

On equicut graphs *

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February 2, 2000

Abstract

The *size* $sz(\Gamma)$ of an ℓ_1 -graph $\Gamma = (V, E)$ is the minimum of n_f/t_f over all the possible ℓ_1 -embeddings f into n_f -dimensional hypercube with scale t_f . The sum of distances between all the pairs of vertices of Γ is at most $sz(\Gamma)\lceil v/2\rceil\lfloor v/2\rfloor$ ($v = |V|$). The latter is an equality if and only if Γ is *equicut graph*, that is, Γ admits an ℓ_1 -embedding f that for any $1 \leq i \leq n_f$ satisfies $\sum_{x \in V} f(x)_i \in \{\lceil v/2\rceil, \lfloor v/2\rfloor\}$. Basic properties of equicut graphs are investigated. A construction of equicut graphs from ℓ_1 -graphs via a natural doubling construction is given. It generalizes several well-known constructions of polytopes and distance-regular graphs. Finally, large families of examples, mostly related to polytopes and distance-regular graphs, are presented.

1 Background

A finite simple graph $\Gamma = (V, E)$ with $v = v(\Gamma)$ vertices has the natural graph distance denoted by $d(x, y) = d_\Gamma(x, y)$. The diameter of Γ , that is, that maximum of $d(x, y)$ over all the pairs of vertices x, y , is denoted by $D(\Gamma)$. Then $\Gamma = (V, E)$ is an ℓ_1 -graph if there exists a mapping $f : V \mapsto R^m$ with $d_\Gamma(x, y)$ being equal to the ℓ_1 -distance between $f(x)$ and $f(y)$ for any two $x, y \in V$ or, equivalently, if there exists a mapping $g : V \mapsto H^n$ (the latter being the vertex-set of the n -cube) such that for any $x, y \in V$ we have $td(x, y) = d_H(g(x), g(y))$ for some integer $t = t_g = t_f$. Recall that $d_H(g(x), g(y))$ is the Hamming distance, that is, the size of the symmetric difference of the sets with characteristic vectors $g(x)$ and $g(y)$. The number t is called the *scale* of the embedding f . For more information on ℓ_1 -graphs we refer the reader to [DeLa97, Chapt. 21].

In fact, such t should be 1 or even number that is at most $v - 2$ (assuming $v \geq 4$). Call the smallest such $t = t_f$ (if any) the *scale* $t(\Gamma)$ of Γ . Call the the minimum of n/t the *size* $sz(\Gamma)$ of ℓ_1 -graph Γ . (In general, $sz(\Gamma)$ is larger than minimal m , the dimension of the host ℓ_1 -space into which the shortest path metric of Γ embeds isometrically.)

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$W(\Gamma)$ denotes the sum of all $v(v-1)/2$ pairwise distances between vertices of Γ ; chemists call it the *Wiener number*. In [Ple84] this number is called *transmission*.

A powerful tool to deal with ℓ_1 -graphs is via linear programming. Namely, consider the cone $Met(V)$ of *semimetrics* on V . The elements of $Met(V)$ are symmetric $v \times v$ -matrices with 0 diagonal that satisfy the triangle inequalities. A *metric* is a semimetric that does not have 0's outside the main diagonal. The distance matrix $Dist(\Gamma)$ of a graph Γ is a point in $Met(V)$.

The set of *cuts* S of V consists of partitions of V into two parts. Each $s \in S$ defines an extremal ray of the cone $Met(V)$. Namely, s defines a semimetric δ_s on V , so that $\delta_s(u, v) = 0$ or 1 depending on $u, v \in V$ being in same or different parts of s . Then one may ask whether $Dist(\Gamma)$ lies in the cone $Cut(V)$ that is generated by the cuts, that is

$$Dist(\Gamma) = \sum_{s \in S} \lambda_s \delta_s \quad (1)$$

for the appropriate choice of λ_s . Hence $Dist(\Gamma) \in Cut(V)$ implies that

$$t \cdot Dist(\Gamma) = \sum_{s \in S} \lambda_s \delta_s \quad (2)$$

for an integer $t > 0$ and an nonnegative integer vector $\lambda = (\lambda_s \mid s \in S)$.

Hence the third, in addition to the two mentioned earlier, definition of the ℓ_1 -graphs: a graph Γ is ℓ_1 -graph if and only if $Dist(\Gamma) \in Cut(V)$. Each equality (2) for a given t and λ defines an t -scale embedding f of Γ into the hypercube of dimension $\sum_{s \in S} \lambda_s$.

An advantage of using (2) is that it allows to classify ℓ_1 -embeddings of Γ up to equivalence: different solutions to (2) with nonnegative integer unknowns λ such that $g.c.d.(t, \lambda_i) = 1$ correspond to different embeddings. If such a solution is unique Γ is called *rigid*. Any rigid ℓ_1 -graph has scale 1 or 2; any ℓ_1 -graph of scale 1 is rigid. If an ℓ_1 -graph does not contain K_4 then it is rigid. Cf. [DeLa97, Sect. 21.4] for further information.

It is easy to check the following reformulation of [DeLa97, (4.3.6)].

Lemma 1.1 $W(\Gamma)/(\lceil v/2 \rceil \lfloor v/2 \rfloor) \leq sz(\Gamma) \leq W(\Gamma)/(v-1)$.

Proof. Let the (t -scale) embedding f be given by the matrix $F = (F_{iu}) = (f(u)_i)$. The columns of F can be regarded as characteristic vectors of one of the two parts of cuts $s \in S$. Using (2) and summing up the $d_\Gamma(u, v)$'s, we obtain

$$\begin{aligned} W &= W(\Gamma) = \sum_{u, v \in V} d_\Gamma(u, v) = \sum_{u, v \in V} \sum_{s \in S} \lambda_s \delta_s(u, v) \\ &= \sum_{s \in S} \sum_{u, v \in V} \lambda_s \delta_s(u, v) = \sum_{s \in S} \lambda_s |s| (v - |s|) \\ &\leq \lceil v/2 \rceil \lfloor v/2 \rfloor \sum_{s \in S} \lambda_s, \end{aligned}$$

by taking the maximum over $|s|$ in (3). On the other hand, by taking the minimum there, we obtain $W \geq (v-1) \sum_{s \in S} \lambda_s$. \square

When the equality holds in the left-hand side of the inequality in Lemma 1.1, we say that Γ is an **equicut graph**. This means that for such a graph in the formulae (2) every $s \in S$ satisfies $\delta_s \neq 0$ if and only if s partitions V into parts of size $\lceil v/2 \rceil$ and $\lfloor v/2 \rfloor$.

On the other hand, the equality on the other side only happens in very special case.

Lemma 1.2 $sz(\Gamma) = W(\Gamma)/(v-1)$ if and only if Γ is the star $K_{1,v-1}$.

Proof. Without loss in generality, the first row $f(u)$ of the matrix F defined by f consists of 0's. Moreover, all the cuts s that appear in F are partitions of V into part of size one and of size $v-1$. It follows that the any geodesic between $x, y \in V$ must contain u . Hence Γ is a star. \square

Further, the size $sz(\Gamma)$ has the following properties formulated in terms of v .

Theorem 1.3 Let Γ be an ℓ_1 -graph.

- (i) $sz(\Gamma) = 2 - 1/(\lceil v/2 \rceil)$ if and only if $\Gamma = K_v$;
- (ii) $sz(\Gamma) = 2$ if and only if Γ is a non-complete subgraph of a Cocktail-Party graph;
- (iii) $sz(\Gamma) = v - 1$ if and only if Γ is a tree;
- (iv) $2 < sz(\Gamma) < v - 1$ otherwise.

Proof. By taking all the equicuts on V , one obtains a realization of K_v with $n/t = 2 - 1/(\lceil v/2 \rceil)$. By using the left-hand side of the inequality in Lemma 1.1 and the fact that all distances in K_v are 1, we obtain that $n/t = sz(\Gamma)$, as required in (i).

The size of the Cocktail Party graph is 2, as it is easy to construct such a realization (see for instance, Theorem 3.4 below). Then observe that $sz(\Gamma) < 2$ implies that Γ is complete, as a two-path from x to y the vertex x is realized, without loss of generality, by the zero vector, and the vertex y is realized by a vector with $2t$ ones. Hence any non-complete subgraph subgraph of the Cocktail Party graph has size 2, and (ii) is proved.

To prove the last two claims, consider the decomposition (1). Choose an $s^* \in S$ with $\lambda_{s^*} > 0$; it defines the partition $V^+(s^*) \cup V^-(s^*) = V$. It follows that the subgraph of Γ induced on $V^+ = V^+(s^*)$ is isometric. Indeed, consider a shortest path between x^+ and y^+ . Let $k \in V$ lie on this path. Then

$$\begin{aligned} d_\Gamma(x^+, k) + d_\Gamma(k, y^+) - d_\Gamma(x^+, y^+) &= \\ &= \sum_{s \in S} \lambda_s (\delta_s(x^+, k) + \delta_s(k, y^+) - \delta_s(x^+, y^+)) = 0. \end{aligned}$$

As $\lambda \geq 0$, we have $\delta_{s^*}(x^+, k) + \delta_{s^*}(k, y^+) = \delta_{s^*}(x^+, y^+)$.

By definition of cut, right-hand side of the above is 0. Hence both the summands in left-hand side are 0, and $k \in V^+$. as required. Similarly, the subgraph induced on $V^-(s^*)$ is isometric.

For an edge $(x, y) \in V^+ \times V^-$ we have $d_\Gamma(x, y) = 1 \geq \lambda_s$. (Such an edge exists, as Γ is connected.) Hence $\lambda \leq 1$.

Let $\mathcal{S} \subseteq S$ denote the set of $s \in S$ satisfying $\lambda_s > 0$. Now we proceed by induction on v . Let $s' \in \mathcal{S}$. Hence the subgraph Δ^+ of Γ induced on $V^+(s')$ is isometric. Thus

$$Dist(\Delta^+) = \sum_{s \in \mathcal{S} - \{s'\}, V^+(s) \cap V^+(s') \neq \emptyset} \lambda_s \delta_{V^+(s) \cap V^+(s')}.$$

By the induction assumption,

$$X^+ := \sum_{s \in \mathcal{S} - \{s'\}, V^+(s) \cap V^+(s') \neq \emptyset} \lambda_s \leq |V^+(s')| - 1.$$

Similarly, we obtain

$$X^- := \sum_{s \in \mathcal{S} - \{s'\}, V^+(s) \cap V^-(s') \neq \emptyset} \lambda_s \leq |V^-(s')| - 1.$$

Hence

$$\begin{aligned} sz(\Gamma) = \sum_{s \in \mathcal{S}} \lambda_s &= \lambda_{s'} + X^+ + X^- \leq 1 + |V^+(s')| - 1 + |V^-(s')| - 1 = (3) \\ &= v - 1. \end{aligned}$$

Finally, if $sz(\Gamma) = v - 1$ then (3) turns into an equality, and $\lambda_{s'} = 1$. The latter and the induction implies that Γ is a tree. \square

For a tree Γ , one can moreover show the following.

Lemma 1.4 $W(K_{1,v-1}) = (v-1)^2$, $W(P_v) = v(v-1)(v+1)/6$. For any tree Γ that is neither the star $K_{1,v-1}$ nor the path P_v , the value $W(\Gamma)$ lies strictly within the extremities above.

2 Generalities on equicut graphs

Recall that a connected graph $\Gamma = (V, E)$ is called *2-connected* (or *2-vertex-connected*) if it remains connected after deletion of any vertex.

Lemma 2.1 *An equicut graph Γ with a least 4 vertices is 2-connected.*

Proof. Assume that Γ is not 2-connected. Then there exists a vertex $x \in V$ so that the subgraph induced on $V - \{x\}$ has two connected components, V_1 and V_2 . We can assume, without loss of generality, that x is represented by all-0

vector $F(x)$ in the equicut realization F of Γ . Then for any $v_i \in V_i$ ($i = 1, 2$) one has

$$d_\Gamma(v_1, v_2) = d_\Gamma(v_1, x) + d_\Gamma(x, v_2) = |v_1| + |v_2|,$$

where $|v_i|$ denotes the number of 1's in the vector $F(v_i)$. Hence the vectors $F(v_1)$ and $F(v_2)$ have disjoint supports, for any $v_1 \in V_1$ and $v_2 \in V_2$. That is, any column of F either has 1's in V_1 , or in V_2 , but never in both of them. As F is an equicut, we obtain $|V_i| \leq 1$, as required. \square

This implies

Corollary 2.2 *For any equicut graph Γ with $v \geq 4$ vertices, we have*

$$2 - 1/(\lceil v/2 \rceil) \leq sz(\Gamma) \leq v/2$$

with equality on the left if and only if $\Gamma = K_v$ and on the right if and only if $\Gamma = C_v$.

Proof. The left hand side of the inequality follows from Lemma 1.3. Then, to see that K_v is equicut, it suffices to observe that the embedding f given by the matrix F with the columns being the cuts with the smallest part of size $\lfloor v/2 \rfloor$ is an equicut embedding of K_v .

To see that C_v is equicut, observe that f given by the matrix F obtained by the cyclic shifts of the row of the form ($\lceil v/2 \rceil$ times 1, $\lfloor v/2 \rfloor$ times 0) produces an equicut realization.

To show the last claim of the lemma and the right hand inequality, it suffices to use Lemma 2.1 and the result from [Ple84] that the Wiener number $W(\Gamma)$ of a 2-connected graph on v vertices is maximal for the circuit C_v (see [Ple84, Th. 5]). \square

The condition $v \geq 4$ is necessary in the statements above. Indeed, $sz(P_3) = 2 > sz(C_3) = 3/2$. Note that P_2 and P_3 are the only equicut trees. Also, $W(C_5) = 15 < W(P_{123452})$ and $sz(C_5) = 5/2 < sz(P_{123452})$, where P_{123452} denotes the circle on $2 \dots 5$ with an extra edge attached to the vertex 2.

Remark. Γ is equicut graph if there is a realization with the binary matrix F with the column sums $\lceil v/2 \rceil$ or $\lfloor v/2 \rfloor$. If, instead of this condition, we asked that any *row* of F has exactly k 1's, then we obtain other special ℓ_1 -graph. Namely, one which embeds isometrically up to scale t , into the Johnson graph $J(v, k)$. It is observed in [Shp98] that such graphs can be recognized in polynomial time using the algorithm in [Shp93]; the ‘‘atom graph’’ of Γ is bipartite.

However, we are not aware of any similar characterization of the equicut graphs.

3 Doublings of ℓ_1 -graphs

Call an equicut ℓ_1 -graph an *antipodal doubling* if its realization (i.e. above $(0, 1)$ -matrix) has form $F = \begin{pmatrix} A & O \\ J-A & J' \end{pmatrix}$, where A is a $v/2 \times n'$ $(0,1)$ -matrix, J, J' are matrices consisting of 1's only and O is $v/2 \times (n - n')$ matrix consisting of 0's.

If moreover the matrix A is a realization, with the same scale t , of a graph Γ' , then it is straightforward to check that J' has $t(D(\Gamma') + 1) - n$ columns. Note that Double Odd graph DO_{2s+1} (see [BCN89]) with $s \geq 3$ is an example of antipodal doubling with the matrix A not corresponding to the realization of a graph Γ' , for any decomposition of F of the above form.

Remark. An antipodal doubling is exactly an ℓ_1 -graph that admits an antipodal isomorphism, i.e. it has a central symmetry (for any vertex, there is exactly one other on the distance equal to the diameter) and the mapping of all vertices into their antipodes is an isomorphism. Antipodal extensions of arbitrary ℓ_1 -metrics was considered in [DeLa97, Sect. 7.2].

In order to investigate when one can construct an ℓ_1 -graph from an ℓ_1 -graph via the antipodal doubling (see Theorem 3.4 below), let us introduce the following definition. For a graph $\Gamma = (V, E)$, define its *diametral doubling* as the graph $\square\Gamma$ with the vertex set $V^+ \cup V^-$ (where V^+ and V^- are copies of V) and the adjacency as follows: x^μ is adjacent to y^ϵ if $\mu = \epsilon$ and $(x, y) \in E$, or if $\mu \neq \epsilon$ and $d_\Gamma(x, y) = D(\Gamma)$.

Lemma 3.1 *The subgraphs of $\square\Gamma$ induced on V^* are isometric to Γ if and only if*

$$d_\Gamma(x, y) \leq 2 + d_\Gamma(z_1, z_2) \text{ for any } x, y, z_1, z_2 \in V \quad (4)$$

satisfying $d_\Gamma(x, z_1) = d_\Gamma(y, z_2) = D(\Gamma)$.

Proof. Clearly, the condition of the lemma is necessary for the subgraphs induced on V^* to be geodesic. Indeed, otherwise there can be a shorter path between, say, x^+ and y^+ that passes through a vertex in V^- .

To prove sufficiency, observe that any geodesic between p^+ and q^+ either lies wholly in V^+ , or contains a path of the form $x^+, z_1^-, \dots, z_2^-, y^+$, with the (sub)path from z_1^- to z_2^- lying wholly in V^- . In the former case there is nothing to prove. In the latter case, by definition of $\square\Gamma$, one has that $d_\Gamma(x, z_1) = d_\Gamma(y, z_2) = D(\Gamma)$. By the assumption, there is a path of length at most $2 + d_\Gamma(z_1, z_2)$ between x and y . Thus the geodesic can be replaced by another geodesic with the path from x^+ to y^+ lying wholly in V^+ . \square

Lemma 3.2 *Let Γ satisfy the condition of Lemma 3.1. Then*

$$d_{\square\Gamma}(x^+, y^-) = D(\Gamma) + 1 - d_\Gamma(x, y) \quad (5)$$

if and only if any geodesic in Γ lies on a geodesic of length $D(\Gamma)$.

Proof. We begin by proving that a geodetic (in Γ) between x and y can be extended to a geodetic of length $D(\Gamma)$. By Lemma 3.1, there is a geodetic from x^+ to y^- of the form $x^+, \dots, z_1^+, z_2^-, \dots, y^-$ such that the path from x^+ to z_1^+ (respectively, from y^- to z_2^-) lies wholly in V^+ (respectively, in V^-).

We claim that there is a geodetic between z_1 and z_2 (by definition of $\square\Gamma$ they are at distance $D(\Gamma)$) that contains x and y . Clearly,

$$d_{\square\Gamma}(x^+, y^-) = d_\Gamma(x, z_1) + 1 + d_\Gamma(z_2, y).$$

Using (5), we derive

$$D(\Gamma) + 1 - d_\Gamma(x, y) = d_\Gamma(x, z_1) + 1 + d_\Gamma(z_2, y).$$

Hence

$$D(\Gamma) = (d_\Gamma(z_1, z_2) =) d_\Gamma(x, z_1) + d_\Gamma(z_2, y) + d_\Gamma(x, y),$$

as claimed.

Now, suppose that a geodetic between x and y can be extended to a geodetic of length $D(\Gamma)$, say, between z_1 and z_2 . By reversing the argument above, we derive (5). \square

Certain properties of Γ are inherited by $\square\Gamma$.

Lemma 3.3 *Let $\Gamma = (V, E)$ be a graph satisfying (4) and (5). Then $\square\Gamma$ satisfies $D(\square\Gamma) = D(\Gamma) + 1$, (4) and (5).*

Proof. As (4) and (5) hold for Γ , we have that each $p^+ \in V^+$ has exactly one vertex p at the maximal distance $D(\square\Gamma) = D(\Gamma) + 1$. Now checking (4) is straightforward.

To check (5), take a geodetic between x^+ and y^- . Note that

$$d_{\square\Gamma}(x^+, y^-) = D(\Gamma) + 1 - d_\Gamma(x, y).$$

Hence

$$d_{\square\Gamma}(x^+, y^-) = d_{\square\Gamma}(y^+, y^-) - d_{\square\Gamma}(x^+, y^+), \quad (6)$$

and the geodetic in question lies on the path from y^+ to y^- , as required. For a geodetic between x^+ and y^+ observe that (6) also proves that it is extendable to the path from y^+ to y^- . \square

If $D(\Gamma) = 2$ then Γ satisfies (4). Moreover, $\Gamma \neq K_v$ satisfies (5), unless it has an edge (x, y) with $\Gamma(x) = \Gamma(y)$. In particular, any strongly regular graph satisfies (4) and (5). Note that ℓ_1 -graphs form a rather small sub-family of strongly regular graphs.

Switchings. Not always $\Delta = \square\Gamma$ is uniquely defining the graph Γ it was constructed from; it can be that $\Delta = \square\Gamma'$ for $\Gamma' \not\cong \Gamma$. See Section 4 for many examples of this situation.

Γ and Γ' are related by the following graph operation. The *diametral switching* of a graph Γ with respect to $S \subset V(\Gamma)$ is a graph Γ' that is obtained from Γ by retaining the edges that lie within $S \times S \cup (V - S) \times (V - S)$ and replacing the set of edges from $S \times (V - S)$ with the set $\{(x, y) \in S \times (V - S) \mid d_\Gamma(x, y) = D(\Gamma)\}$.

Note that *Seidel switching* (see [BCN89]) is an operation that coincides with the diametral switching for graphs of diameter 2.

Theorem 3.4 *Let $\Gamma = (V, E)$ be a ℓ_1 -graph. Then $\square\Gamma$ is an ℓ_1 -graph if Γ satisfies (4), (5) and*

$$sz(\Gamma) \leq D(\Gamma) + 1, \quad (7)$$

Moreover, if $\square\Gamma$ is an ℓ_1 -graph then

- $D(\square\Gamma) = D(\Gamma) + 1$, $\square\Gamma$ satisfies (4), (5) and (7) with equality, and
- All the ℓ_1 -realizations of $\square\Gamma$ are equicut, of the form, up to permutation of rows and columns, and taking complements of columns, $\begin{pmatrix} A & O \\ J^-_A & J' \end{pmatrix}$ with $t(D(\Gamma) + 1)$ columns, so that A is an ℓ_1 -realization of Γ with scale t .

Proof. It is obvious from $sz(\Gamma) \leq D(\Gamma) + 1$ that there exists a $(v \times n, \text{ say})$ matrix A that determines an ℓ_1 -embedding of Γ such that $n/t_A \leq D(\Gamma) + 1$. Let $\lambda = t_A(D(\Gamma) + 1) - n$ and $B = \begin{pmatrix} A & O \\ J^-_A & J' \end{pmatrix}$, where J' has λ columns. Identify the first v rows of B with V^+ and the last v rows of B with V^- .

By construction, the Hamming distances between rows of B are equal, up to scale $t = t_A$, to the corresponding distances in $\square\Gamma$. Hence $\square\Gamma$ is an ℓ_1 -graph. In view of Lemma 3.3, (4) and (5) hold for $\square\Gamma$.

$\square\Gamma$ satisfies (7) with equality, as by construction, $sz(\square\Gamma) = D(\Gamma) + 1$.

Taking an ℓ_1 -realization B' of $\square\Gamma$, we can reorder the rows of B' so that, as in B , the first half of the rows corresponds to V^+ , and the second half corresponds to V^- . By reordering the columns and possibly taking their complements, we reduce B' to the desired form. The rest of the claim follows from the fact that Γ induced on V^+ is isometric in $\square\Gamma$. \square

Remarks. $K_4 - P_3$, $K_4 - P_2$, the Dynkin diagram E_6 are examples of ℓ_1 -graphs satisfying (4), (7), but not (5). *Affine* Dynkin diagram \tilde{E}_6 An example of a graph that does not satisfy (4) and (7), but does satisfy (5).

Note that any Γ with $D(\Gamma) = 2$ satisfies (4) and (5). Certainly, not all of them are ℓ_1 -graphs, for instance, $K_{2,3}$. Also, not all the ℓ_1 -graphs of diameter 2 satisfy (7), for instance, $K_{1,4}$.

In general, for any ℓ_1 -graph $\Gamma = (V, E)$ with $|V| \geq 4$ one has

$$D(\Gamma) \leq sz(\Gamma) \leq D(\Gamma) + |V| - 3. \quad (8)$$

The equality at the right-hand side of (8) holds if and only if Γ is a star, as can be seen by applying Theorem 1.3

Theorem 3.4 generalizes to arbitrary ℓ_1 -graphs Γ the situation for the Cocktail Party graph $K_{n \times 2}$ considered in [DeLa97, Sect. 7.4], see also second part of Lemma 1.3. It implies that the minimal scale of an ℓ_1 -embedding of $\square\Gamma$ equals the minimal $t > 0$ so that the metric $t \cdot \text{Dist}(\Gamma)$ embeds isometrically in $H^{t(D(\Gamma)+1)}$. In particular case of $\Gamma = K_{4a}$ Lemma 7.4.6 of [DeLa97] states that such a minimal $t \geq 2a$, with the equality holding if and only if there exists a Hadamard matrix of order $4a$.

4 Examples

In this section we list many examples in no particular order. An equicut ℓ_1 -graph Γ is called a *doubling* if it is an antipodal doubling; moreover, if it is obtained from a graph Δ as in Theorem 3.4, we give such a representation $\Gamma = \square\Delta$.

All equicut graphs with at most 6 vertices are C_v ($2 \leq v \leq 6$), K_4 , K_5 , K_6 , P_3 , 4-wheel and the octahedron. Among those 11 graphs only K_4 , K_5 , K_6 are not rigid and only C_4 , C_6 and the octahedron are doublings.

Cartesian product $\Gamma \times \Gamma'$ of two ℓ_1 -graphs Γ and Γ' is ℓ_1 -graph if and only if both Γ , Γ' are (see [DeLa97, Sect. 7.5]); the scale will be least common multiple of their scales and the size will be the sum of their sizes. Moreover, $\Gamma \times \Gamma'$ is rigid if and only if they are.

Lemma 4.1 *If Γ , Γ' have even number of vertices each, then $\Gamma \times \Gamma'$ is an equicut graph if and only if they are.*

Proof. Let F and F' be ℓ_1 -realizations of Γ and Γ' , respectively. Then

$$B = \begin{pmatrix} F_1 & F' \\ \dots & \dots \\ F_1 & F' \\ F_2 & F' \\ \dots & \dots \\ F_2 & F' \\ \dots & \dots \\ F_v & F' \\ \dots & \dots \\ F_v & F' \end{pmatrix}, \text{ where } F_i \text{'s are rows of } F, \text{ is an } \ell_1\text{-realization of } \Gamma \times \Gamma'. \text{ Now}$$

let F and F' be equicut realizations. It is straightforward that B is an equicut realization when Γ and Γ' both have even number of vertices. \square

A *Doob* graph (the Cartesian product of a number of copies of *Shrikhande graph* and a number of copies of K_4 , see e.g. [BCN89, p.27]) is an example of an equicut graph obtained via Lemma 4.1. It is a (non-rigid and non-doubling) ℓ_1 -graph of scale 2.

Distance-regular graphs. Here we freely use notation from [BCN89]. Also, a significant use is made of [KoSh94]. The Petersen graph and the Shrikhande graph are both equicut graphs of scale 2 and size 3; both are rigid and are not

doublings. The Double Odd graph DO_{2s+1} is an equicut graph of scale 1 and size $2s + 1$. The half- s -cube $\frac{1}{2}H(s, 2)$ is an equicut graph of scale 2 and size s . It is not rigid only for $s = 3, 4$; it is a doubling if and only if s is even. The Johnson graph $J(2s, s)$ is a non-rigid doubling of scale 2 and size s .

Further, the following are distance-regular equicut graphs.

1. All Taylor ℓ_1 -graphs - $\frac{1}{2}H(6, 2)$, $J(6, 3)$, C_6 , $H(3, 2)$, the Icosahedron. They are all doublings of diameter 3 and size 3 that can be constructed using Theorem 3.4 above.
2. All strongly regular ℓ_1 -graphs, except $J(s, 2)$ ($s \geq 5$) and grids $H(2, s)$ with s odd. That is, C_5 , Petersen, $\frac{1}{2}H(5, 2)$, Shrikhande, $H(2, s)$ with s even, and $K_{s \times 2}$.
3. Among distance regular graphs Γ with $D(\Gamma) > 2$ and $\mu > 1$: $\frac{1}{2}H(s, 2)$ with $s > 5$, $H(s, d)$, $J(s, t)$ with $t > 2$, the Icosahedron, and Doob graphs.
4. All Coxeter ℓ_1 -graphs except $J(s, t)$ with $t < s/2$: $J(2s, s)$, the Icosahedron, the Dodecahedron, $K_{s \times 2}$, $\frac{1}{2}H(s, 2)$, $H(s, 2)$, C_s ($s \geq 5$).
5. All cubic distance-regular ℓ_1 -graphs: K_4 , Petersen, $H(3, 2)$, DO_5 , the Dodecahedron.

Also all amply regular ℓ_1 -graphs with $\mu > 1$ are equicut graphs. Yet another example is given by the 12-vertex co-edge regular subgraph of the Clebsch graph $\frac{1}{2}H(5, 2)$, see [BCN89, Sect. 3.11, p. 104]; it is an equicut graph of size $5/2$, scale 2, non-doubling.

Some equicut graphs which are doublings of ℓ_1 -graphs (see some in Section 7.2 of [DeLa97]):

1. $C_{2s} = \square P_s$.
2. $K_{s \times 2} = \square K_s$.
3. $H(s, 2) = \square H(s - 1, 2)$.
4. $J(2s, s) = \square J(2s - 1, s)$.
5. $\frac{1}{2}H(2s, 2) = \square \frac{1}{2}H(2s - 1, 2)$.
6. $Prism_{2s} = \square C_{2s} = \square L_{2s}$; here L_{2s} is the "ladder" on $2s$ vertices. L_{2s} is the diametral switching of C_{2s} with respect to the vertex set of a path $P_s \subset C_{2s}$.
7. $APrism_{2s+1} = \square C_{2s+1}$.
8. The Dodecahedron is the doubling of the 9-circle $(1, \dots, 9)$ with extra vertex connected to the vertices 3, 6, 9 of the circle.

9. The Icosahedron is the doubling of 5-wheel; as well, it is the doubling of the graph obtained from hexagon $(1, \dots, 6)$ by adding edges $(2i, 2j)$ for $0 < i < j < 4$.
10. $J(6, 3)$ is the doubling of Petersen graph, in addition to 4.
11. $\frac{1}{2}H(6, 2)$ is the doubling of Shrikhande graph, and the doubling of $H(2, 4)$ (more precisely, of its realization in half-6-cube) in addition to 5.

In 9, 10, 11 we have (diametral) switching-equivalent graphs Γ such that $\square\Gamma$ is a Taylor ℓ_1 -graph; see remark preceding Theorem 3.4. This situation in general is well-known; for instance, the Gosset graph, a Taylor graph that is not an ℓ_1 -graph, can be obtained as the diametral doubling of one of 5 non-isomorphic, but switching-equivalent, graphs. For definitions and discussion of this situation in more general setting, see e.g. [BCN89, pp.103-105].

Equicut polytopes. The skeletons of many nice polytopes are equicut graphs. Below we list several such examples. We follow the terminology from [Joh66, Cox73].

All five Platonic solids have equicut skeletons; all, except the Tetrahedron, are rigid. All but the cube (of scale 1), have scale 2. The sizes for the Tetrahedron, the Octahedron, the Cube, the Icosahedron and the Dodecahedron are $3/2$, 2, 3, 3 and 5.

The skeleton of any *zonotope* (see e.g. [Zie95] for a definition of zonotope) is a doubling and it has scale 1, so it is rigid.

Among the Archimedean ℓ_1 -polytopes

1. all zonohedra (i.e. 3-dimensional zonotopes) are as follows: the truncated Octahedron, the truncated Cuboctahedron, the truncated Icosidodecahedron and $Prism_{2s}$ ($s > 2$) with sizes 6, 9, 15 and $s + 1$, respectively;
2. all the other doublings are as follows: the Rhombicuboctahedron, the Rhombicosidodecahedron and $Aprism_{2s+1}$ ($s > 1$) with scale 2 and sizes 5, 8 and $s + 1$, respectively;
3. all remaining equicut polytopes - the snub Cube, the snub Dodecahedron and $Aprism_{2s}$ ($s > 1$) - all have scale 2 and sizes $9/2$, $15/2$ and $s + 1/2$, respectively;
4. the remaining, $Prism_{2s+1}$ ($s \geq 1$) has scale 2 and size $s + 3/2$; it is not an equicut graph.

Among all Catalan (dual Archimedean) ℓ_1 -polyhedra:

1. all the zonohedra are as follows: dual Cuboctahedron and dual Icosidodecahedron of sizes 4 and 6, respectively;
2. the only other doubling is dual truncated Icosahedron (also known as dual football) of scale 2 and size 5;

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