

Metric pattern spaces

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Abstract

On a collection of subsets of a space, fundamentally different metrics may be defined. In pattern matching, it is often required that a metric is invariant for a given transformation group. In addition, a pattern metric should be robust for defects in patterns caused by discretisation and unreliable feature detection. We formalise these properties by presenting axioms. Finding invariant metrics without requiring such axioms is a trivial problem. Using our axioms, we analyse various pattern metrics, including the Hausdorff distance and the symmetric difference. Finally, we present the *reflection metric*. This metric is defined on finite unions of $(n - 1)$ -dimensional hyper-surfaces in \mathbb{R}^n . The reflection metric is affine invariant and satisfies our axioms.

1 Introduction

Pattern matching is the problem of finding transformations that make one pattern resemble another pattern. It depends on a measure of resemblance, which should be invariant for the transformations under consideration. The *shape* of a pattern is the pattern modulo the action of a transformation group. Minimisation of a pattern metric under a transformation group for which it is invariant gives a metric on shapes, see Rucklidge [13]. Shape recognition involves measuring the resemblance of shapes corresponding to patterns.

Pattern metrics can be classified using various criteria. For example, each pattern metric is defined on a particular collection of patterns, such as convex polygons, simple curves, simple polygons, polygons, unions of curves, and unions of surfaces. In addition, each pattern metric uniquely determines a maximal geometric transformation group under which it is invariant. Besides that, the behaviour of distinct pattern metrics with respect to pattern-manipulations, such as deformation, blur, cracks, noise, and extension, can be different.

The aim of our research was to find an affine invariant metric defined on finite unions of $(n - 1)$ -dimensional hypersurfaces in n -dimensional Euclidean

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space. Such a metric can be used as a coordinate independent similarity measure for patterns that are not necessarily boundaries of open sets.

Finding invariant metrics on a given collection of patterns is a trivial problem: simply choose the discrete metric (defined in Section 3.1). The problem becomes interesting when the metric is required to have additional properties. We express such properties using axioms, and define an affine invariant metric for unions of hypersurfaces, satisfying these axioms. Before we present the new metric, we evaluate a number of pattern metrics known from literature. For each, we examine the domain of definition (type of patterns), the maximal invariance group, and satisfaction of the axioms.

Many pattern matching algorithms are based on the Hausdorff metric, see [4, 11, 1]. This metric is defined on the collection of closed, bounded (nonempty) subsets of any underlying metric space. The Hausdorff metric is not robust for outliers. In fact, one might even say that the Hausdorff metric is defined in terms of outliers. In addition, the Hausdorff metric is only invariant for isometries. The partial Hausdorff distance is a non-metric variant of the Hausdorff metric that is more robust for outliers and noise, see [12, 7]. It is based on a parameter that estimates the amount of outliers.

The Fréchet distance, see Alt and Godau [3], is a pattern metric defined on simple closed curves. Other resemblance measures for simple curves are based on turning angle, see Cohen and Guibas [5], and normalised affine arc-length, see Huttenlocher and Kedem [10].

For solid patterns, i.e. patterns equal to the closure of some open set, suitable affine invariant metrics exist. Examples include the normalised volume of symmetric difference, see Alt et al. [2], and the difference of normalised indicators, see Hagedoorn and Veltkamp [8].

Here, we introduce the *reflection metric*. This pattern metric is robust, sensitive, and affine invariant. It is defined on finite disjoint unions of hypersurfaces.

Section 2 presents axioms describing properties of pattern metrics. The axioms express robustness for “deformation”, “blur”, “cracks”, and “noise”. The general formulation allows the axioms to be verified for pattern metrics defined on subsets of any Euclidean, affine, projective, or any other topological space. Section 3 evaluates various pattern metrics with respect to the domain of definition, the invariance group, and each of the five axioms.

2 Axioms

Let S be any set. A metric on S is a function $d : S \times S \rightarrow \mathbb{R}$ satisfying the following two conditions for all $x, y, z \in S$:

- (i) $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(y, z) \leq d(x, y) + d(x, z)$.

Non-negativity and symmetry follow from these two conditions. Given two elements x and y of S , the value $d(x, y)$ is called the *distance* between x and y . Consider a weaker version of property (i):

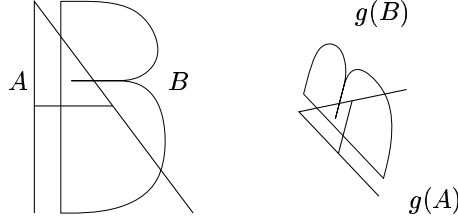


Figure 1: Invariance.

(i)' $d(x, x) = 0$,

A function satisfying (i)' and (ii) is called a *semimetric*.

The following definition describes a special type of metric space.

Definition 1 Let X be a topological space, \mathcal{P} a collection of subsets of X , and d a metric on \mathcal{P} . We call the structure (X, \mathcal{P}, d) a metric pattern space.

The elements of \mathcal{P} will be called *patterns*.

Let $\text{Hom}(X)$ be the class of all homeomorphisms on X . A collection of patterns \mathcal{P} uniquely determines a maximal subgroup T of $\text{Hom}(X)$ under which \mathcal{P} is closed. The transformation group T consists of all $t \in \text{Hom}(X)$ such that both the image $t(A)$ and the inverse image $t^{-1}(A)$ are members of \mathcal{P} for all $A \in \mathcal{P}$.

The metric pattern space (X, \mathcal{P}, d) is invariant for a transformation $g \in T$ if $d(g(A), g(B))$ equals $d(A, B)$ for all $A, B \in \mathcal{P}$. We define the *invariance group* G of a metric pattern space to be the set of all transformations in T for which the space is invariant.

Figure 1 shows planar patterns A and B , and their image patterns $g(A)$ and $g(B)$ under an affine transformation g . Invariance for affine transformations is very useful for patterns in \mathbb{R}^n . It makes the distance between two patterns independent of the choice of coordinate system.

Note that a metric pattern space (X, \mathcal{P}, d) , with \mathcal{P} not containing the empty set \emptyset , can be extended with \emptyset , by defining

$$\rho(A, B) = \frac{d(A, B)}{1 + d(A, B)},$$

$\rho(\emptyset, \emptyset) = 0$, and $\rho(A, \emptyset) = 1$ for $A, B \in \mathcal{P}$. This gives a bounded metric pattern space $(X, \mathcal{P} \cup \{\emptyset\}, \rho)$, such that the restriction of ρ to \mathcal{P} is topologically equivalent to d . In addition, the transformation group T and the invariance group G are unchanged in the extended space.

Below, we discuss axioms for metric pattern spaces. The axioms themselves obey a number of principles. For example, the axioms remain satisfied if the collection of patterns is restricted. The axioms are also preserved under complementation: If one of the axioms holds for (X, \mathcal{P}, d) , then it also holds for

(X, \mathcal{P}^c, ρ) , with patterns $\mathcal{P}^c = \{X - A \mid A \in \mathcal{P}\}$, and metric $\rho(X - A, X - B) = d(A, B)$. Furthermore, extension of a collection of patterns with the empty set, explained above, preserves each of the axioms.

First, an axiom expressing robustness for “deformation” will be discussed. This axiom states that choosing transformations $t \in T$ sufficiently “close” to the identity, makes the distance $d(A, t(A))$ arbitrarily small. In a topological space, such “closeness” can be described by considering increasingly smaller open neighbourhoods of a point. For this purpose, the group of transformations T is given the compact-open topology, which is defined in terms of the topology on X . This way, T can be considered a topological space in which transformations are points, and each transformation has open neighbourhoods that are sets of transformations.

The compact-open topology is determined by its collection of open sets. A collection of subsets \mathcal{S} of a space S whose union equals S , is called a subbasis for a topology on a space S . The topology generated by the subbasis \mathcal{S} consists of all unions of finite intersections of elements of \mathcal{S} . The (relative) compact-open topology on the set of transformations T is generated by the subbasis consisting of all sets of the form

$$S(K, U) = \{t \in T \mid t(K) \subseteq U\},$$

where $K \subseteq X$ is compact, and $U \subseteq X$ is open.

The metric pattern space (X, \mathcal{P}, d) is called *deformation robust* if it satisfies the following axiom:

Axiom 1 *For each $A \in \mathcal{P}$ and $\epsilon > 0$, an open neighbourhood $I \subseteq T$ of the identity exists such that $d(A, t(A)) < \epsilon$ for all $t \in I$.*

Deformation robustness is equivalent to saying that for each pattern $A \in \mathcal{P}$, the map $t \mapsto t(A)$ with domain the group of transformations T and range the collection of patterns \mathcal{P} is continuous.

Figure 2 shows the image of A under a transformation t contained in an open neighbourhood of the identity. This neighbourhood is a finite intersection of subbasis elements $S(K_i, U_i)$ generated by compact segments K_i of A and open balls U_i containing A_i .

Let $\text{Cl}(A)$ and $\text{Int}(A)$ denote the closure and the interior of A in X , respectively. Define the boundary of a subset A of X by $\text{Bd}(A) = \text{Cl}(A) \cap \text{Cl}(X - A)$. We call a metric pattern space *blur robust* if the following holds:

Axiom 2 *For each $A \in \mathcal{P}$ and $\epsilon > 0$, an open neighbourhood U of $\text{Bd}(A)$ exists, such that $d(A, B) < \epsilon$ for all $B \in \mathcal{P}$ satisfying $B - U = A - U$ and $\text{Bd}(A) \subseteq \text{Bd}(B)$.*

The axiom says that additions to the boundary of A close to the boundary do not cause discontinuities. Figure 3 shows a neighbourhood U of $\text{Bd}(A)$ in which parts of $\text{Bd}(B)$ occur that are not in $\text{Bd}(A)$.

A *crack* of A is a closed subset R of $\text{Bd}(A)$ consisting entirely of limit points of $\text{Bd}(A) - R$. This means that all open neighbourhoods of a point $x \in R$

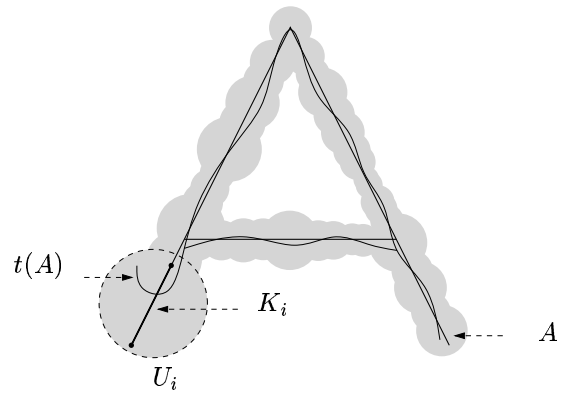


Figure 2: Deformation robust.

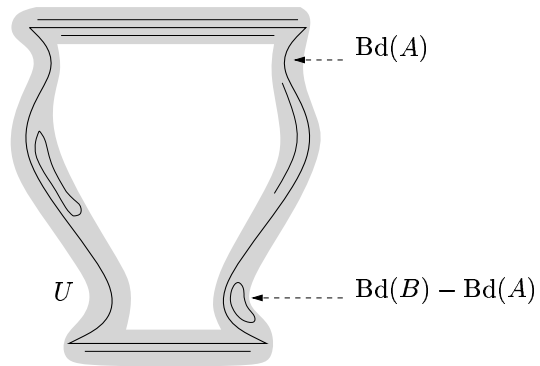


Figure 3: Blur robust.

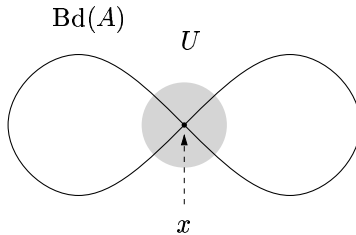


Figure 4: Crack robust.

intersect $\text{Bd}(A) - R$. Cracks can be seen as parts of the boundary that can be “restored” after they have been removed from the boundary, by forming the closure of the remaining boundary. Changing a pattern in neighbourhoods of a crack may cause the pattern (or its complement) to become separated or connected.

Figure 4 shows a pretzel, consisting of two topological 1-spheres glued together at a point x . The singleton set $R = \{x\}$ is a crack of A .

We say (X, \mathcal{P}, d) is *crack robust* if the next axiom holds:

Axiom 3 *For each $A \in \mathcal{P}$, each crack R of A , and $\epsilon > 0$, there exists an open neighbourhood U of R such that $A - U = B - U$ implies $d(A, B) < \epsilon$ for all $B \in \mathcal{P}$.*

The axiom says that applying changes to A within a small enough neighbourhood of a crack of A results in a pattern B close to A in pattern space. Whether the connectedness is preserved does not matter.

If the following axiom is satisfied, we call a metric pattern space *noise robust*:

Axiom 4 *For each $A \in \mathcal{P}$, $x \in X$, and $\epsilon > 0$, an open neighbourhood U of x exists such that $B - U = A - U$ implies $d(A, B) < \epsilon$ for all $B \in \mathcal{P}$.*

This axiom says that changes in patterns do not cause discontinuities in pattern distance, provided the changes happen within small regions. By means of the triangle inequality, we obtain an equivalent axiom when neighbourhoods of finite point sets instead of singletons are considered.

Figure 5 shows a pattern A and a point x . Addition of noise $B - A$ within a neighbourhood U of x results in a new pattern B . Axiom 4 says that the distance between A and B can be made smaller by making U smaller.

3 Examples

This section analyses various metric pattern spaces using the theory from the previous section. For each example, the space X and the collection of patterns \mathcal{P} are fixed. For convenience, we say a metric d satisfies an axiom, when we actually mean the metric pattern space (X, \mathcal{P}, d) . We discuss the discrete metric, the

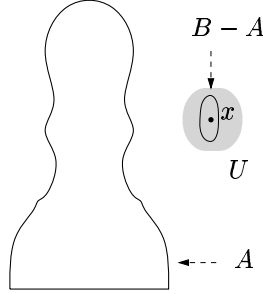


Figure 5: Noise robust.

Hausdorff metric, the volume of symmetric difference, the normalised volume of symmetric difference, and the difference of normalised indicators. After that, we introduce the reflection metric.

3.1 Discrete metric

The following example, *the discrete metric*, illustrates that it is no problem finding a metric defined on any collection of patterns and invariant under any transformation group. It proves much more interesting to find a model for our axioms, given a collection of patterns and a class of transformations.

We analyse the special case in which the underlying space X equals \mathbb{R}^n . Consider $\wp(\mathbb{R}^n)$, the power set of \mathbb{R}^n , as the collection of patterns.

Definition 2 *The discrete metric on $\wp(\mathbb{R}^n)$ is given by*

$$d_{\mathfrak{D}}(A, B) = \begin{cases} 0 & \text{if } A = B \\ 1 & \text{otherwise.} \end{cases}$$

The discrete metric is invariant for all homeomorphisms on X . It satisfies none of our axioms. We provide a simple counterexample for each axiom.

Deformation robustness, Axiom 1, fails because non-empty open subsets of \mathbb{R}^n always contain more than one point. Choose $A = \{x\}$, where x is an arbitrary point in \mathbb{R}^n , and choose $\epsilon = 1$. Then, $d(A, t(A)) < \epsilon$ if and only if $t(x) = x$. It can be shown that any open I containing the identity, contains a transformation t for which $t(x) \neq x$, implying $d(A, t(A)) = 1$.

The following example contradicts blur robustness, Axiom 2. Choose $A = \{x\}$ and $\epsilon = 1$. For any neighbourhood U of x , choose $B = \{y, x\}$, where $y \neq x$ is in U . Then $A \neq B$, so $d_{\mathfrak{D}}(A, B) = 1$.

Using the unit sphere S^1 in \mathbb{R}^2 , we construct a counterexample for crack robustness, Axiom 3. Choose $A = S^1$, and let R consist of a single point on S^1 . Clearly $R \subseteq \text{Cl}(\text{Bd}(A) - R)$. Let $\epsilon = 1$. For any open neighbourhood U of R , choose $B = A - R$, giving $d_{\mathfrak{D}}(A, B) = 1$.

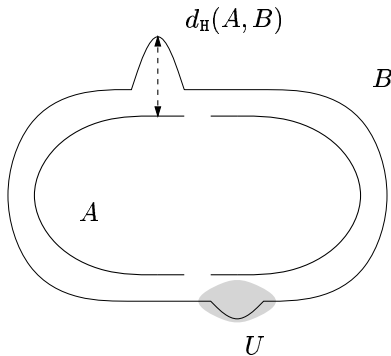


Figure 6: Hausdorff metric.

We obtain a counterexample for noise robustness, Axiom 4, by choosing $A = \{x\}$ and $\epsilon = 1$. For any open neighbourhood U of x , choosing $B = \emptyset$ gives $d_b(A, B) = 1$.

3.2 Hausdorff metric

The Hausdorff metric can be defined on any collection of (nonempty) bounded, closed subsets of an underlying metric space X . Using the construction from Section 2, the Hausdorff metric can be extended to include the empty set without changing the transformation group T and the invariance group G , and keeping the satisfied axioms. Here, the Hausdorff metric is considered for Euclidean space $X = \mathbb{R}^n$. The collection of patterns \mathcal{K}^n consists of all nonempty compact subsets of \mathbb{R}^n .

Let $B(x, \epsilon)$ denote the open Euclidean ball in \mathbb{R}^n centred at $x \in \mathbb{R}^n$ with radius $\epsilon > 0$. For any subset S of \mathbb{R}^n , define the ϵ -neighbourhood of S in \mathbb{R}^n as

$$S^\epsilon = \bigcup_{x \in S} B(x, \epsilon).$$

The following definition uses these ϵ -neighbourhoods.

Definition 3 *The Hausdorff metric on \mathcal{K}^n is given by*

$$d_H(A, B) = \inf\{\epsilon > 0 \mid A \subseteq B^\epsilon \text{ and } B \subseteq A^\epsilon\}.$$

Figure 6 depicts the Hausdorff metric for two planar patterns A and B in \mathcal{K}^2 . The length of the dotted line represents the distance between A and B . Observe that any change of B within the grey area U does not change the value of the Hausdorff metric.

The Hausdorff metric is a real metric in the mathematical sense. Consider metric property (i). Since $A \subseteq A^\epsilon$ for all $\epsilon > 0$, $d_H(A, A) = 0$. Now, let A and B be bounded, closed subsets of \mathbb{R}^n . For convenience, suppose a lies in

the difference $A - B$. There is an open ball $B(a, r)$ disjoint with B . Otherwise the point a would have been a limit point of B , and therefore an element of B . Since A cannot be a subset of B^r , the distance $d_{\mathbb{H}}(A, B)$ is at least r , a positive number. So property (i) holds. Consider metric property (ii). For any $A \subseteq \mathbb{R}^n$, the following equality holds:

$$(A^\gamma)^\delta = A^{\gamma+\delta}.$$

This means $B \subseteq A^\gamma$ and $A \subseteq C^\delta$ imply $B \subseteq C^{\gamma+\delta}$. Now $A \subseteq B^\gamma$, $B \subseteq A^\gamma$, $A \subseteq C^\delta$, and $C \subseteq A^\delta$ imply $B \subseteq C^{\gamma+\delta}$ and $C \subseteq B^{\gamma+\delta}$. As a result $d_{\mathbb{H}}(B, C) \leq d_{\mathbb{H}}(A, B) + d_{\mathbb{H}}(A, C)$.

The invariance group for the Hausdorff metric equals $\text{Iso}(\mathbb{R}^n)$, the group of Euclidean isometries (rigid motions combined with reflection). For isometries $g \in \text{Iso}(\mathbb{R}^n)$, the containment $A \subseteq B^\epsilon$ is equivalent to $g(A) \subseteq g(B)^\epsilon$. Substitution gives $d_{\mathbb{H}}(A, B) = d_{\mathbb{H}}(g(A), g(B))$. Now suppose the homeomorphism g is not a Euclidean isometry. Then there must exist two points $x, y \in \mathbb{R}^n$ such that $\|x - y\| \neq \|g(x) - g(y)\|$. Choosing $A = \{x\}$ and $B = \{y\}$ gives

$$d_{\mathbb{H}}(A, B) = \|x - y\| \neq \|g(x) - g(y)\| = d_{\mathbb{H}}(g(A), g(B)).$$

The Hausdorff metric is deformation robust. Let $A \in \mathcal{K}^n$ and $\epsilon > 0$ be given. Cover A with a finite number of balls $\{B(c_1, \delta), \dots, B(c_k, \delta)\}$ each having radius $\delta < \frac{1}{2}\epsilon$. Choose an open set of transformations containing the identity as a finite intersection of subbasis elements:

$$I = \bigcap_{i=1}^k \{t \in \text{Hom}(\mathbb{R}^n) \mid t(\text{Cl}(B(c_i, \delta))) \subseteq B(c_i, 2\delta)\}.$$

If $t \in I$, then $t(A) \subseteq A^{2\delta}$ and $A \subseteq t(A)^{2\delta}$. Therefore $d_{\mathbb{H}}(A, t(A)) \leq 2\delta < \epsilon$.

Now, blur-robustness will be shown for the Hausdorff metric. Let $A \in \mathcal{K}^n$ and $\epsilon > 0$. Choose $\delta < \epsilon$ and $U = \text{Bd}(A)^\delta$. Suppose $B \in \mathcal{K}^n$ satisfies $B - U = A - U$ and $\text{Bd}(A) \subseteq \text{Bd}(B)$. We claim $A \subseteq B^\delta$. If $a \in A \cap U$, then $a \in U = \text{Bd}(A)^\delta \subseteq \text{Bd}(B)^\delta \subseteq B^\delta$. We claim $B \subseteq A^\delta$. If $b \in B \cap U$, then $b \in U = \text{Bd}(A)^\delta \subseteq A^\delta$. Since $A \subseteq B^\delta$ and $B \subseteq A^\delta$, $d_{\mathbb{H}}(A, B) \leq \delta < \epsilon$.

The Hausdorff metric is crack robust. Let A be a nonempty compact subset of \mathbb{R}^n , R a crack of A , and $\epsilon > 0$. Set $\delta < \frac{1}{2}\epsilon$. Since R is a closed subset of a compact set, the boundary of A , R is itself compact. As a result, R can be covered by a finite collection $\{B_1, \dots, B_k\}$ of open balls centred at points of R and having radius δ . Since R consists of limit points of $\text{Bd}(A) - R$, each B_i intersects $\text{Bd}(A) - R$. For each ball B_i choose a point x_i in the intersection with $\text{Bd}(A) - R$. Construct an open neighbourhood U of R :

$$U = \bigcup_{i=1}^k B_i - \{x_1, \dots, x_k\}.$$

Suppose $B \in \mathcal{K}^n$ is such that $B - U = A - U$. We claim $A \subseteq B^{2\delta}$. Choose $a \in A \cap U$. Then a lies in some ball B_i . By the triangle inequality, a is contained

in the ball $B(x_i, 2\delta)$. Because $x_i \in \text{Bd}(A) - R \subseteq A$ and $x_i \notin U$, it follows that $x_i \in A - U = B - U \subseteq B$. Now $a \in B(x_i, 2\delta) \subseteq B^{2\delta}$. We claim $B \subseteq A^\delta$. Choose $b \in B \cap U$. Then b lies in some ball B_i . Since B_i is centred at a point of $R \subseteq A$, $b \in A^\delta$. Therefore $d_{\mathbb{H}}(A, B) \leq 2\delta < \epsilon$.

A counterexample for noise robustness can be constructed in the real line. Choose $A = \{0\}$, $x = 1$, and $\epsilon = 1$. For any neighbourhood U of 1, choosing $B = \{0, 1\}$ gives $A - U = B - U$ and $d_{\mathbb{H}}(A, B) = \epsilon$.

Before we continue with discussing the volume of symmetric difference, its normalised version, and other pattern metrics, we need a number of results. These will be discussed in the following section.

3.3 Imbedding patterns

We discuss two ways of constructing semimetrics on patterns. Both techniques map patterns to integrable functions. The results will be used in proving properties of metric pattern spaces further on. The following observation is the main ingredient.

Observation 1 *If $f : S \rightarrow T$ is a function and σ is a semimetric on T , then $d(x, y) = \sigma(f(x), f(y))$ is a semimetric on S .*

Let S be a collection of patterns \mathcal{P} . By varying the mapping f and the semimetric σ , we can form distinct semimetrics on \mathcal{P} . This technique will be used for patterns that are subsets of \mathbb{R}^n , and mappings f from \mathcal{P} into the space of integrable functions. On the latter space, we consider two semimetrics: an “absolute” one and a “normalised” one. We show how semimetrics invariant for a given transformation group can be constructed using any of the former two semimetrics.

Let $\mathbf{I}(\mathbb{R}^n)$ be the vector space of real-valued Lebesgue-integrable functions on \mathbb{R}^n , with scalar multiplication and vector addition defined pointwise. Define the L^1 seminorm on $\mathbf{I}(\mathbb{R}^n)$:

$$|\mathbf{a}| = \int_{\mathbb{R}^n} |\mathbf{a}(x)| dx.$$

The abbreviation $|\mathbf{a}|_D$ is used for integration over a subset D of \mathbb{R}^n . The L^1 seminorm induces a semimetric σ_a given by:

$$\sigma_a(\mathbf{a}, \mathbf{b}) = |\mathbf{a} - \mathbf{b}|.$$

We call this semimetric the *absolute difference*.

For a differentiable function g , let $D_g^x : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the derivative of g in x , a linear function. The Jacobi-determinant is the determinant of the derivative at a given point. We use $j_g(x) = |\det(D_g^x)|$, to denote the absolute value of the Jacobi-determinant of g in x . Let $C^1(\mathbb{R}^n)$ be the group of C^1 diffeomorphisms acting on \mathbb{R}^n .

The following lemma tells us how to apply Observation 1 in constructing an “invariant” semimetric for patterns based on the absolute difference σ_a .

Lemma 1 Let \mathcal{P} be a collection of subsets of \mathbb{R}^n . Let each $A \in \mathcal{P}$ define a unique function $\mathbf{n}_A : \mathbb{R}^n \rightarrow \mathbb{R}$ in $\mathbf{I}(\mathbb{R}^n)$. If $g \in C^1(\mathbb{R})$ satisfies

$$\mathbf{n}_{g(A)}(g(x)) = j_g(x)^{-1} \mathbf{n}_A(x)$$

for all $A \in \mathcal{P}$ and $x \in \mathbb{R}^n$, then

$$\sigma_a(\mathbf{n}_{g(A)}, \mathbf{n}_{g(B)}) = \sigma_a(\mathbf{n}_A, \mathbf{n}_B)$$

for all $A, B \in \mathcal{P}$.

proof. Set $\mathbf{u} = \mathbf{n}_A - \mathbf{n}_B$, and $\mathbf{v} = \mathbf{n}_{g(A)} - \mathbf{n}_{g(B)}$. Now $\mathbf{v}(g(x)) = j_g(x)^{-1} \mathbf{u}(x)$. Substitution of variables in an n -dimensional integral gives:

$$|\mathbf{v}| = \int_{\mathbb{R}^n} j_g(x) \mathbf{v}(g(x)) dx = |\mathbf{u}|.$$

□

For real-valued functions \mathbf{a} and \mathbf{b} , let $\mathbf{a} \sqcap \mathbf{b}$ and $\mathbf{a} \sqcup \mathbf{b}$ denote the point-wise minimum and maximum, respectively. This notation is analogous to set intersection and union. Define the *normalised difference* of two functions with non-zero integrals by

$$\sigma_n(\mathbf{a}, \mathbf{b}) = \frac{|\mathbf{a} - \mathbf{b}|}{|\mathbf{a} \sqcup \mathbf{b}|}.$$

Lemma 2 The normalised difference σ_n is a semimetric on the set of non-negative functions with non-zero integrals.

proof. Property (i)' is trivial. Property (ii): Let \mathbf{a} , \mathbf{b} , and \mathbf{c} be non-negative functions with non-zero integrals. We need to prove:

$$\frac{|\mathbf{b} - \mathbf{c}|}{|\mathbf{b} \sqcup \mathbf{c}|} \leq \frac{|\mathbf{a} - \mathbf{b}|}{|\mathbf{a} \sqcup \mathbf{b}|} + \frac{|\mathbf{a} - \mathbf{c}|}{|\mathbf{a} \sqcup \mathbf{c}|}.$$

Since $|\cdot|$ is a seminorm the inequality $|\mathbf{b} - \mathbf{c}| \leq |\mathbf{u} - \mathbf{b}| + |\mathbf{u} - \mathbf{c}|$ holds, implying:

$$\frac{|\mathbf{b} - \mathbf{c}|}{|\mathbf{b} \sqcup \mathbf{c}|} \leq \frac{|\mathbf{u} - \mathbf{b}|}{|\mathbf{c} \sqcup \mathbf{b}|} + \frac{|\mathbf{u} - \mathbf{c}|}{|\mathbf{b} \sqcup \mathbf{c}|}.$$

Choosing $\mathbf{u} = \mathbf{a} \sqcap (\mathbf{b} \sqcup \mathbf{c})$, both terms on the right side of this inequality can be bounded, obtaining the triangle inequality. We show it only for the first term, since the procedure for the second one is analogous.

$$\begin{aligned} \frac{|\mathbf{u} - \mathbf{b}|}{|\mathbf{c} \sqcup \mathbf{b}|} &\leq \frac{|\mathbf{u} - \mathbf{b}|}{|\mathbf{u} \sqcup \mathbf{b}|} \\ &\leq \frac{|\mathbf{a} - \mathbf{u}| + |\mathbf{u} - \mathbf{b}|}{|\mathbf{a} - \mathbf{u}| + |\mathbf{u} \sqcup \mathbf{b}|} \\ &= \frac{|(\mathbf{a} - \mathbf{u}) + (\mathbf{u} - \mathbf{b})|}{|(\mathbf{a} - \mathbf{u}) + (\mathbf{u} \sqcup \mathbf{b})|} \\ &= \frac{|\mathbf{a} - \mathbf{b}|}{|\mathbf{a} \sqcup \mathbf{b}|}. \end{aligned}$$

□

Let $\text{CJ}(\mathbb{R}^n)$ be the subgroup of $\text{C}^1(\mathbb{R}^n)$ consisting of those g for which the Jacobi-determinant $j_g(x)$ is constant in $x \in \mathbb{R}^n$. The next lemma shows that a large class of mappings from patterns to integrable functions result in “invariant” semimetrics based on the normalised difference σ_n .

Lemma 3 *Let \mathcal{P} be a collection of subsets of \mathbb{R}^n . Let each $A \in \mathcal{P}$ define a unique function $\mathbf{n}_A : \mathbb{R}^n \rightarrow \mathbb{R}$ in $\mathbf{I}(\mathbb{R}^n)$. If $g \in \text{CJ}(\mathbb{R})$ determines a number $\delta > 0$ such that*

$$\mathbf{n}_{g(A)}(g(x)) = \delta \mathbf{n}_A(x)$$

for all $A \in \mathcal{P}$ and $x \in \mathbb{R}^n$, then

$$\sigma_n(\mathbf{n}_{g(A)}, \mathbf{n}_{g(B)}) = \sigma_n(\mathbf{n}_A, \mathbf{n}_B)$$

for all $A, B \in \mathcal{P}$.

proof. Apply substitution of variables using the constant $j = j_g(x)$:

$$\begin{aligned} & \sigma_n(\mathbf{n}_{g(A)}, \mathbf{n}_{g(B)}) \\ &= \frac{|\mathbf{n}_{g(A)} - \mathbf{n}_{g(B)}|}{|\mathbf{n}_{g(A)} \sqcup \mathbf{n}_{g(B)}|} \\ &= \frac{j |\mathbf{n}_{g(A)} \circ g - \mathbf{n}_{g(B)} \circ g|}{j |\mathbf{n}_{g(A)} \circ g \sqcup \mathbf{n}_{g(B)} \circ g|} \\ &= \sigma_n(\mathbf{n}_{g(A)} \circ g, \mathbf{n}_{g(B)} \circ g) \\ &= \sigma_n(\delta \mathbf{n}_A, \delta \mathbf{n}_B) \\ &= \sigma_n(\mathbf{n}_A, \mathbf{n}_B). \end{aligned}$$

□

3.4 Volume of symmetric difference

Let $S = \{v_0, \dots, v_k\}$ be a pointwise independent subset of \mathbb{R}^n . The *closed k -simplex* determined by S is the set of points in the k -dimensional hyperplane containing S having nonnegative barycentric coordinates with respect to S . Two simplices are *properly joined* if their intersection is a subsimplex of both simplices. We define \mathcal{S}^n as the collection of subsets of \mathbb{R}^n that are C^1 -diffeomorphic with a nonempty finite union of properly joined n -simplices.

For compact sets $A \subseteq \mathbb{R}^n$, define $\text{vol}(A)$ as the Lebesgue-integral of the indicator function $\mathbf{1}_A$. For any bounded set $A \subseteq \mathbb{R}^n$, we define $\text{vol}(A)$ as the volume of the closure of A . The symmetric difference of two sets A and B is defined as

$$A \dot{-} B = (A - B) \cup (B - A).$$

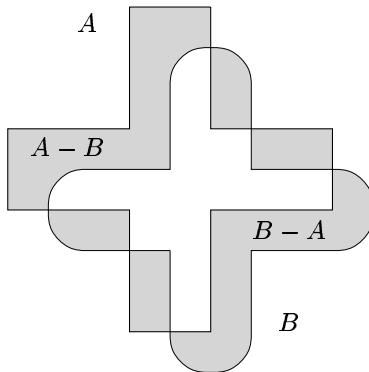


Figure 7: Volume of symmetric difference.

Definition 4 *The volume of symmetric difference on S^n is given by*

$$d_S(A, B) = \text{vol}(A \dot{-} B).$$

Figure 7 shows the symmetric difference of two patterns A and B in S^2 . The volume of the grey region equals $d_S(A, B)$.

Observation 1 tells us that d_S is a semimetric: We map each element $A \in S^n$ to its indicator function 1_A and find $d_S(A, B) = \sigma_a(1_A, 1_B)$. Now we show $d_S(A, B) > 0$ for all distinct $A, B \in S^n$. Assume A is not a subset of B , otherwise exchange A and B . Clearly, the difference $\text{Int}(A) - B$ is non-empty and open, giving $d_S(A, B) > 0$. As a result d_S is a metric on S^n .

The collection of patterns S^n is closed under $C^1(\mathbb{R}^n)$, the group of diffeomorphisms on \mathbb{R}^n . Let $\text{UJ}(\mathbb{R}^n)$ be the class of diffeomorphisms with unit magnitude Jacobi-determinant everywhere. These transformations preserve volume. The affine transformations g with $j_g(x)$ constant 1 form a proper subgroup of $\text{UJ}(\mathbb{R}^n)$. For example, the following C^1 diffeomorphism has constant Jacobi-determinant, but is not an affine transformation.

$$g : (x_1, x_2) \mapsto (x_1, x_2 + x_1^2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2.$$

Lemma 1 says that d_S is invariant under each transformation in $\text{UJ}(\mathbb{R}^n)$. Actually, $\text{UJ}(\mathbb{R}^n)$ is the maximal subgroup of $C^1(\mathbb{R}^n)$ for which d_S is invariant. Suppose $g \in \text{Hom}(\mathbb{R}^n) - \text{UJ}(\mathbb{R}^n)$. Then there is a point $x \in \mathbb{R}^n$ such that $j_g(x)$ does not equal 1. We assume $j_g(x) < 1$, the case $j_g(x) > 1$ is analogous. Define an open neighbourhood U of x by setting

$$U = \{y \in \mathbb{R}^n \mid j_g(y) < (j_g(x) + 1)/2\}.$$

Choose disjoint closed n -simplices A and B within U . This gives:

$$d_S(g(A), g(B)) = \text{vol}(g(A \cup B)) < \text{vol}(A \cup B) = d_S(A, B).$$

The following lemma will be used to prove that the deformation, blur, and crack robustness axioms are satisfied.

Lemma 4 *Let $A \in \mathcal{S}^n$. For any $\epsilon > 0$, there is an open neighbourhood U of the boundary $\text{Bd}(A)$ such that $\text{vol}(U) < \epsilon$.*

proof. Write $A = \phi(\bigcup S_i)$, where $\phi \in C^1(\mathbb{R}^n)$ and $\{S_1, \dots, S_c\}$ is a (nonempty) finite collection of properly joined n -simplices. Let C be the closure of an open ball containing A . Let $\gamma = \max_{x \in C} j_\phi(x)$. Each n -simplex S_i has $n + 1$ subsimplices of dimension $n - 1$. Let $\{R_1, \dots, R_k\}$ be the collection of all $(n - 1)$ -subsimplices. There are at most $c(n + 1)$ of these. For each R_i , choose an open neighbourhood N_i having volume strictly smaller than $\frac{\epsilon}{c(n+1)\mu}$. Choose $V = \text{Int}(C) \cap \bigcup_{i=1}^k N_i$. The set $U = \phi(V)$ is an open neighbourhood of $\text{Bd}(A)$ satisfying $\text{vol}(U) \leq \mu \text{vol}(V) < \epsilon$. \square

The volume of symmetric difference is deformation robust. Let $A \in \mathcal{S}^n$ and $\epsilon > 0$. Using Lemma 4, choose a neighbourhood U of $\text{Bd}(A)$ with volume strictly smaller than ϵ . An open neighbourhood of the identity transformation is given by

$$I = \{t \in C^1(\mathbb{R}^n) \mid t(\text{Bd}(A)) \subseteq U\}.$$

Suppose $t \in I$. Since both $A - t(A)$ and $t(A) - A$ are contained in U , it follows that $d_S(A, t(A)) \leq \text{vol}(U) < \epsilon$.

The symmetric difference is blur robust. For any $A \in \mathcal{S}^n$ and $\epsilon > 0$, choose some open neighbourhood U of $\text{Bd}(A)$ with volume smaller than ϵ , using Lemma 4. Clearly, $d_S(A, B) < \epsilon$ for any $B \in \mathcal{S}^n$ satisfying $B - U = A - U$.

Crack robustness can also be shown by choosing neighbourhoods with sufficiently small volumes. For $A \in \mathcal{S}^n$, crack R of A , and $\epsilon > 0$, choose an open neighbourhood U of R with volume strictly less than ϵ . Any $B \in \mathcal{S}^n$ satisfying $B - U = A - U$ gives $d_S(A, B) < \epsilon$.

Proving noise robustness is almost the same. Let $A \in \mathcal{S}^n$, $x \in \mathbb{R}^n$, and $\epsilon > 0$. Choose a neighbourhood U of x with volume smaller than ϵ , then $A - U = B - U$ implies $d_S(A, B) < \epsilon$ for all $B \in \mathcal{S}^n$.

3.5 Normalised volume of symmetric difference

Definition 5 *The normalised volume of symmetric difference on \mathcal{S}^n is given by*

$$d_{\text{NS}}(A, B) = \frac{\text{vol}(A \dot{-} B)}{\text{vol}(A \cup B)}.$$

Observation 1 can be used to show that d_{NS} is a semimetric: We map each element $A \in \mathcal{S}^n$ to its indicator function 1_A in $\mathbf{I}(\mathbb{R}^n)$. The normalised volume of symmetric difference can be written as

$$d_{\text{NS}}(A, B) = \sigma_n(1_A, 1_B).$$

Since by Lemma 2, the normalised difference σ_n is a semimetric, d_{NS} must be a semimetric. The symmetric difference of any two distinct sets A and B in \mathcal{S}^n has non-zero volume, giving $d_{\text{NS}}(A, B) > 0$. We conclude that d_{NS} is a metric.

The following lemma will be used to establish the invariance group for d_{NS} .

Lemma 5 *If $g \in C^1(\mathbb{R}^n) - \text{CJ}(\mathbb{R}^n)$, then there exist disjoint, nonempty open sets $U, V \subseteq \mathbb{R}^n$ satisfying*

$$\frac{\text{vol}(g(U))}{\text{vol}(U)} \neq \frac{\text{vol}(g(V))}{\text{vol}(V)}.$$

proof. Choose points $x, y \in \mathbb{R}^n$ such that $j_g(x) > j_g(y)$. Let $\epsilon < \frac{1}{2}(j_g(x) - j_g(y))$. Using continuity of j_g , define open sets U and V containing x and y , respectively:

$$\begin{aligned} U &= j_g^{-1}((j_g(x) - \epsilon, j_g(x) + \epsilon)), \\ V &= j_g^{-1}((j_g(y) - \epsilon, j_g(y) + \epsilon)). \end{aligned}$$

Substitution of variables gives a non-zero lower bound for the difference in volume ratios:

$$\frac{\text{vol}(g(U))}{\text{vol}(U)} - \frac{\text{vol}(g(V))}{\text{vol}(V)} \geq (j_g(x) - j_g(y)) - 2\epsilon > 0.$$

□

Using Lemma 3 we find that d_{NS} is invariant under all transformations in $\text{CJ}(\mathbb{R}^n)$. We show that $\text{CJ}(\mathbb{R}^n)$ is actually the largest transformation group for which d_{NS} is invariant. Let $g \in C^1(\mathbb{R}^n) - \text{CJ}(\mathbb{R}^n)$. Let U and V as in Lemma 5. Choose closed n -simplices S_1 and S_2 lying in U and V , respectively. Choosing $A = S_1$ and $B = S_1 \cup S_2$ gives

$$d_{\text{NS}}(g(A), g(B)) = \frac{\text{vol}(g(S_2))}{\text{vol}(g(S_1 \cup S_2))} \neq \frac{\text{vol}(S_2)}{\text{vol}(S_1 \cup S_2)} = d_{\text{NS}}(A, B).$$

For proving deformation, blur, crack and noise robustness, it suffices to show that the topology induced by the metric d_{S} is finer than that induced by that of d_{NS} . This is done in the following lemma.

Lemma 6 *For each $A \in \mathcal{S}^n$ and each $\epsilon > 0$, there exists a $\delta > 0$ such that $d_{\text{S}}(A, B) < \delta$ implies $d_{\text{NS}}(A, B) < \epsilon$ for all $B \in \mathcal{S}^n$.*

proof. Choosing $\delta = \epsilon \text{vol}(A)$ gives:

$$d_{\text{NS}}(A, B) \leq \frac{d_{\text{S}}(A, B)}{\text{vol}(A)} < \epsilon.$$

□

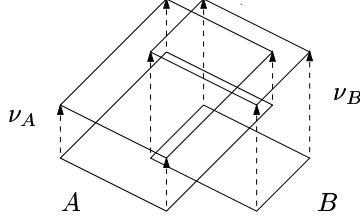


Figure 8: Difference of normalised indicators.

3.6 Difference of normalised indicators

For $A \in \mathcal{S}^n$, define a function $\nu_A : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\nu_A(x) = \text{vol}(A)^{-1}1_A(x)$.

Definition 6 The difference of normalised indicators d_{NI} for \mathcal{S}^n is given by

$$d_{\text{NI}}(A, B) = \sigma_a(\nu_A, \nu_B).$$

Figure 8 shows how two patterns A and B in \mathcal{S}^2 are “elevated” to their corresponding functions ν_A and ν_B , respectively. Integrating the region enclosed between these two functions gives $d_{\text{NI}}(A, B)$.

Observation 1 tells us that d_{NI} is a semimetric: The patterns \mathcal{S}^n are imbedded in the function space $\mathbf{I}(\mathbb{R}^n)$ with semimetric σ_a , by the mapping $f : A \mapsto \nu_A$. That d_{NI} is a metric follows from the fact that the patterns in \mathcal{S}^n are equal to the closure of their interior.

The invariance group of d_{NI} is $\text{CJ}(\mathbb{R}^n)$. That d_{NI} is invariant under each element of $\text{CJ}(\mathbb{R}^n)$ follows using Lemma 1. For $g \in \text{C}^1(\mathbb{R}^n) - \text{CJ}(\mathbb{R}^n)$, we construct patterns A and B in \mathcal{S}^n , contradicting invariance. Now choose disjoint, non-empty open sets U and V as in Lemma 5, and choose closed n -simplices $S_1 \subseteq U$ and $S_2 \subseteq V$. Choose patterns in \mathcal{S}^n by setting $A = \text{Cl}(U)$ and $B = \text{Cl}(U \cup V)$. Observe that any $K, L \in \mathcal{S}^n$ with $K \subseteq L$:

$$d_{\text{NI}}(K, L) = 2 - 2 \frac{\text{vol}(K)}{\text{vol}(L)}.$$

Substituting the inequality $\text{vol}(A)/\text{vol}(B) \neq \text{vol}(g(A))/\text{vol}(g(B))$ in the above equation gives that $d_{\text{NI}}(A, B) \neq d_{\text{NI}}(g(A), g(B))$.

The following lemma is the analogue of Lemma 6 for the difference of normalised indicators. It shows that the topology $d_{\mathcal{S}}$ induces on \mathcal{S}^n is finer than that induced by d_{NI} . This immediately gives deformation, blur, crack, and noise robustness for d_{NI} .

Lemma 7 For each $A \in \mathcal{S}^n$ and each $\epsilon > 0$, there exists a $\delta > 0$ such that $d_{\mathcal{S}}(A, B) < \delta$ implies $d_{\text{NI}}(A, B) < \epsilon$ for all $B \in \mathcal{S}^n$.

proof. Let $A, B \in \mathcal{S}^n$ and $D = A \dot{-} B$. The value of $d_{\text{NI}}(A, B)$ can be decomposed into the sum of $p_1 = |\nu_A - \nu_B|_{A-D}$ and $p_2 = |\nu_A - \nu_B|_D$. Write $c = \text{vol}(A)$.

Assume $\text{vol}(D) < \delta$, where $0 < \delta < c$. We derive an upper bound for p_1 in terms of δ and c :

$$\begin{aligned}
p_1 &= |\text{vol}(A)^{-1} - \text{vol}(B)^{-1}| \text{vol}(A - D) \\
&\leq |\text{vol}(A)^{-1} - \text{vol}(B)^{-1}| \text{vol}(A) \\
&\leq (\text{vol}(A \cap B)^{-1} - \text{vol}(A \cup B)^{-1}) \text{vol}(A) \\
&= (\text{vol}(A - D)^{-1} - \text{vol}(A \cup D)^{-1}) \text{vol}(A) \\
&< ((c - \delta)^{-1} - (c + \delta)^{-1}) c.
\end{aligned}$$

Similarly, an upper bound for p_2 in terms of δ and c is given by:

$$\begin{aligned}
p_2 &\leq \max(\text{vol}(A)^{-1}, \text{vol}(B)^{-1}) \text{vol}(D) \\
&\leq \text{vol}(A \cap B)^{-1} \text{vol}(D) \\
&= \text{vol}(A - D)^{-1} \text{vol}(D) \\
&< (c - \delta)^{-1} \delta.
\end{aligned}$$

Both upper bounds are continuous in δ and have zero value for $\delta = 0$. Therefore $\delta > 0$ can be chosen as desired. \square

3.7 Reflection metric

Let \mathcal{R}^n be the collection of subsets of \mathbb{R}^n (not contained in any $(n-1)$ -dimensional hyperplane) that are C^1 -diffeomorphic to a properly joined union of closed $(n-1)$ -simplices. Formally, we write each pattern $A \in \mathcal{R}^n$ as $A = \phi(\bigcup_{i=1}^k R_i)$, where R_1, \dots, R_k are properly joined closed $(n-1)$ -simplices and $\phi \in C^1(\mathbb{R}^n)$.

We use the notation \overline{xy} for the open line segment connecting two distinct points $x, y \in \mathbb{R}^n$. We say that a point $y \in \mathbb{R}^n$ is visible (in A) from a point $x \in \mathbb{R}^n$ if $A \cap \overline{xy} = \emptyset$. For $A \in \mathcal{R}^n$ and $x \in \mathbb{R}^n$, the *visibility star* V_A^x is defined as the set of open line segments connecting points of A that are visible from x :

$$V_A^x = \bigcup \{ \overline{ax} \mid a \in A \text{ and } A \cap \overline{ax} = \emptyset \}.$$

We define the *reflection star* R_A^x by intersecting V_A^x with its reflection in x :

$$R_A^x = \{ x + v \in \mathbb{R}^n \mid x - v \in V_A^x \text{ and } x + v \in V_A^x \}.$$

Each pattern $A \in \mathcal{R}^n$ determines a function $\rho_A : \mathbb{R}^n \rightarrow \mathbb{R}$ given by $\rho_A(x) = \text{vol}(R_A^x)$. Note that ρ_A is zero outside the convex hull of A . Figure 9 shows the visibility star V_A^x and the corresponding reflection star R_A^x for a pattern $A \in \mathcal{R}^2$, and a point $x \in \mathbb{R}^2$.

Definition 7 The reflection metric $d_{\mathbb{R}}$ for \mathcal{R}^n is given by

$$d_{\mathbb{R}}(A, B) = \sigma_n(\rho_A, \rho_B).$$

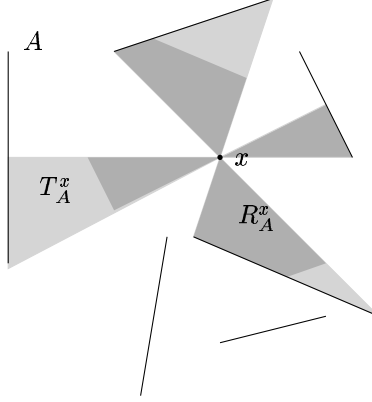


Figure 9: Reflected visibility.

Lemma 3 tells us that d_r is invariant under the group of affine transformations $\text{Af}(\mathbb{R}^n)$.

Observe that for any two patterns $A, B \in \mathcal{R}^n$ and any point $x \in \mathbb{R}^n$:

$$\text{vol}(R_A^x - R_B^x) \leq 2 \text{vol}(V_A^x - V_B^x).$$

From this, we find that

$$|\rho_A(x) - \rho_B(x)| \leq 2 \text{vol}(V_A^x - V_B^x). \quad (1)$$

Thus, we can prove the first four axioms by bounding the change in the visibility star for deformation, blur, crack and noise.

The metric d_r is deformation robust. Let $A \in \mathcal{R}^n$, where $A = \phi(\bigcup_{i=1}^k S_i)$. Let Q_1, \dots, Q_l be the set of $(n-2)$ -subsimpllices of the $(n-1)$ -simpllices S_1, \dots, S_k . Let U_1 be an open neighbourhood of A with volume strictly smaller than a given number $\delta_1 > 0$. Let $L = \phi(\bigcup_{i=1}^l Q_i)$ and let U_2 be an open neighbourhood of L with volume smaller than a given number $\delta_2 > 0$. Define an open neighbourhood of the identity:

$$I = \{ t \in C^1(\mathbb{R}^n) \mid t(A) \subseteq U_1 \text{ and } t(L) \subseteq U_2 \} .$$

Given any $\epsilon > 0$, we apply Eq. 1, in choosing $\delta_1 > \delta_2 > 0$ small enough so that $d_r(A, t(A)) < \epsilon$.

The reflection metric is blur robust. Let $A \in \mathcal{R}^n$ and $\epsilon > 0$ be given. Choose an open neighbourhood U of A with $\text{vol}(U) < \delta$ for some given $\delta > 0$. Using Eq. 1, choose $\delta > 0$ small enough so that for all $B \in \mathcal{R}^n$ satisfying $B - U = A - U$ and $A \subseteq B$, the distance $d_r(A, B)$ is smaller than ϵ .

The reflection metric is crack robust. Let $A \in \mathcal{R}^n$, R be a crack of A , and $\epsilon > 0$. By means of Eq. 1, we can choose a sufficiently small open neighbourhood U of the crack R such that $d_r(A, B) < \epsilon$ for patterns $B \in \mathcal{R}^n$ satisfying $B - U = A - U$.

Table 1: Patterns, metrics, invariance, and axioms.

\mathcal{P}	d	G	Defo.	Blur	Crack	Noise
$\wp(\mathbb{R}^n)$	d_D	$\text{Hom}(\mathbb{R}^n)$	no	no	no	no
\mathcal{K}^n	d_H	$\text{Iso}(\mathbb{R}^n)$	yes	yes	yes	no
\mathcal{S}^n	d_S	$\text{UJ}(\mathbb{R}^n)$	yes	yes	yes	yes
\mathcal{S}^n	d_{NS}	$\text{CJ}(\mathbb{R}^n)$	yes	yes	yes	yes
\mathcal{S}^n	d_{NI}	$\text{CJ}(\mathbb{R}^n)$	yes	yes	yes	yes
\mathcal{R}^n	d_R	$\text{Af}(\mathbb{R}^n)$	yes	yes	yes	yes

The reflection metric is noise robust. Let $A \in \mathcal{R}^n$ and $x \in \mathbb{R}^n$ be given. Using Eq. 1, we can choose an open neighbourhood U of x small enough such that $d_R(A, B) < \delta$ for all $B \in \mathcal{R}^n$ satisfying $B - U = A - U$.

4 Conclusion

Given a collection of patterns \mathcal{P} , and a group of transformations G , finding a metric d on \mathcal{P} invariant under G is trivial; we can always choose the discrete metric. The real problem is finding robust metrics for a class of patterns and an invariance group. Until now, robust affine invariant pattern metrics were only known for solid sets or simple closed curves. In this paper, we introduced a robust metric, called the reflection metric, which is invariant under affine transformations, and is defined on both solid and non-solid boundaries.

We have formalised our notion of resemblance using five axioms. The axioms can be verified for any metric defined on a collection of subsets of a topological space. Axioms 1–4 express robustness for deformation, blur, cracks and noise, respectively.

We have used the axioms as a guiding line in analysing various metric pattern spaces. We discussed the discrete metric, the Hausdorff metric, the volume of symmetric difference and a normalised variant, the difference of normalised indicators, and the new reflection metric. Table 1 shows for collection of patterns, metrics, invariance groups, and satisfaction of axioms for each of the metric pattern spaces considered in this paper.

The reflection metric is defined on the class of finite unions of hyper-surfaces, is affine invariant, and satisfies all our axioms. This makes the reflection metric especially suitable for matching patterns obtained using image processing techniques. The reflection metric between two finite segment unions, can be computed in $O((s + k) \log(s + k) + v)$ time, where s is the total number of segments, k is the total number of edges in both visibility graphs, and v the number of vertices in the reflection-visibility arrangement (at most $O(s^2 + k^2)$), see [9].

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