Non-Linear Scale-Spaces Isomorphic to the Linear Case*

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Abstract

An infinite dimensional class of isomorphisms is considered, relating a particular class of nonlinear scale-spaces to the well-established linear case. The nonlinearity pertains to an invertible mapping of grey-values, which can be adapted so as to account for external knowledge. This is particularly interesting for applications such as segmentation in medical imaging, whereby one is in possession of a model relating tissue types to image grey-values. It is moreover of interest in defining a consistent scale-space representation of vector-valued and multispectral images.

Keywords: linear/nonlinear/morphological scale-space theory.

1 Introduction

We consider an infinite dimensional class of *pseudo-linear scale-space representations*, in which the members are isomorphically related to the linear case by a transformation of grey-values. Although the nonlinearity can be "transformed away" in theory, it may be prudent not to do so in a practical situation whereby one is in possession of *a priori* knowledge about signal production details. For instance, in the case of medical imaging one may have knowledge (from theory or phantom studies) of how tissue types relate to grey-values. In this case the nonlinearity can be adapted so as to reflect this knowledge in an optimal way relative to the task at hand.

Moreover, nonlinearities of the type considered in this paper may be necessary in order to define a consistent multiscale representation of vector-valued and multispectral images.

Finally, the construction is of theoretical interest as it provides a methodology for establishing relationships between different multiresolution frameworks. This has been illustrated elsewhere for the case of linear scale-space theory generated by a normalised Gaussian *versus* morphological erosion and dilation scale-spaces based on a quadratic structuring function [6].

By construction all nonlinear scale-spaces inherit a number of desirable properties from the linear case, such as the scale-space axiom of "causality" (or non-enhancement principle), *cf.* Koenderink [10]. For details of the causality principle the reader is referred to the appendix.

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2 Theory

Consider the linear isotropic diffusion equation with initial condition f.

$$\left\{ \begin{array}{lcl} \partial_s u & = & \Delta u \,, \\ \lim_{s \downarrow 0} u & = & f \,. \end{array} \right. \tag{1}$$

If f represents a raw image, then u is its linear scale-space representation according to the well-known Gaussian scale-space paradigm. Although it is convenient for the present discussion to consider scale-space theory from the point of view of its p.d.e.-formulation[1] it should be noted that Eq. (1) actually has a closed-form solution, *viz.* u can be obtained from f simply by convolving it with a normalised Gaussian of width $\sigma = \sqrt{2s}$. This is an important observation that will be exploited in all subsequent considerations.

Subjecting u to an arbitrary transformation

$$u \stackrel{\text{def}}{=} \gamma(v) \quad \text{with} \quad \gamma' > 0 \tag{2}$$

yields the following nonlinear initial value problem for v:

$$\left\{ \begin{array}{lcl} \partial_s v & = & \Delta v + \mu \|\nabla v\|^2 \,, \\ \lim_{s \downarrow 0} v & = & g \,. \end{array} \right. \tag{3}$$

in which the nonlinearity is defined by

$$\mu \stackrel{\text{def}}{=} (\ln \gamma')' \,, \tag{4}$$

and the initial condition by

$$g \stackrel{\text{def}}{=} \gamma^{-1}(f) \,. \tag{5}$$

We have the following commuting diagram:

$$\begin{array}{ccc} u & \stackrel{\gamma}{\longleftarrow} & v \\ {\scriptstyle Eq.\ (1)} \big\uparrow & & \big\uparrow {\scriptstyle Eq.\ (3)} \\ f & \stackrel{\gamma^{-1}}{\longrightarrow} & g \end{array} \tag{6}$$

Note that if γ tends to an affine transformation, *i.e.* $\mu \to 0$ implying $\gamma(v) = \alpha + \beta v$, one reobtains the linear equation. This is as one would expect, since the affine group is in fact precisely the invariance group of Eq. (1) under grey-scale point mappings.

One might conjecture that Eq. (3) is no longer invariant under affine grey-scale transformations. However, one should not fail to notice that the parameter μ and the image v are dimensionally dependent, so that the affine group also affects μ. Indeed, $(v; \mu) \to (\alpha + \beta v; \mu/\beta)$ with $\beta \neq 0$ is the full invariance group corresponding to affine grey-scalings in the new representation defined by Eq. (2).

In order to cope with the nonlinear schemes one may exploit the commuting diagram of Eq. (6).

[1] p.d.e.= "partial differential equation".

Proposition 1 *Let $\star$ denote correlation, i.e.*

$$f \star \phi(x) \stackrel{\text{def}}{=} \int dz\, f(x+z)\, \phi(z)\,.$$

If ϕ is the Green's function of Eq. (1) then the solution of Eq. (3) is given by

$$v \stackrel{\text{def}}{=} \gamma^{-1}\left(\gamma(g) \star \phi\right)\,.$$

Recall that the Green's function of Eq. (1) is the normalised Gaussian,

$$\phi(z;\sigma) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi\sigma^2}^{\,n}} \exp\left(-\frac{1}{2}\frac{\|z\|^2}{\sigma^2}\right)\,, \tag{7}$$

in which the *inner scale parameter* σ is related to the evolution parameter s of Eq. (1) by $\sigma = \sqrt{2s}$.

We henceforth assume that the initial image f is normalised to the unit interval, and that $\gamma : [0,1] \to [0,1]$ preserves this range. It is then easily seen that confinement to the unit interval is preserved under evolution by Eq. (3).

The fact that Eq. (3) is a generalisation of linear scale-space defined in terms of an infinite-dimensional function space adds more flexibility to the way Gaussian scale-space theory can be used in practical applications. As opposed to general nonlinear diffusion schemes all rigorous results known for the linear case still hold in some precise form. We discuss a number of concrete possibilities below.

Scalar Images

We first consider the case in which we are given a single image, and discuss the relevance of the theory for image processing, front-end vision, and image analysis.

Image Processing

Of all possible non-affine transformations, one class is particularly simple and somewhat special, *viz.* the one for which the coefficient μ of Eq. (4) is a global constant. This transformation is determined up to a couple of integration constants, $\alpha = \gamma_\mu(0)$ and $\beta = \gamma'_\mu(0)$, and is given by

$$\gamma_\mu(v) = \begin{cases} \beta\mu^{-1}\left(\exp(\mu v) - 1\right) + \alpha & \text{if} \quad \mu \neq 0\,, \\ \beta\, v + \alpha & \text{if} \quad \mu = 0\,. \end{cases} \tag{8}$$

This case has been discussed in detail elsewhere [6], where it is also shown that one can take the limits $\mu \to \pm\infty$, producing the morphological counterparts of Eq. (1) in the form of erosion and dilation scale-spaces based on a quadratic structuring function. See *e.g.* Boomgaard *et al.* [1, 2, 4, 3] and Dorst *et al.* [5] for details on the morphological limits. Fig. 1 illustrates the resulting scale-spaces that one obtains if one inserts finite values for μ.

Front-End Vision

Apart from the fact that Eqs. (3), (4) and (8) yield scale-spaces that are in some precise sense "in-between" the familiar linear and morphological ones, which is of interest in its own right, they may also provide a valuable model for front-end visual processing, as they account for a logarithmic compression of the raw input distribution f (the photon flux impinging onto the retina), as well as for multiple scales. (Note that, for instance, Weber's and Fechner's laws [7] arise naturally for values of u that are well above threshold.)

Image Analysis

The general case, Eq. (3), is of interest in the development of specialised multiscale techniques. More specifically, since the mapping introduces a particular bias in the grey-value domain, it is interesting for its potential role in accounting for *a priori* knowledge that warrants such a bias.

One way to obtain a monotonic mapping is as follows:

$$\gamma(v) = \int_0^v dy\, \varrho(y)\,, \tag{9}$$

in which $\varrho(y)$ is some positive measure, normalised such that $\gamma(1) = 1$ (say). This measure could be the image's grey-value histogram, or any other histogram inspired by some tissue/image model (*v.i.*). However, one should be a bit careful and take into account that error propagation is governed by the Jacobian of the mapping, *i.e.* $\gamma'(v) = \varrho(v)$, so that an error δu in the u-domain corresponds to an error $\delta v = \delta u/\varrho(v)$ in the v-domain. If for some values of v the measure $\varrho(v)$ is nearly zero this obviously becomes problematic. A hack around this problem could be to replace $\varrho(y)$, if it is problematic, by $\varrho_\varepsilon(y) = \varrho(y) + \varepsilon$, for some $0 < \varepsilon \ll 1$ (of course one should then renormalise $\varrho_\varepsilon(y)$ again to unit weight). This guarantees that the Jacobian always exceeds ε, thus tempering the errors. A similar effect is obtained if one restricts grey-values to a sub-interval in which $\varrho(y) > \varepsilon$ (although one may then lose information it may not be a problem depending on one's task, *e.g.* segmentation of one particular tissue type in medical imaging). A similar regularisation effect could be obtained by blurring the measure $\varrho(y)$ in the y-domain [11]. A natural scale for this could be the quantisation error of the initial image or the noise amplitude in the case of additive noise. Perhaps a better way to avoid problems is to simply remain cautious and take error propagation into account throughout the analysis.

Applying Eq. (9) to Eq. (4) one obtains a nonlinearity coefficient that is given by

$$\mu = (\ln \varrho)'\,. \tag{10}$$

From this we see that at critical points of the histogram measure, *i.e.* if $\varrho' = 0$ (assuming $\varrho \neq 0$), the nonlinearity coefficient vanishes and the blurring becomes linear. On the other hand, if $\varrho = 0$ (while $\varrho' \neq 0$), the nonlinearity coefficient becomes degenerate.

A possible way of exploiting Eq. (10) is to enhance image evidence for a particular tissue type of which the *a priori* histogram is known. If the appropriate modality is used such a histogram is typically unimodular, so that Eq. (9) will indeed produce an

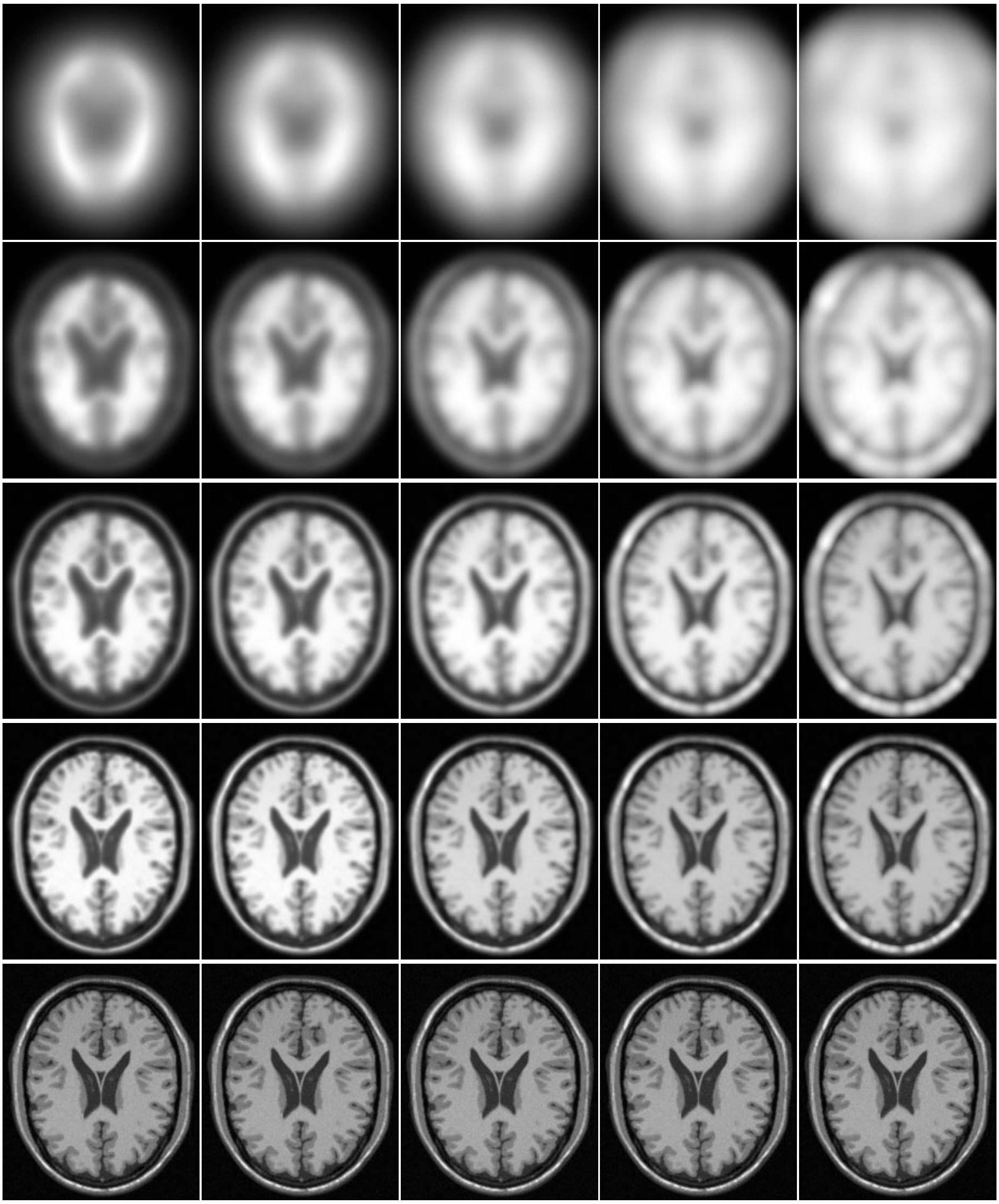

Figure 1: Pseudo-linear scale-space representations of MRI image, obtained according to Eqs. (3–4) for constant μ. Scale $\sigma = \sqrt{2s}$ varies exponentially in vertical direction: $\sigma = 2^k$ pixels, with $k = 0, 1, 2, 3$ (bottom up). The parameter μ varies in horizontal direction: from left to right we have $\mu = -8, -4, 0, 4, 8$, respectively. Dark regions are pronounced in erosion-like schemes ($\mu < 0$) whereas bright regions are emphasised in dilation-like schemes ($\mu > 0$).

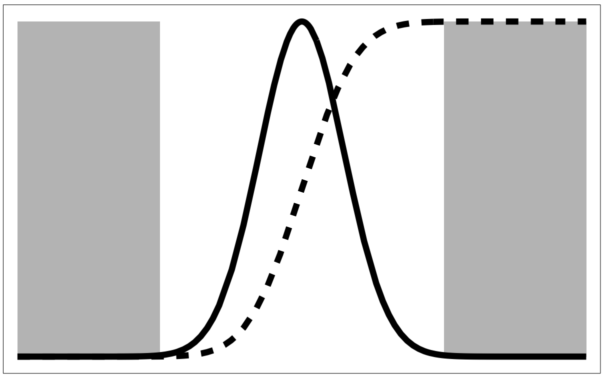

Figure 2: The prior distribution $\varrho(v)$ of grey-values (solid curve) induced by a particular tissue type is usually unimodular, at least for bulk material. Its primitive $\gamma(v)$ (dashed curve) is consequently strictly monotonic within the grey-value interval of interest (*i.e.* in-between the shaded regions), and may therefore serve as an admissible transformation. Information pertaining to the shaded regions is lost, while uncertainty increases towards the boundaries.

invertible mapping within a subinterval of grey-values. Of course the bias introduced by the mapping will also enhance other tissue types insofar as their grey-value histograms overlap with that of the desired one. See Figure 2.

Multi-Component Images

Next we consider a possible extension to deal with multi-component images. One has to distinguish between vector-valued images and multispectral images (which may not be part of a vector, *i.e.* linear space).

Vector-Valued Images

Suppose we have a vector-valued raw image $\mathbf{g}$ with components g^μ relative to a coordinate basis, *i.e.*[2] $\mathbf{g} = g^\mu \mathbf{e}_\mu$, for which we would like to define a scale-space extension $\mathbf{v} = v^\mu \mathbf{e}_\mu$. Let us furthermore assume that u is some scalar field obtained from $\mathbf{v}$, *e.g.* the scalar product $\mathbf{v} \cdot \mathbf{v}$. Since this is a scalar it is not unreasonable to require that it satisfies the linear diffusion equation, Eq. (1), although any other scalar that can be constructed from $\mathbf{v}$ would be an equally legitimate choice (*i.c.* any power of the scalar product, such as the magnitude $\|\mathbf{v}\|$). For reasons of generality let us assume that the scalar of interest is given by

$$u = \gamma(\mathbf{v}) \,. \tag{11}$$

By substitution into Eq. (1) one then finds that the components of $\mathbf{v}$ must satisfy

$$\gamma_\mu \partial_s v^\mu = \gamma_\mu \Delta v^\mu + \gamma_{\nu\rho} \nabla_\alpha v^\nu \nabla^\alpha v^\rho \,, \tag{12}$$

[2]Summation convention is used throughout: Repeated spatial indices—with values in the range $1, \ldots, n$ in n-dimensional space—are dummies over which a summation is implied.

in which γ_μ and $\gamma_{\nu\rho}$ are first and second order derivatives of γ. If the Jacobian has maximal rank we can use the γ_μ, $\mu = 1, \ldots, n$, as a basis, and we can write

$$\gamma_{\nu\rho} = \mu^{\mu}_{\nu\rho} \gamma_\mu \,, \tag{13}$$

for some $(2+1)$-tensor $\mu(\mathbf{v})$. Substituting this into Eq. (12) we then obtain

$$\begin{cases} \partial_s v^\mu & = & \Delta v^\mu + \mu^\mu_{\nu\rho} \nabla_\alpha v^\nu \nabla^\alpha v^\rho \\ \lim_{s\downarrow 0} v^\mu & = & g^\mu \,. \end{cases} \tag{14}$$

which prescribes the scale-space representation for the individual components g^μ of the raw vector field $\mathbf{g}$, consistent with the linear representation of Eq. (1) for the scalar $f = \gamma(\mathbf{g})$. Eqs. (13–14) are the vector analogues of Eqs. (3–4), with the scalar nonlinearity $\mu(v)$ replaced by the nonlinearity tensor $\mu(\mathbf{v})$.

As an example, suppose that $u = \gamma(\mathbf{v}) = \sqrt{\mathbf{v} \cdot \mathbf{v}}$, *i.e.* we take the magnitude of the vector field as the scalar that we wish to subject to a linear scale-space representation. In this case the components of the nonlinearity tensor are given by

$$\mu^\mu_{\nu\rho}(\mathbf{v}) = \frac{1}{||\mathbf{v}||^2} \left(g_{\nu\rho} \delta^\mu_\sigma - \frac{1}{2} (g_{\nu\sigma} \delta^\mu_\rho + g_{\rho\sigma} \delta^\mu_\nu) \right) v^\sigma \,, \tag{15}$$

in which $g_{\mu\nu}$ are the components of the spatial metric tensor (in a Cartesian coordinate system equal to 1 iff $\mu = \nu$, otherwise 0), and δ^ν_μ are the invariant components of the Kronecker tensor (similarly defined in an arbitrary coordinate system). It should be noted that the tensor is completely parameter free.

Multispectral Images

Now suppose we have a collection of multispectral images g_α, $\alpha = 1, \ldots, N$, for which we would like to define a consistent multiscale representation. Consistency pertains to the presumption that all these images have been obtained by sampling a single source spectrum ϱ according to N different protocols. If we now postulate that Eq. (1) should hold for the unbiased source field

$$u = \int d\lambda \, \varrho(\lambda) \,, \tag{16}$$

then the multiscale representations v_α of g_α must be constrained accordingly.

To be more specific, let us assume that each component v_α is the output of a probing procedure in λ-space that entails two steps: (i) a linear λ-superposition of the spectrum, and (ii) a nonlinear monotonic resampling of the result. That is, if γ_α is the inverse of the latter resampling function, then[3]

$$\gamma_\alpha(v_\alpha) = \int d\lambda \, \varrho_\alpha(\lambda) \,. \tag{17}$$

It is convenient to introduce the efficiency function η_α for channel α, which takes values in the unit interval and together with the other channels constitutes a partition of unity:

$$\varrho_\alpha(\lambda) = \eta_\alpha(\lambda) \varrho(\lambda) \quad \text{and} \quad \sum_{\alpha=1}^{N} \eta_\alpha(\lambda) = 1 \,. \tag{18}$$

[3] In expressions with only lower indices summation convention does *not* apply.

Consequently, the "unbiased" field u is essentially the result of a superposition:

$$u = \gamma(\mathbf{v}) \stackrel{\text{def}}{=} \sum_{\alpha=1}^{N} \gamma_\alpha(v_\alpha)\,, \tag{19}$$

and we can apply the techniques of the previous section. Since the derivative of $\gamma(\mathbf{v})$ with respect to v_α depends only on v_α, the channels are effectively decoupled:

$$\left\{ \begin{array}{lcl} \partial_s v_\alpha & = & \Delta v_\alpha + \mu_\alpha \|\nabla v_\alpha\|^2\,, \\ \lim_{s\downarrow 0} v_\alpha & = & g_\alpha\,. \end{array} \right. \tag{20}$$

in which μ_α corresponds to the only nontrivial component $\mu^\alpha_{\alpha\alpha}$ of the tensor of Eq. (15):

$$\mu_\alpha = \frac{\gamma''_\alpha}{\gamma'_\alpha}\,, \tag{21}$$

i.e. we have N equations similar to Eqs. (3) and (4) for the scalar case.

To determine the form of the nonlinearity coefficients μ_α one needs to have knowledge of image formation details for the corresponding imaging modalities or protocols, so that one can fill in the missing details of Eqs. (16–19).

3 Conclusion and Discussion

We have constructed a class of mutually diffeomorphic "pseudo-linear" scale-spaces depending on a particular function μ. The resulting construct resembles *reaction-diffusion* schemes reported in the literature by Kimia and Siddiqi [8], Kimia *et al.* [9], among others; *cf.* also Smoller [13]. Indeed, the principle is quite similar, as is easily appreciated from the p.d.e. formulation; Eqs. (3), (14) and (20) all describe evolution schemes driven by an elliptic (or diffusion) and a parabolic (or reaction) term. However, intrinsic to the way we have arrived at our scheme there exists, from the outset, only a *single* control parameter to play with, *viz.* μ, in other words, the ratio of "reaction/diffusion" is predetermined, whereas in conventional reaction-diffusion schemes the amount of reaction and diffusion can in principle be independently controlled.

From Eqs. (2) and (4) and similar ones for the generalised cases it follows that one always has a reaction term of the form $\mu(v)\|\nabla v\|^2$. Since $\gamma(v)$ is a *point mapping—i.e.* depends only on v but not on its derivatives—one cannot get rid of the factor $\|\nabla v\|^2$. It remains an unsolved problem whether it might be possible to extend the class of isomorphisms to mappings that also depend on derivatives of v akin to Lie-Bäcklund transformations.

As for applications one should think of multiresolution techniques for scalar, vector and multispectral images in the same context in which the familar linear and morphological schemes for grey-value images are being employed with success, such as in multiscale segmentation algorithms. The choice of the mapping $\gamma(v)$ introduces a grey-value bias that can be exploited in the presence of *a priori* knowledge.

A suggestion for future work is therefore to take the parameter μ into account in such multiresolution segmentation schemes [12, 14, 15]. The reason is that segments are always characterised by a grey-value histogram of finite spread. For instance, in medical imaging definite tissue types will map to segments of which the *a priori* grey-value histogram (for bulk matter) may be known either by theoretical prediction or from phantom studies. Whereas the geometry of individual iso-contours is invariant by construction, it is precisely this histogram that is affected by the pseudo-linearity in a way that can be controlled by μ. Thus it provides a well-understood point of departure for bringing in *a priori* knowledge on the relation between tissue types and signal formation. It is feasible to tune the choice of μ to the tissue type of interest and the noise characteristics.

Apart from considering more general mappings of the type $\gamma(v, \nabla v, \ldots)$ as suggested above, one could admit *complex* functions μ, the simplest case being that in which $\mu \in \mathbb{C}$ is a purely imaginary or complex constant.

A Causality in the Resolution Domain

Consider functions of the type $u(x; s)$ with $(x; s) \in \mathbb{R}^n \times \mathbb{R}$. If u is continuously differentiable with respect to s and twice continuously differentiable with respect to x, then we can represent the *osculating paraboloid* to an iso-surface in scale-space at a spatial extremum—the origin, say—in which the tangent plane is a plane of constant scale, by a second order "Monge patch" parametrisation:

$$s = \frac{1}{2} x^{\mathrm{T}} \mathbf{Q} x \quad \text{with} \quad \mathbf{Q} = -\frac{1}{u_s} \mathbf{H} . \tag{22}$$

Here, $\mathbf{H}$ is the Hessian matrix of u evaluated at the origin, with entries u_{ij}, $i, j = 1, \ldots, n$ (second order *spatial* derivatives of u), and u_s is the scale derivative of u. Note that in an extremum the Hessian eigenvalues are either all positive or all negative, so that the corresponding surface is indeed a paraboloid. At the origin its normal coincides with the normal to the original iso-surface, and points towards increasing scale iff (note that the uniform sign of the Hessian eigenvalues equals that of their sum, *i.e.* the Laplacean)

$$u_s \Delta u > 0 . \tag{23}$$

The enforcement of this demand is known as the *causality principle*. The simplest *linear* p.d.e. that realises this is the isotropic diffusion equation

$$u_s = \Delta u . \tag{24}$$

Now suppose $u = \gamma(v)$, where γ is a strictly monotonic grey-scale mapping ($\gamma' \neq 0$). It is easily verified that at the location of an extremum the matrix $\mathbf{Q}$ is invariant under this mapping, so that we may conclude that at extrema the convex side of a scale-space isosurface still points in the direction of increasing scale. In other words, monotonic grey-scale mappings of the type $u = \gamma(v)$ preserve the causality property.

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