \circ \text{equiv).t)} \\ \Leftarrow & (\text{Modus ponens assumption } \mathbf{A}_9 \text{ and the Induction Hypothesis (} \mathbf{A}_5)) \\ & \exists r :: (r = \text{equiv.t}) \wedge (Q.t) \\ = & (\text{rewriting assumption } \mathbf{A}_7 \text{ with } Q, \text{ and assumption } \mathbf{A}_9) \\ & \exists r :: (r = \text{equiv.t}) \end{aligned}$$

Instantiating with equiv t proves this case. As already indicated the other cases (where ISR h) are similar, the **NUM** case is trivial, and for the **LIST** and **TREE** cases, theorems similar to 4.18 had to be proved.

Now that an induction theorem on **Val** is available, it is straightforward to prove the uniqueness. Assume the premises of proof obligation (4.3). We have to prove:

$$\begin{aligned}
& x = y \\
= & \text{(function equality)} \\
& \forall p :: (x.p) = (y.p) \\
\Leftarrow & \text{(Val Induction, } H = (\lambda p. (x.p) = (y.p))) \\
& \forall n :: (x.(NUM.n)) = (y.(NUM.n)) \\
& \wedge \\
& \forall s :: ((\text{finite}.s) \wedge (\forall p :: (p \in s) \Rightarrow ((x.p) = (y.p)))) \Rightarrow ((x.(SET.s)) = (y.(SET.s))) \\
& \wedge \\
& \forall l :: (\text{every}.\lambda p. (x.p) = (y.p)).l \Rightarrow ((x.(LIST.l)) = (y.(LIST.l))) \\
& \wedge \\
& \forall t :: (\text{every_tree}.\lambda p. (x.p) = (y.p)).t \Rightarrow ((x.(TREE.t)) = (y.(TREE.t)))
\end{aligned}$$

The first conjunct immediately follows from the premises of (4.3). We shall continue to prove the **SET** case, the **LIST** and **TREE** cases are similar. Suppose, for an arbitrary set s with **Val** typed values:

$$\begin{aligned}
& \mathbf{A}'_1) \text{ finite}.s \\
& \mathbf{A}'_2) \forall p :: (p \in s) \Rightarrow ((x.p) = (y.p))
\end{aligned}$$

we have to prove that: $(x.(SET.s)) = (y.(SET.s))$. From the premises of (4.3), we can deduce that:

$$\begin{aligned}
& \mathbf{A}'_3) (x.(SET.s)) = f_s.\text{(image.(split}.x).s) \\
& \mathbf{A}'_4) (y.(SET.s)) = f_s.\text{(image.(split}.y).s) \\
& \\
& (x.(SET.s)) = (y.(SET.s)) \\
= & \text{(assumptions } \mathbf{A}'_3 \text{ and } \mathbf{A}'_4) \\
& f_s.\text{(image.(split}.x).s) = f_s.\text{(image.(split}.y).s) \\
\Leftarrow & \\
& \text{(image.(split}.x).s) = \text{(image.(split}.y).s) \\
\Leftarrow & \text{(Theorem C.6)} \\
& \forall p :: (p \in s) \Rightarrow ((\text{split}.x.p) = (\text{split}.y.p)) \\
= & \text{(Definition 4.9 of split)} \\
& \forall p :: (p \in s) \Rightarrow (((x.p), p) = ((y.p), p)) \\
= & \text{(pairs)} \\
& \forall p :: (p \in s) \Rightarrow (x.p) = (y.p)
\end{aligned}$$

Assumption \mathbf{A}'_2 proves this **SET** case, and, as indicated, the **LIST** and **TREE** cases are similar. This completes the outline of the uniqueness part, and consequently the entire proof of the abstract characterisation theorem of **Val** (Theorem 4.8).

5 Defining functions and operations on Val

In [Mee90] it is proved that all functions with source type σ are expressible in the form of a paramorphism, i.e. are paramorphisms. Consequently, functions on **Val** are defined in HOL as paramorphisms on **Val**. For example, a destructor function `evaln`, that unpacks the value `(NUM n)`, is added to HOL as follows. First, using Hilbert's choice operator, `evaln` is defined as a function that has the desired behaviour:

Definition 5.1 `evaln` *evaln_DEF*
`evaln = $\varepsilon g. (\forall n. (g.(\text{NUM}.n) = n))$`

Then we prove that the function `evaln` is a paramorphism:

Theorem 5.2 `evaln` IS A PARAMORPHISM ON **Val** *evaln*
 `$\forall n. \text{evaln}.\text{NUM}.n = n$`

proof of 5.2

For arbitrary n we have to prove that:

$$\begin{aligned} & \text{evaln}.\text{NUM}.n = n \\ &= (\text{Definition 5.1}) \\ & \quad (\varepsilon g. (\forall n. (g.(\text{NUM}.n) = n))).(\text{NUM}.n) = n \\ & \Leftarrow (\text{Hilbert's } \varepsilon \text{ (Theorem A.1)}) \\ & \quad \exists g. g.(\text{NUM}.n) = n \end{aligned}$$

Specialising the conclusion of Theorem 4.8, by substituting $(\lambda n. n)$ for f_n , we can conclude the (unique) existence of a paramorphism `para` for which it holds that `para.(NUM. n) = n` . Now the existentially quantified proof obligation from above can be proved by reducing it with the witness `para`.

end proof of 5.2

Theorem 5.2 above is exactly desired the definition of the destructor function `evaln`. Note that it has been defined as a partial function by leaving the results of values not in the correct domain unspecified. If one wants to define `evaln` such that “undefinedness” is explicitly dealt with, first some constant of type **Val** has to be defined about which nothing can be proved, e.g.

```
new_constant {Name = "undef", Ty = ==':Val'==} ;
```

Then `evaln` can be proved to be the paramorphism:

$$\begin{aligned} & \forall n. \text{evaln}.\text{NUM}.n = n \\ & \forall \text{set}. (\text{FINITE}.\text{set}) \Rightarrow \text{evaln}.\text{SET}.\text{set} = \text{undef} \\ & \forall l. \text{evaln}.\text{LIST}.l = \text{undef} \\ & \forall t. \text{evaln}.\text{TREE}.t = \text{undef} \end{aligned}$$

by extending the **proof of 5.2** with specialising the universally quantified functions f_{set} , f_l and f_t in Theorem 4.8 with functions that given an argument of the correct type return the value `undef`.

6 Conclusions

The contents of this report serves as a justification for adding the abstract characterisation theorem of the following data type as an axiom to the HOL theorem proving environment.

```

VAL = NUM num
      | BOOL bool
      | REAL real
      | STR string
      | SET (VAL)set
      | LIST (VAL)list
      | TREE (VAL)ltree

```

Although the data type `Val` is somewhat simpler than `VAL`, one can see that it contains the same problematic aspects as `VAL` (i.e. the constructors `SET`, `LIST` and `TREE`). From the verification activities described in this report, it becomes clear that manually adding the recursive data type `VAL` to HOL can be done analogous to the way `Val` is added. The representation, abstract characterisation and proof obligations for the additional constructors `BOOL`, `REAL` and `STRING`, will be analogous to those of `NUM`. However, as the number of constructors increases the proofs become long and tedious. Since all formal proofs necessary to prove the abstract characterisation theorem of the subtype `Val` have been verified in HOL, we are convinced that the abstract characterisation theorem of `VAL` can also be proved. Therefore, we have added it to HOL as an axiom, saving time that was spent on proving theorems of which we were not yet convinced that they held.

7 Acknowledgements

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Appendices

A Hilbert's ε -operator

Hilbert's ε -operator is a primitive constant of higher order logic [Mel89, GM93]. Informally, its syntax and semantics are as follows. If $P[x]$ is a boolean term involving a variable x of type α , then $(\varepsilon x. P[x])$ denotes some value, say v , of type α such that $P[v]$ is true. If there is no such value (i.e. $\forall v \in \alpha : \neg P[v]$) then $(\varepsilon x. P[x])$ denotes some fixed but arbitrary chosen value of α ². For example:

- $\varepsilon n. 4 < n \wedge n < 6$ denotes the value 5
- $\varepsilon n. (\exists m :: n = 2 \times m)$ denotes an unspecified even natural number
- $\varepsilon n. n < n$ denotes an arbitrary natural number

The formalisation of the ε -operator in HOL is by the following theorem:

Theorem A.1

SELECT_AX

$$\forall P :: (\exists x :: P.x) \Rightarrow (P.(\varepsilon x. P.x))$$

Consequently, ε can be used to obtain a logical term which provably denotes a value with a given property P from a theorem merely stating the existence of such a value.

B Lists

In the built-in theory list, a type `('a)list` is defined to denote the set of all finite lists having elements of type `'a`. The constructor functions that are used to construct any list-structured value of type `('a)list` are:

$$\begin{aligned} [] &\in ('a)\text{list} \\ \text{CONS} &\in 'a \rightarrow ('a)\text{list} \rightarrow ('a)\text{list} \end{aligned}$$

Below definitions and theorems are stated. In these definitions, `0` and `SUC` refer to the constructor functions that are used to construct any natural number in `num`.

Theorem B.1 LIST ELEMENT

IS_EL

$$(\forall x :: \neg \text{is_el}.x.[]) \wedge (\forall x y l :: \text{is_el}.y.(\text{CONS}.x.l) = (y = x) \vee \text{is_el}.y.l)$$

Throughout this technical report, when it is clear from the context that lists are used, `is_el` will be denoted by \in .

Definition B.2 MAP

MAP

$$(\forall f :: \text{map}.f.[] = []) \wedge (\forall f x l :: \text{map}.f.(\text{CONS}.x.l) = \text{CONS}.(f x).(\text{map}.f.l))$$

²A consequence, in HOL, types must be non-empty.

Definition B.3 ZIP ZIP

$$(\text{zip}([], []) = []) \wedge (\forall x_1 l_1 x_2 l_2 :: \text{zip}(\text{CONS}.x_1.l_1, \text{CONS}.x_2.l_2) = \text{CONS}.(x_1, x_2).(\text{zip}.(l_1, l_2)))$$

Definition B.4 FOLDR FOLDR

$$(\forall f e :: \text{foldr}.f.e.[] = e) \wedge (\forall f e x l :: \text{foldr}.f.e.(\text{CONS}.x.l) = f.x.(\text{foldr}.f.e.l))$$

Definition B.5 LENGTH LENGTH

$$(\text{length}([] = 0) \wedge (\forall x l :: \text{length}(\text{CONS}.x.l) = (\text{length}.l) + 1$$

Definition B.6 EVERY EVERY

$$(\forall P :: \text{every}.P.[] = \text{true}) \wedge (\forall P h t :: \text{every}.P.(\text{CONS}.h.t) = P.h \wedge \text{every}.P.t)$$

Definition B.7 SUM SUM

$$(\text{sum}([] = 0) \wedge (\forall x l :: \text{sum}(\text{CONS}.x.l) = x + (\text{sum}.l))$$

Theorem B.8 MAP COMPOSTION MAP_o

$$\forall f g l :: \text{map}.f.(\text{map}.g.l) = \text{map}.(f \circ g).l$$

Theorem B.9 ZIP_MAP_EQ_MAP_split

$$\forall f l :: \text{zip}((\text{map}.f.l), l) = \text{map}(\text{split}.f).l$$

Theorem B.10 IS_EL_MAP

$$\forall Q f l :: (\forall x : \text{is_el}.x.(\text{map}.f.l) : Q.x) = (\forall x : \text{is_el}.x.l : Q.(f.x))$$

Definition B.11 interval

$$\text{interval}.0 = [] \wedge \text{interval}(\text{SUC}.n) = \text{CONS}.n.(\text{interval}.n)$$

Theorem B.12 APPENDING LISTS APPEND

$$\forall l :: ([] ++ l) = l \wedge \forall l_1 l_2 x :: ((\text{CONS}.x.l_1) ++ l_2) = (\text{CONS}.x.(l_1 ++ l_2))$$

Definition B.13 DELETING AN ELEMENT DEL

$$(\forall y :: \text{del}.y.[] = []) \wedge (\forall y l x :: \text{del}.y.(\text{CONS}.x.l) = ((x = y) \rightarrow l \mid \text{CONS}.x.(\text{del}.y.l)))$$

Definition B.14 FIRST ELEMENT OF A LIST HD

$$\forall x l :: \text{hd}(\text{CONS}.x.l) = x$$

Definition B.15 TAIL OF A LIST TL

$$\forall x l :: \text{tl}(\text{CONS}.x.l) = l$$

Definition B.16 INDEXED ELEMENTS EL

$$\forall l :: \text{el}.0.l = \text{hd}.l \wedge \forall n l :: \text{el}(\text{SUC}.n).l = \text{el}.n.(\text{tl}.l)$$

Definition B.17 LIST CONTAINS NO DUPLICATE ELEMENTS NO_DUPLICATES

$$\forall l :: \text{no_duplicates}.l = \forall n k : n < \text{length}.l \wedge k < \text{length}.l \wedge n \neq k : \text{el}.n.l \neq \text{el}.k.l$$

C Sets

Definition C.1 CHARACTERISTIC SET PREDICATE *IN_DEF*

$$\forall s x :: \text{CHF}.s.x = (x \in s)$$

Definition C.2 IMAGE *IMAGE_DEF*

$$\forall f s :: \text{image}.f.s = \{f.x \mid x \in s\}$$

Definition C.3 INSERT *INSERT_DEF*

$$\forall x s :: x \text{ insert } s = \{y \mid (y = x) \vee y \in s\}$$

Definition C.4 COMPLEMENT OF A SET

$$\forall s :: s^c = \{x \mid x \notin s\}$$

Theorem C.5 *IN_IMAGE*

$$\forall y s f :: y \in \text{image}.f.s = (\exists x. (y = (f.x)) \wedge x \in s)$$

Theorem C.6 *IMAGE_EQ*

$$\forall f g s :: \frac{\forall x. (f.x) = (g.x)}{\text{image}.f.s = \text{image}.g.s}$$

The predicate FINITE is true of finite sets and false for infinite ones.

Definition C.7 CARDINALITY OF SETS *set*

$$\begin{aligned} &(\text{card.}\{\} = 0) \wedge \\ &\forall s :: \text{FINITE}.s \\ &\quad \Rightarrow \\ &\quad (\forall x :: \text{card.}(x \text{ insert } s) = ((x \in s) \Rightarrow \text{card}.s \mid \text{card}.s + 1)) \end{aligned}$$

Definition C.8 CONVERTING LISTS TO SETS *L2S_DEF*

$$\begin{aligned} &(\text{l2s}.\square = \{\}) \wedge \\ &(\forall x l :: \text{l2s}.\text{CONS}.x.l = x \text{ insert } (\text{l2s}.l)) \end{aligned}$$

Theorem C.9 *L2S_MAP_EQ_IMAGE*

$$\forall f l :: \text{l2s}.\text{map}.f.l = \text{image}.f.(\text{l2s}.l)$$

Theorem C.10 *L2S_FINITE*

$$\forall l :: \text{FINITE}.\text{l2s}.l$$

Theorem C.11 *IN_L2S_IS_EL*

$$\forall l x :: (x \in (\text{l2s}.l)) = (\text{is_el}.x.l)$$

D Converting (finite) sets to lists

Because sets are unordered, a function that converts sets to lists *cannot* be defined by set-induction on finite sets, like e.g. the definition of function `card`:

$$\begin{aligned} \text{s2l.}\{\} &= [] \wedge \\ \forall s :: \text{FINITE.}s & \\ \Rightarrow & \\ (\forall x :: \text{s2l.}(x \text{ insert } s) &= ((x \in s) \Rightarrow \text{s2l.}s \mid \text{CONS.}x.(\text{s2l.}s))) \end{aligned}$$

Defining `s2l` this way, results in that `s2l.{2, 3, 4}` is not equal to `s2l.{4, 3, 2}`, and this is clearly not what we want.

The correct definition of the function `s2l` has been constructed as follows. First, we define when a function $f : \text{num} \rightarrow \alpha$ is a bijection from a subset of `num` to a set containing elements of type α :

Definition D.1 BIJECTION FROM SUBSET OF `num` TO A SET *is_BIJ_NUM_to*

$$\begin{aligned} f \text{ is_BIJ_NUM_to } s & \\ = \forall n \ m : n < \text{card.}s \wedge m < \text{card.}s : (f.n = f.m) &\Rightarrow n = m \\ \wedge s = \{f.n \mid n < \text{card.}s\} & \end{aligned}$$

Then we define such a bijection for a set `s` using Hilbert's choice operator (Section A):

Definition D.2 A BIJECTION FOR A SET *set_BIJ*

$$\text{set_BIJ.}s = \varepsilon f. f \text{ is_BIJ_NUM_to } s$$

and prove that it indeed is a bijection as defined in definition D.1:

Theorem D.3 *set_BIJ_DEF*

$$\forall s :: \text{FINITE.}s \Rightarrow \text{set_BIJ.}s \text{ is_BIJ_NUM_to } s$$

Now the idea is, to define the function `s2l.s` by mapping the set bijection `set_BIJ.s` as defined above on a list containing the elements 0 to `card.s`.

Definition D.4 CONVERTING FINITE SETS TO LISTS *S2L_L2S*

$$\text{s2l.}s = \text{map.}(\text{set_BIJ.}s).(\text{interval.}(\text{card.}s))$$

The rest of this section lists some theorems.

Theorem D.5 *IN_IS_EL_S2L*

$$\forall s :: \text{FINITE.}s \Rightarrow \forall x :: (x \in s) = (\text{is_el.}x.(\text{s2l.}s))$$

Theorem D.6 *L2S_S2L_id*

$$\forall s :: \text{FINITE.}s \Rightarrow (s = \text{l2s.}(\text{s2l.}s))$$

Theorem D.7 *NO_DUPLICATES_S2L*

$$\forall s :: \text{FINITE.}s \Rightarrow \text{no_duplicates.}(\text{s2l.}s)$$

Theorem D.8*IS_EL_DEL_S2L*

$$\forall s x y :: \frac{\text{FINITE}.s \wedge \text{is_el}.x.(s2l.s) \wedge \text{is_el}.y.(\text{del}.x.(s2l.s))}{x \neq y}$$

Theorem D.9*S2L_split*

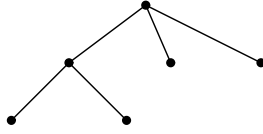
$$\forall s x :: \frac{\text{FINITE}.s \wedge \neg(x \in s)}{\exists l_1 l_2 :: (s2l.(x \text{ insert } s) = (l_1 ++ [x] ++ l_2)) \wedge (s = l2s.(l_1 ++ l_2))}$$

E Labelled trees

In the built-in HOL theory `tree`, a type `tree` is defined to denote the set of all ordered trees of which the nodes can branch any (finite) number of times. A constructor function

$$\text{node} \in (\text{tree})\text{list} \rightarrow \text{tree}$$

can be used to construct any value of type `tree`. For example, the tree:



is denoted by the term:
`node.[node.[node.[]; node.[]]; node.[]; node.[]]`

Figure 1: Some tree

The size of a tree is defined to be the number of nodes in that tree:

Definition E.1 SIZE OF TREES*size*

$$\forall tl :: \text{size}.\text{(node } tl) = \text{sum}.\text{(map.size.tl)} + 1$$

In the built-in HOL theory `ltree` a type of *labelled* trees (called `('a)ltree`) is defined by a type definition, following the way this is described in Section 2 and [Mel89]. Labelled trees have the same structure as values of the defined type `tree`. The only difference is that a labelled tree of type `('a)ltree` has a value associated with each of its nodes. The constructor:

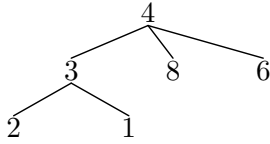
$$\text{Node} \in 'a \rightarrow ((\text{'a)ltree})\text{list} \rightarrow (\text{'a)ltree}$$

can be used to construct any value of type `('a)ltree`. For example, There is a function available that given an labelled tree of type `('a)ltree` returns the shape of the tree:

$$\text{shape}.t \in (\text{'a)ltree} \rightarrow \text{tree}$$

For example, applying `shape` to the labelled tree in Figure 2, returns the tree in Figure 1. The function that returns the list of values that are associated with the nodes of an labelled tree of type `('a)ltree` is:

$$\text{values}.t \in (\text{'a)ltree} \rightarrow (\text{'a})\text{list}$$



is denoted by the term:
 $\text{Node.4.}[\text{Node.3.}[\text{Node.2.}[]; \text{Node.1.}[]]; \text{Node.6.}[]; \text{Node.8.}[]]$

Figure 2: Some labelled tree

A pair $(t, l) \in (\text{tree} \times (\text{'a})\text{list})$ for which it holds that the size of t equals the length of l can be used to create a labelled tree:

Definition E.2 *IS_LTREE* *Is_ltree*

$\forall t l :: \text{Is_ltree.}(t, l) = (\text{length.}l = \text{size.}t)$

Theorem E.3 *CAN CREATE LTREE FROM shape AND values* *Is_ltree_REP_ltree_lemma*

$\forall t :: \text{Is_ltree.}(\text{shape.}t, \text{values.})$

There is an induction principle on labelled trees:

Theorem E.4 *LTREE INDUCTION* *ltree_Induct*

$\forall P :: \frac{(\forall t :: \text{every.P.t} \Rightarrow (\forall h :: P.(\text{Node.h.t})))}{\forall l :: P.l}$

Finally, analogous to the functions on lists, we have (defined in the theory `more_ltrees`):

Definition E.5 *MAP ON TREES* *MAP_TREE_DEF*

$\forall v t :: \text{map_tree.f.}(\text{Node.v.t}) = \text{Node.f.v.}(\text{map.map_tree.f.t})$

Definition E.6 *ZIP ON TREES* *ZIP_TREE_DEF*

For all v_1, v_2, t_1, t_2 :

$\text{length.t}_1 = \text{length.t}_2$

$\text{zip_tree.}(\text{Node.v}_1.t_1, \text{Node.v}_2.t_2) = \text{Node.v}_1.v_2.(\text{map.zip_tree.zip.t}_1.t_2))$

Definition E.7 *EVERY ON TREES* *EVERY_TREE_DEF*

$(\forall P h t :: \text{every_tree.P.}(\text{Node.h.t}) = P.h \wedge \text{every.every_tree.P.t})$

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