On the Fatness of Minkowski Sums

Mark de Berg A. Frank van der Stappen

Department of Computer Science, Utrecht University,
P.O.Box 80089, 3508 TB Utrecht, the Netherlands.

Abstract

Let $\mathcal{A}$ and $\mathcal{B}$ be two connected, closed, and bounded sets in $\mathbb{E}^d$. Let $\mathcal{A} \oplus \mathcal{B}$ denote the Minkowski sum (that is, the vector sum) of $\mathcal{A}$ and $\mathcal{B}$. We prove two results concerning the fatness of $\mathcal{A} \oplus \mathcal{B}$. First, we prove that

$$\text{fatness}(\mathcal{A} \oplus \mathcal{B}) \geqslant \min(\text{fatness}(\mathcal{A}), \text{fatness}(\mathcal{B})).$$

In addition, we show that if $\text{diam}(\mathcal{A}) \geqslant \text{diam}(\mathcal{B})$, where diam() denotes the diameter, then we have

$$\text{fatness}(\mathcal{A} \oplus \mathcal{B}) \geqslant \text{fatness}(\mathcal{A}) \cdot \left(\frac{\text{diam}(\mathcal{A})}{\text{diam}(\mathcal{A}) + \text{diam}(\mathcal{B})} \right)^{d-1}.$$

Both bounds are tight in the worst case.

Keywords: Computational geometry, Minkowski sums, fatness.

1 Introduction

The *Minkowski sum* [4] of two sets $\mathcal{A}$ and $\mathcal{B}$, which we denote by $\mathcal{A} \oplus \mathcal{B}$, is the vector sum of the two sets:

$$\mathcal{A} \oplus \mathcal{B} := \{a + b : a \in \mathcal{A} \text{ and } b \in \mathcal{B}\},$$

where $a+b$ denotes the vector sum of a and b. Minkowski sums play an important role in many areas. One example is motion planning [7], where the forbidden space of a translating robot R can be described as the union of the Minkowski sums of the obstacles in the scene with $-R$ (a copy of R reflected in the origin). Another example comes from image processing [9, 10]: the *dilation* of a set A with a so-called structuring element B is just the Minkowski sum of A and B. The structuring element is usually a simple figure like (in the plane) a square or a disc. In Geographic Information Systems (GIS) the term *buffer* is often used to denote the Minkowski sum of a given set with a disc [5].

The goal of taking Minkowski sums with a disc can be to 'round' objects, that is, to get rid of small details like slivers in the image. In this case the Minkowski sum operation is usually followed by an operation called *erosion* to reduce the size of the shape to roughly its original size. The motivation for our research stems from this type of applications: we want to quantify the statement that taking the Minkowski sum with a disc rounds an object. As a measure for the roundness of an object we use the concept of *fatness*. There are many different definitions of fatness [1, 2, 6, 8], which are all more or less equivalent—at least for convex objects. We shall use the definition proposed by de Berg et al. [3], which is similar to the one used in van der Stappen's thesis [11]. The reason to choose this definition is that it is one of the most general ones: it can be used for objects in any dimension, and the objects need not be convex or polygonal.

The definition is as follows. Let $\mathrm{vol}(\mathcal{A})$ denote the volume of an object $\mathcal{A}$. Here and in the sequel we use the term *object* to denote a connected, closed, and bounded set in $\mathbb{E}^d$.

Definition 1.1 *Let $\mathcal{A} \subseteq \mathbb{E}^d$ be an object. Define $U(\mathcal{A})$ as the set of all balls centered inside $\mathcal{A}$ whose boundary intersects $\mathcal{A}$. The* fatness *of $\mathcal{A}$, denoted* $\mathrm{fatness}(\mathcal{A})$, *is defined as*

$$\mathrm{fatness}(\mathcal{A}) := \min_{B \in U(\mathcal{A})} \frac{\mathrm{vol}(\mathcal{A} \cap B)}{\mathrm{vol}(B)}.$$

In words, the fatness of an object is determined by the emptiest ball centered inside the object and not fully containing it in its interior. The fatness of an object is invariant under translation and rotation; it is purely a function of the shape of the object. The fatness of an object can be seen as a measure for its 'sliveriness': if it contains thin slivers its fatness will generally be close to zero, whereas fairly round objects have large fatness. (In the plane 'large' means 'close to 1/4', because the fattest object in the plane, the disc, has fatness 1/4.)

Our results show that the fatness of an object indeed increases when we take the Minkowski sum with a disc. In fact, our result is a lot more general. For two connected, closed, and bounded sets $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{E}^d$ we prove that

$$\mathrm{fatness}(\mathcal{A} \oplus \mathcal{B}) \geqslant \min(\mathrm{fatness}(\mathcal{A}), \mathrm{fatness}(\mathcal{B})).$$

Since the global shape of $A \oplus B$ is to a large extent determined by the shape of the larger of $\mathcal{A}$ and $\mathcal{B}$, we also study how the fatness of $\mathcal{A} \oplus \mathcal{B}$ depends on the fatness of the larger of $\mathcal{A}$ and $\mathcal{B}$. In particular, we show that if $\mathrm{diam}(\mathcal{A})$, the diameter of $\mathcal{A}$ is larger than or equal to $\mathrm{diam}(\mathcal{B})$, the diameter of $\mathcal{B}$, then we have

$$\mathrm{fatness}(\mathcal{A} \oplus \mathcal{B}) \geqslant \mathrm{fatness}(\mathcal{A}) \cdot \left(\frac{\mathrm{diam}(\mathcal{A})}{\mathrm{diam}(\mathcal{A}) + \mathrm{diam}(\mathcal{B})} \right)^{d-1}.$$

As a result we have $\mathrm{fatness}(\mathcal{A} \oplus \mathcal{B}) \geqslant \mathrm{fatness}(\mathcal{A})/2^{d-1}$. All bounds are tight in the worst case.

2 A bound in terms of the least fat object

Let $\mathcal{A}$ and $\mathcal{B}$ be two objects—connected, closed bounded sets—in $\mathbb{E}^d$. We want to prove that the minimum of the fatness of $\mathcal{A}$ and the fatness of $\mathcal{B}$ is a lower bound on the fatness of their Minkowski sum. Before we can do so, we need to introduce some notation and give some preliminary lemmas.

We let the diameter of an object $\mathcal{A}$, denoted by $\mathrm{diam}(\mathcal{A})$, be the largest distance between two points in $\mathcal{A}$. Throughout the entire paper, the diameter of an object will serve as a measure for its size. We define ω_d to be the volume of the d-dimensional unit ball. (For even dimensions $\omega_d = \pi^{d/2}/(d/2)!$ and for odd dimensions $\omega_d = 2(2\pi)^{\lfloor d/2 \rfloor}/d!!$, where $d!! = d \cdot (d-2) \cdot \ldots \cdot 3 \cdot 1$.)

Lemma 2.1 gives a bound on the volume of an object given its fatness and diameter.

Lemma 2.1 *Let $\mathcal{A}$ be a d-dimensional object. Then* $\mathrm{vol}(\mathcal{A}) \geqslant \omega_d \cdot \mathrm{fatness}(\mathcal{A}) \cdot \mathrm{diam}(\mathcal{A})^d$, *with equality for convex $\mathcal{A}$.*

Proof: There exist two points $m, m' \in \mathcal{A}$ that are a distance $\mathrm{diam}(\mathcal{A})$ apart. The boundary of the ball B with radius $\mathrm{diam}(\mathcal{A})$ centered at m intersects A (in m'). As a result, the ball B belongs to $U(\mathcal{A})$ and therefore satisfies $\mathrm{fatness}(\mathcal{A}) \leqslant \mathrm{vol}(\mathcal{A} \cap B)/\mathrm{vol}(B)$. We have that $\mathrm{vol}(\mathcal{A}) \geqslant \mathrm{vol}(\mathcal{A} \cap B) \geqslant \mathrm{fatness}(\mathcal{A}) \cdot \mathrm{vol}(B) = \mathrm{fatness}(\mathcal{A}) \cdot \omega_d \cdot \mathrm{diam}(\mathcal{A})^d$. For convex $\mathcal{A}$, the ball B turns out to be the emptiest ball [11]. $\square$

The shape of the Minkowski sum depends on the shapes of $\mathcal{A}$ and $\mathcal{B}$ and on their orientations, but it is invariant under translation: the Minkowski sum of translated copies of $\mathcal{A}$ and $\mathcal{B}$ is a translated copy of the Minkowski sum of $\mathcal{A}$ and $\mathcal{B}$. Since we are interested in the fatness of Minkowski sums, which is invariant under translations, we can (and will throughout the entire paper) assume that objects $\mathcal{A}$ and $\mathcal{B}$ are placed such that they both contain the origin, which we denote by O. We denote by $\mathcal{A}(p)$ the copy of $\mathcal{A}$ obtained by translating A along a vector $p \in \mathbb{E}^d$, so $\mathcal{A}(p) = \mathcal{A} \oplus p$. (Given a set $\mathcal{A}$ and a point (or vector) p, we write $\mathcal{A} \oplus p$ as a shorthand for $\mathcal{A} \oplus \{p\}$.) Similarly, $\mathcal{B}(p)$ is obtained by translating $\mathcal{B}$ along p. With this notation we have the following observation, which follows immediately from the definition of Minkowski sums.

Observation 2.2 $\mathcal{A} \oplus \mathcal{B} = \bigcup_{p \in \mathcal{B}} \mathcal{A}(p) = \bigcup_{q \in \mathcal{A}} \mathcal{B}(q)$

This observation implies that for any point $m \in \mathcal{A} \oplus \mathcal{B}$ we can place a copy $\mathcal{A}(p)$ of $\mathcal{A}$ with $p \in \mathcal{B}$ (or a copy $\mathcal{B}(q)$ of $\mathcal{B}$ with $q \in \mathcal{A}$) that contains m and is contained in $\mathcal{A} \oplus \mathcal{B}$. Given a copy $\mathcal{A}(p)$ with $p \in \mathcal{B}$, we observe that $\mathcal{B}(-p)$ satisfies $\mathcal{A}(p) \oplus \mathcal{B}(-p) = \mathcal{A} \oplus \mathcal{B}$ as well as $O \in \mathcal{B}(-p)$.

Our proof relies on the Brunn-Minkowski Inequality [12] which is given below as Lemma 2.3 for the particular case of two objects; it provides a lower bound on the volume of a Minkowski sum in terms of the volumes of its constituents.

Lemma 2.3 *Let $\mathcal{A}$ and $\mathcal{B}$ be objects in $\mathbb{E}^d$. Then*

$$\sqrt[d]{\mathrm{vol}(\mathcal{A} \oplus \mathcal{B})} \geqslant \sqrt[d]{\mathrm{vol}(\mathcal{A})} + \sqrt[d]{\mathrm{vol}(\mathcal{B})}.$$

We are now ready to bound the fatness of the Minkowski sum $\mathcal{A} \oplus \mathcal{B}$.

Theorem 2.4 *Let $\mathcal{A}$ and $\mathcal{B}$ be objects in $\mathbb{E}^d$. Then*

$$\mathrm{fatness}(\mathcal{A} \oplus \mathcal{B}) \geqslant \min(\mathrm{fatness}(\mathcal{A}), \mathrm{fatness}(\mathcal{B})).$$

This bound is tight in the worst case.

Proof: Let $B \in U(\mathcal{A} \oplus \mathcal{B})$ be such that $\mathrm{fatness}(\mathcal{A} \oplus \mathcal{B}) = \mathrm{vol}((\mathcal{A} \oplus \mathcal{B}) \cap B)/\mathrm{vol}(B)$ and let r be the radius of B. The center m of B is contained in a copy $\tilde{\mathcal{A}} = \mathcal{A}(p)$ of $\mathcal{A}$ for some $p \in \mathcal{B}$. From Observation 2.2 we know that there exists a copy $\tilde{\mathcal{B}}$ of $\mathcal{B}$ satisfying $O \in \tilde{\mathcal{B}}$ and $\tilde{\mathcal{A}} \oplus \tilde{\mathcal{B}} = \mathcal{A} \oplus \mathcal{B}$—see the text below the observation. The latter equality implies that $\mathrm{fatness}(\mathcal{A} \oplus \mathcal{B}) = \mathrm{vol}((\tilde{\mathcal{A}} \oplus \tilde{\mathcal{B}}) \cap B)/\mathrm{vol}(B)$.

If the boundary of B intersects $\tilde{\mathcal{A}}$ then $B \in U(\tilde{\mathcal{A}})$. We have

$$\mathrm{fatness}(\tilde{\mathcal{A}} \oplus \tilde{\mathcal{B}}) = \frac{\mathrm{vol}((\tilde{\mathcal{A}} \oplus \tilde{\mathcal{B}}) \cap B)}{\mathrm{vol}(B)} \geqslant \frac{\mathrm{vol}(\tilde{\mathcal{A}} \cap B)}{\mathrm{vol}(B)} \geqslant \mathrm{fatness}(\tilde{\mathcal{A}}) = \mathrm{fatness}(\mathcal{A})$$

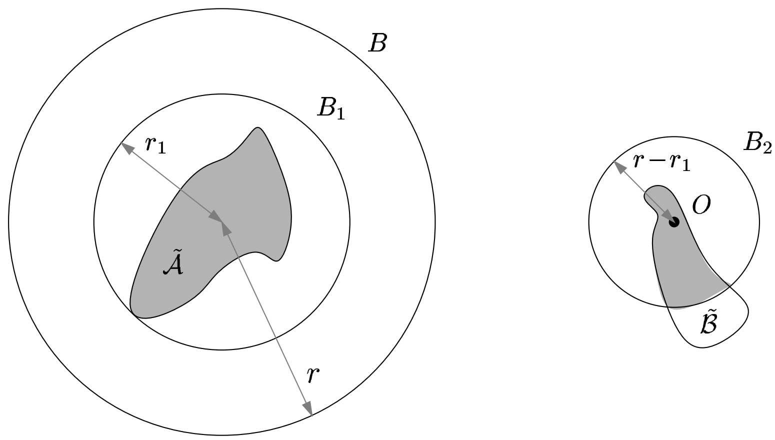

Figure 1: The Minkowski sum of the shaded shapes $\mathcal{A}$ and $\mathcal{B}' = \mathcal{B} \cap B_2$ fits completely inside the emptiest ball B. The lower bounds on the volumes of $\mathcal{A}$ and $\mathcal{B}'$ and the Brunn-Minkowski Inequality imply a lower bound on the volume of $\mathcal{A} \oplus \mathcal{B}'$, and, hence, of $(\mathcal{A} \oplus \mathcal{B}) \cap B$.

and we are done.

Alternatively, $\tilde{\mathcal{A}}$ is completely contained in B. We define B_1 to be the largest ball concentric with B whose boundary intersects $\tilde{\mathcal{A}}$. Notice that no point of $\tilde{\mathcal{A}}$ lies outside B_1—see Figure 1. Let r_1 be the radius of B_1. We have $r_1 \leqslant \operatorname{diam}(\tilde{\mathcal{A}}) = \operatorname{diam}(\mathcal{A})$.

We consider $\tilde{\mathcal{B}}$, which is known to contain O. Let B_2 be the ball with center O and radius $r - r_1$. The object $\tilde{\mathcal{B}}$ cannot lie entirely in the interior of B_2, because then $\tilde{\mathcal{A}} \oplus \tilde{\mathcal{B}}$ would be completely in the interior of $B_1 \oplus B_2$ and thus be completely in the interior of B, contradicting $B \in U(\tilde{\mathcal{A}} \oplus \tilde{\mathcal{B}}) = U(\mathcal{A} \oplus \mathcal{B})$. We conclude that $B_2 \in U(\tilde{\mathcal{B}})$. Defining $\tilde{\mathcal{B}}' = \tilde{\mathcal{B}} \cap B_2$, we obtain

$$\operatorname{vol}(\tilde{\mathcal{B}}') = \operatorname{vol}(\tilde{\mathcal{B}} \cap B_2) \geqslant \operatorname{fatness}(\mathcal{B}) \cdot \operatorname{vol}(B_2) = \omega_d \cdot \operatorname{fatness}(\mathcal{B}) \cdot (r - r_1)^d.$$

The Minkowski sum $\tilde{\mathcal{A}} \oplus \tilde{\mathcal{B}}'$ is clearly a subset of $\tilde{\mathcal{A}} \oplus \tilde{\mathcal{B}}$. Since it lies entirely inside $B_1 \oplus B_2$, the Minkowski sum $\tilde{\mathcal{A}} \oplus \tilde{\mathcal{B}}'$ is also completely contained in B. Using Lemmas 2.1 and 2.3, and letting $f_{\mathcal{A}} := \operatorname{fatness}(\mathcal{A})$ and $f_{\mathcal{B}} := \operatorname{fatness}(\mathcal{B})$, we have

$$\begin{aligned}
\operatorname{fatness}(\tilde{\mathcal{A}} \oplus \tilde{\mathcal{B}}) &= \operatorname{vol}((\tilde{\mathcal{A}} \oplus \tilde{\mathcal{B}}) \cap B)/\operatorname{vol}(B) \\
&\geqslant \operatorname{vol}(\tilde{\mathcal{A}} \oplus \tilde{\mathcal{B}}')/\operatorname{vol}(B) \\
&\geqslant \left(\sqrt[d]{\operatorname{vol}(\tilde{\mathcal{A}})} + \sqrt[d]{\operatorname{vol}(\tilde{\mathcal{B}}')} \right)^d /\operatorname{vol}(B) \\
&\geqslant \left(\sqrt[d]{\omega_d \cdot f_{\mathcal{A}} \cdot \operatorname{diam}(\mathcal{A})^d} + \sqrt[d]{\omega_d \cdot f_{\mathcal{B}} \cdot (r - r_1)^d} \right)^d \cdot \left(\omega_d r^d\right)^{-1} \\
&\geqslant \left(\sqrt[d]{\omega_d \cdot \min(f_{\mathcal{A}}, f_{\mathcal{B}}) \cdot r_1^d} + \sqrt[d]{\omega_d \cdot \min(f_{\mathcal{A}}, f_{\mathcal{B}}) \cdot (r - r_1)^d} \right)^d \cdot \left(\omega_d r^d\right)^{-1} \\
&= r^d \cdot \left(\sqrt[d]{\omega_d \cdot \min(f_{\mathcal{A}}, f_{\mathcal{B}})} \right)^d \cdot \left(\omega_d r^d\right)^{-1} \\
&= \min(f_{\mathcal{A}}, f_{\mathcal{B}}),
\end{aligned}$$

which is the desired result.

An easy example shows that the bound is tight: the Minkowski sum of two balls $\mathcal{A}$ and $\mathcal{B}$ is again a ball and as such equally fat as both $\mathcal{A}$ and $\mathcal{B}$. $\square$

3 A bound in terms of the largest object

We now deduce a bound on the fatness of the Minkowski sum of two objects $\mathcal{A}$ and $\mathcal{B}$ that depends only on the fatness of the larger of $\mathcal{A}$ and $\mathcal{B}$. The constant of proporionality in this bound is determined by the diameters of $\mathcal{A}$ and $\mathcal{B}$. By leaving the fatness of the smaller of $\mathcal{A}$ and $\mathcal{B}$ out of our considerations, we can no longer use Lemma 2.1 to get a lower bound on the volume of this object. Instead we will use the elementary observation that any connected part of the smaller object contains a curve whose diameter is at least the diameter of the part. Lemma 3.1 provides a lower bound on the volume of the Minkowski sum of an object and a curve in terms of the diameter of the curve and the volume, diameter, and fatness of the object.

Lemma 3.1 *Let $\mathcal{A}$ be an object and s be a curve in $\mathbb{E}^d$. Then*

$$\mathrm{vol}(\mathcal{A} \oplus s) \geqslant \mathrm{vol}(\mathcal{A}) + \omega_d \cdot \mathrm{fatness}(\mathcal{A}) \cdot \mathrm{diam}(s) \cdot \mathrm{diam}(\mathcal{A})^{d-1}.$$

Proof: Let p and q be two points on s that are $\mathrm{diam}(s)$ apart. For ease of discussion we simultaneously rotate and translate s and $\mathcal{A}$ until $p = (0, 0, \ldots, 0)$ and $q = (\mathrm{diam}(s), 0, \ldots, 0)$; this does not affect the fatness of $\mathcal{A} \oplus s$.

Let $P[c]$ denote the plane $x_1 = c$ and let I be the interval on the x_1-axis consisting of all c for which $\mathcal{A} \cap P[c] \neq \emptyset$. Note that $\mathrm{length}(I) \leqslant \mathrm{diam}(\mathcal{A})$. We claim that there exists a $c \in I$ such that the $(d-1)$-dimensional volume of $\mathcal{A} \cap P[c]$, denoted by $\mathrm{vol}_{d-1}(A \cap P[c])$, satisfies

$$\mathrm{vol}_{d-1}(A \cap P[c]) \geqslant \omega_d \cdot \mathrm{fatness}(\mathcal{A}) \cdot \mathrm{diam}(\mathcal{A})^{d-1}.$$

Assume, for a contradiction that $\mathrm{vol}_{d-1}(A \cap P[x_1]) < \omega_d \cdot \mathrm{fatness}(\mathcal{A}) \cdot \mathrm{diam}(\mathcal{A})^{d-1}$, for all $x_1 \in I$. Then

$$\begin{aligned} \mathrm{vol}(\mathcal{A}) &= \int_I \mathrm{vol}_{d-1}(A \cap P[x_1]) dx_1 \\ &< \omega_d \cdot \mathrm{fatness}(\mathcal{A}) \cdot \mathrm{diam}(\mathcal{A})^{d-1} \cdot \int_I dx_1 \\ &\leqslant \omega_d \cdot \mathrm{fatness}(\mathcal{A}) \cdot \mathrm{diam}(\mathcal{A})^{d}, \end{aligned}$$

contradicting the lower bound on $\mathrm{vol}(\mathcal{A})$ given by Lemma 2.1.

We now translate $\mathcal{A}$ such that

$$\mathrm{vol}_{d-1}(\mathcal{A} \cap P[0]) \geqslant \omega_d \cdot \mathrm{fatness}(\mathcal{A}) \cdot \mathrm{diam}(\mathcal{A})^{d-1}.$$

See Figure 2(a). Let $\mathcal{A}^- = \{(x_1, \ldots, x_d) \in \mathcal{A} | x_1 < 0\}$ and $\mathcal{A}^+ = \{(x_1, \ldots, x_d) \in \mathcal{A} | x_1 > 0\}$. We partition $\mathbb{E}^d$ into a slice $S^- : x_1 < 0$, a slice $S : 0 \leqslant x_1 \leqslant \mathrm{diam}(s)$, and a slice $S^+ : x_1 > \mathrm{diam}(s)$, and bound the volume of the intersection of $\mathcal{A} \oplus s$ with each of S^-, S, and S^+—see Figure 2(b).

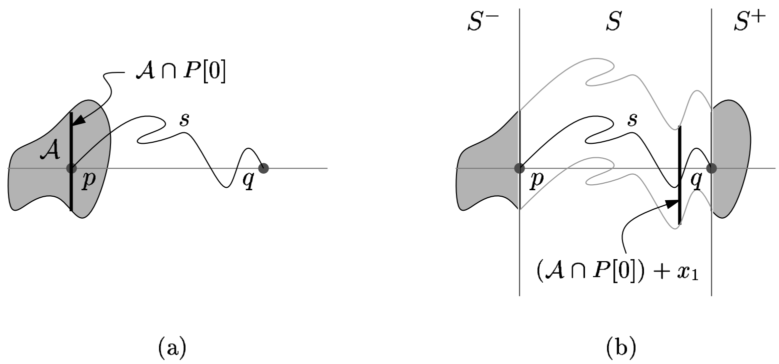

Figure 2: (a) The bold vertical segment, which corresponds to $\mathcal{A} \cap P[0]$, partitions $\mathcal{A}$ into a left part A^- and a right part $\mathcal{A}^+$. (b) The volume of the Minkowski sum $\mathcal{A} \oplus s$ is at least the sum of the volume of the grey shapes inside S^- and S^+ and the volume bounded by the upper and lower curves inside S.

Both the sets $\mathcal{A}$ and $\mathcal{A} \oplus q$ are by definition completely contained in $\mathcal{A} \oplus s$. We have that $\mathcal{A} \cap S^- = \mathcal{A}^-$ and $(\mathcal{A} \oplus q) \cap S^+ = \mathcal{A}^+ \oplus q$. Hence,

$$\mathrm{vol}((\mathcal{A} \oplus s) \cap (S^- \cup S^+)) \geqslant \mathrm{vol}(\mathcal{A}^-) + \mathrm{vol}(\mathcal{A}^+ \oplus q) = \mathrm{vol}(\mathcal{A}^-) + \mathrm{vol}(\mathcal{A}^+) = \mathrm{vol}(\mathcal{A}).$$

The Minkowski sum of the $(d-1)$-dimensional set $\mathcal{A} \cap P[0]$ and s lies entirely inside S. Since any intersection of $\mathcal{A} \oplus s$ with a plane $P[x_1]$ for $0 \leqslant x_1 \leqslant \mathrm{diam}(s)$ contains $(\mathcal{A} \cap P[0]) \oplus x_1$ we have

$$\begin{aligned}
\mathrm{vol}((\mathcal{A} \oplus s) \cap S) &\geqslant \int_0^{\mathrm{diam}(s)} \mathrm{vol}_{d-1}((\mathcal{A} \oplus s) \cap P[x_1]) dx_1 \\
&\geqslant \int_0^{\mathrm{diam}(s)} \mathrm{vol}_{d-1}((\mathcal{A} \cap P[0]) \oplus x_1) dx_1 \\
&= \int_0^{\mathrm{diam}(s)} \mathrm{vol}_{d-1}(\mathcal{A} \cap P[0]) dx_1 \\
&= \omega_d \cdot \mathrm{fatness}(\mathcal{A}) \cdot \mathrm{diam}(\mathcal{A})^{d-1} \cdot \mathrm{diam}(s)
\end{aligned}$$

Combining the bounds for $S^- \cup S^+$ and S yields the claimed result. □

The role of Lemma 3.1 in the proof of Theorem 3.2 is similar to that of Lemma 2.3 in the proof of Theorem 2.4.

Theorem 3.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be objects in $\mathbb{E}^d$ with $\mathrm{diam}(\mathcal{A}) \geqslant \mathrm{diam}(\mathcal{B})$. Then*

$$\mathrm{fatness}(\mathcal{A} \oplus \mathcal{B}) \geqslant \mathrm{fatness}(\mathcal{A}) \cdot \left(\frac{\mathrm{diam}(\mathcal{A})}{\mathrm{diam}(\mathcal{A}) + \mathrm{diam}(\mathcal{B})} \right)^{d-1}.$$

This bound is tight in the worst case.

Proof: A large part of the proof proceeds in the same way as the proof of Theorem 2.4. That is, we take a ball B determining the fatness of $\mathcal{A} \oplus \mathcal{B}$, and we let $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ be such that $\mathcal{A} \oplus \mathcal{B} = \tilde{\mathcal{A}} \oplus \tilde{\mathcal{B}}$, the center of B is contained in $\tilde{\mathcal{A}}$, and $O \in \tilde{\mathcal{B}}$. If the boundary of B

intersects $\tilde{\mathcal{A}}$ we are again done. Otherwise $\tilde{\mathcal{A}}$ is completely contained in B, and we define B_1 to be the largest ball concentric with B whose boundary intersects $\tilde{\mathcal{A}}$. We also define B_2 to be the ball with center O and whose radius is $r - r_1$, where r and r_1 are the radii of B and B_1, respectively. We now consider a subshape of $\tilde{\mathcal{B}}$ contained in B_2. The definition of the subshape is the first point where we deviate from the proof of Theorem 2.4: instead of using $\tilde{\mathcal{B}} \cap B_2$, we now take a curve $s \subset \tilde{\mathcal{B}}$ connecting O to the boundary of B_2. Such a curve exists because $B_2 \in U(\tilde{\mathcal{B}})$ and $\tilde{\mathcal{B}}$ is connected. Notice that $\mathrm{diam}(s) \geqslant r - r_1$.

The Minkowski sum $\tilde{\mathcal{A}} \oplus s$ is clearly a subset of both $\tilde{\mathcal{A}} \oplus \tilde{\mathcal{B}}$ and B. Let $f_{\mathcal{A}} := \mathrm{fatness}(\mathcal{A})$. Using Lemmas 2.1 and 3.1 along with the inequality $r \leqslant \mathrm{diam}(\mathcal{A}) + \mathrm{diam}(\mathcal{B})$, we find

$$\begin{aligned}
\mathrm{fatness}(\tilde{\mathcal{A}} \oplus \tilde{\mathcal{B}}) &= \mathrm{vol}((\tilde{\mathcal{A}} \oplus \tilde{\mathcal{B}}) \cap B)/\mathrm{vol}(B) \\
&\geqslant \mathrm{vol}(\tilde{\mathcal{A}} \oplus s)/\mathrm{vol}(B) \\
&\geqslant (\mathrm{vol}(\tilde{\mathcal{A}}) + \omega_d \cdot \mathrm{fatness}(\tilde{\mathcal{A}}) \cdot \mathrm{diam}(s) \cdot \mathrm{diam}(\tilde{\mathcal{A}})^{d-1})/\mathrm{vol}(B) \\
&\geqslant \left(\omega_d \cdot f_{\mathcal{A}} \cdot \mathrm{diam}(\mathcal{A})^d + \omega_d \cdot f_{\mathcal{A}} \cdot \mathrm{diam}(s) \cdot \mathrm{diam}(\mathcal{A})^{d-1}\right) \cdot \left(\omega_d r^d\right)^{-1} \\
&\geqslant \omega_d \cdot f_{\mathcal{A}} \cdot \mathrm{diam}(\mathcal{A})^{d-1} \cdot (\mathrm{diam}(\mathcal{A}) + \mathrm{diam}(s)) \cdot \left(\omega_d r^d\right)^{-1} \\
&\geqslant \omega_d \cdot f_{\mathcal{A}} \cdot \mathrm{diam}(\mathcal{A})^{d-1} \cdot r \cdot \left(\omega_d r^d\right)^{-1} \\
&\geqslant f_{\mathcal{A}} \cdot \mathrm{diam}(\mathcal{A})^{d-1} \cdot r^{-(d-1)} \\
&\geqslant f_{\mathcal{A}} \cdot \mathrm{diam}(\mathcal{A})^{d-1} \cdot (\mathrm{diam}(\mathcal{A}) + \mathrm{diam}(\mathcal{B}))^{-(d-1)}.
\end{aligned}$$

This proves the upper bound.

To show that the bound is tight we let $\mathcal{A} = [0, L_{\mathcal{A}}] \times [0,1]^{d-1}$ and $\mathcal{B} = [0, L_{\mathcal{B}}] \times 0^{d-1}$, leading to $\mathcal{A} \oplus \mathcal{B} = [0, L_{\mathcal{A}} + L_{\mathcal{B}}] \times [0,1]^{d-1}$. Note that $\mathrm{diam}(\mathcal{A})^2 = L_{\mathcal{A}}^2 + d - 1$ and $\mathrm{diam}(\mathcal{B}) = L_{\mathcal{B}}$. Furthermore $\mathrm{diam}(\mathcal{A} \oplus \mathcal{B})^2 = (L_{\mathcal{A}} + L_{\mathcal{B}})^2 + d - 1$. By Lemma 2.1 we have that

$$\mathrm{fatness}(\mathcal{A}) = L_{\mathcal{A}}/(\omega_d \cdot \mathrm{diam}(\mathcal{A})^d)$$

and

$$\mathrm{fatness}(\mathcal{A} \oplus \mathcal{B}) = (L_{\mathcal{A}} + L_{\mathcal{B}})/(\omega_d \cdot \mathrm{diam}(\mathcal{A} \oplus \mathcal{B})^d).$$

When $L_{\mathcal{A}}$ tends to infinity, then $L_{\mathcal{A}}$ approaches $\mathrm{diam}(\mathcal{A})$ and $\mathrm{diam}(\mathcal{A} \oplus \mathcal{B})$ goes to $\mathrm{diam}(\mathcal{A}) + \mathrm{diam}(\mathcal{B})$. As a result, $\mathrm{fatness}(\mathcal{A})$ and $\mathrm{fatness}(\mathcal{A} \oplus \mathcal{B})$ go to $\omega_d \cdot \mathrm{diam}(\mathcal{A})^{-(d-1)}$ and $\omega_d \cdot (\mathrm{diam}(\mathcal{A}) + \mathrm{diam}(\mathcal{B}))^{-(d-1)}$ respectively. □

The ratio of diameters attains its smallest value when $\mathrm{diam}(\mathcal{B}) = \mathrm{diam}(\mathcal{A})$. As a consequence, we have $\mathrm{fatness}(\mathcal{A} \oplus \mathcal{B}) \geqslant \mathrm{fatness}(\mathcal{A})/2^{d-1}$.

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