On the Behaviour of Critical Points under Gaussian Blurring*

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Abstract

The level of detail of an image can be expressed in terms of its topology, *i.e.* the distribution of Morse critical points and their types, which in turn is governed by resolution. We study the behaviour of critical points as a function of resolution for Gaussian scale-space images using catastrophe theory. Unlike existing literature, in which one employs local, so-called canonical coordinates for theoretical convenience, we state results in terms of a global, user-defined Cartesian coordinate system. This enables a fairly straightforward implementation of these results in practice.

1 Introduction

A fairly well understood way to endow an image with a topology is to embed it into a one-parameter family of images known as a "scale-space image". The parameter encodes "scale" or "resolution" (coarse/fine scale means low/high resolution, respectively).

Among the simplest is the linear or Gaussian scale-space model. Proposed by Iijima [10] in the context of pattern recognition it went largely unnoticed for a couple of decades, at least outside the Japanese scientific community. Another early Japanese contribution is due to Otsu [26]. The Japanese accounts are quite elegant and can still be regarded up-to-date in their way of motivating Gaussian scale-space; for a translation, the reader is referred to Weickert, Ishikawa, and Imiya [33]. The earliest accounts in the English literature are due to Witkin [34] and Koenderink [15]. In view of ample literature on the subject we will henceforth assume familiarity with the basics of Gaussian scale-space theory [4, 9, 23, 28].

In their original accounts both Koenderink as well as Witkin proposed to investigate the "deep structure" of an image, *i.e. structure at all levels of resolution simultaneously*. Today, the handling of deep structure is still an outstanding problem in applications of scale-space theory. Nevertheless, many heuristic approaches have been developed for specific purposes that do appear promising. These typically utilise some form of scale selection and/or linking scheme, *cf.* Bergholm's edge focusing scheme [1], Lindeberg's feature detection method [23, 24], the scale optimisation criterion used by Niessen *et al.* [25] and Florack *et al.* [7] for motion extraction, Vincken's hyperstack segmentation algorithm [32], *etc.*. Encouraged by the results in specific image analysis applications an increasing interest has recently emerged trying to establish a generic underpinning of deep structure. Results from this could serve as a common basis for a diversity of multiresolution schemes. Such bottom-up approaches invariably rely on *catastrophe theory*.

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Catastrophe theory in the context of the scale-space paradigm is now fairly well-established. It has been studied, among others, by Damon [2]—probably the most comprehensive account on the subject—as well as by Griffin [8], Johansen [11, 12, 13], Lindeberg [21, 22, 23], and Koenderink [16, 17, 18, 19, 20]. An algorithmic approach has been described by Tingleff [31]. Closely related to the present article is the work by Kalitzin [14], who pursues a nonperturbative topological approach.

2 Theory

Catastrophe theory is the study of how the critical points change as the control parameters change.

While varying a control parameter in a continuous fashion, a *Morse critical point* (extremum or saddle in a "typical" image) will move along a *critical curve*. At isolated points on such a curve one of the eigenvalues of the Hessian may become zero, so that the Morse critical point turns into a *non-Morse critical point*.

The *Morse Lemma* states that image topology in a neighbourhood of a Morse critical point is essentially determined by the second order Taylor expansion. In a neighbourhood of a non-Morse critical point one generally needs a Taylor polynomial of order 3 or higher, details of which are described by *Thom's Theorem*.

If one slightly perturbs the image, Morse critical points will generally not be affected to the extent that although they may undergo a small spatial displacement as well as a change of intensity, nothing will happen to them qualitatively. Non-Morse parts, on the other hand, do change qualitatively upon perturbation. In general, a non-Morse critical point will split into a number of Morse critical points. This state of events is called *morsification*. The Morse saddle types of the isolated Morse critical points involved in this process are characteristic for the catastrophe. Thom's Theorem provides an exhaustive list of "elementary catastrophes" ($1, \ldots, 5$ control parameters), with canonical formulas for the catastrophe germs as well as for the perturbations needed to describe their morsification [29, 30].

For a rigorous account of catastrophe theory applied to scale-space images, *cf.* Damon [2, 3]. Below we present a summary, whereby we concentrate on the case that interests us, *viz.* that in which we presume that the initial image is a "typical" one in the sense that there are no special mutual dependencies between derivatives. This is always the case if the image is contaminated by noise, be it ever so small, provided quantisation and discretisation effects are negligible.

The only generic morsifications in scale-space are *creations* and *annihilations* of pairs of Morse hypersaddles of opposite Hessian signature[1]: Fig. 1 (for a proof, see Damon [2]). Everything else can be expressed as a compound of isolated events of either of these two types (although one may not always be able to segregate the elementary events due to numerical limitations).

In order to facilitate the description of topological events, Damon's account, following the usual line of approach in the literature, relies on a slick choice of coordinates, which is possible by virtue of the so-called *Thom Splitting Lemma*. However, these so-called "canonical coordinates" are inconvenient in practice, unless one provides an operational scheme relating them to user-defined coordinates. Mathematical accounts fail to be operational in the sense that—in typical cases—canonical coordinates are at best proven to exist.

The line of approach that exploits suitably chosen coordinates is known as the *canonical formalism*. It provides the most parsimonious way to approach topology if neither metrical relations nor numerical computations are of interest. Adhering to the canonical formalism for the moment let us define the

[1]"Hessian signature" means "sign of the Hessian determinant evaluated at the location of the critical point", also referred to as the critical point's "(topological) charge".

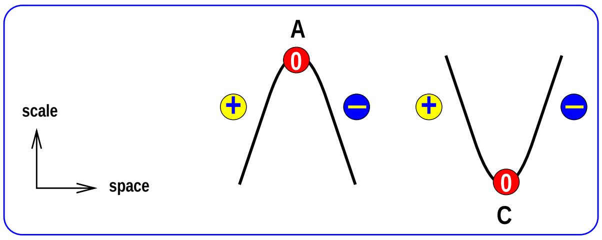

Figure 1: The generic catastrophes in isotropic scale-space. Left: annihilation of a pair of Morse critical points. Right: creation of a pair of Morse critical points. In both cases the points involved have opposite Hessian signature. In 1D, positive signature signifies a minimum, while a negative one indicates a maximum; creation is prohibited by the diffusion equation. In multidimensional spaces creations do occur generically, but are typically not as frequent as annihilations.

following *catastrophe germs* in 2D:

$$g^{\mathrm{A}}(x;t) \stackrel{\mathrm{def}}{=} x^3 + 6xt\,, \tag{1}$$

and

$$g^{\mathrm{C}}(x,y;t) \stackrel{\mathrm{def}}{=} x^3 - 6x(y^2+t)\,, \tag{2}$$

together with their perturbations

$$f^{\mathrm{A}}(x,y;t) \stackrel{\mathrm{def}}{=} g^{\mathrm{A}}(x;t) + Q(x,y;t)\,, \tag{3}$$

$$f^{\mathrm{C}}(x,y;t) \stackrel{\mathrm{def}}{=} g^{\mathrm{C}}(x,y;t) + Q(x,y;t)\,, \tag{4}$$

in which the perturbation term is a quadric $Q(x,y;t) = \pm(y^2+2t)$. Germs as well as perturbations satisfy the diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u\,. \tag{5}$$

In the canonical formalism it is conjectured that, given a generic event in scale-space, one can always set up coordinates in such a way that qualitative behaviour is summarised by one of the two "canonical forms" given above. As one may easily verify, the forms $f^{\mathrm{A}}(x,y;t)$ and $f^{\mathrm{C}}(x,y;t)$ correspond to an *annihilation* and a *creation* event at the origin, respectively (*v.i.*). Both events are so-called "fold catastrophes". The diffusion equation imposes a constraint that manifests itself in the asymmetry of these two forms.

Morsification of the A-germ entails an annihilation of two critical points of opposite charge (recall Footnote 1) as resolution is diminished. For $t < 0$ we have two Morse-critical points carrying opposite charge, for $t > 0$ there are none. At $t = 0$ the two critical points collide and annihilate. See Fig. 2. Morsification of the C-germ entails a creation of two critical points of opposite charge as resolution is diminished. For $t < 0$ there are no Morse-critical points in the immediate neighbourhood of the origin. At $t = 0$ two critical points of opposite charge emerge producing two critical curves for $t > 0$.

Genericity implies that annihilations and creations will persist despite perturbations, and will suffer at most a small displacement in scale-space. Together these two types of events exhaust the list of possible generic catastrophes.

To summarise, the canonical formalism enables a fairly simple description of what can happen topologically. However, in practice the separation into "bad" and "nice" coordinates exploited in the canonical formalism is not given. Moreover, canonical coordinates do not permit us to compute metrical

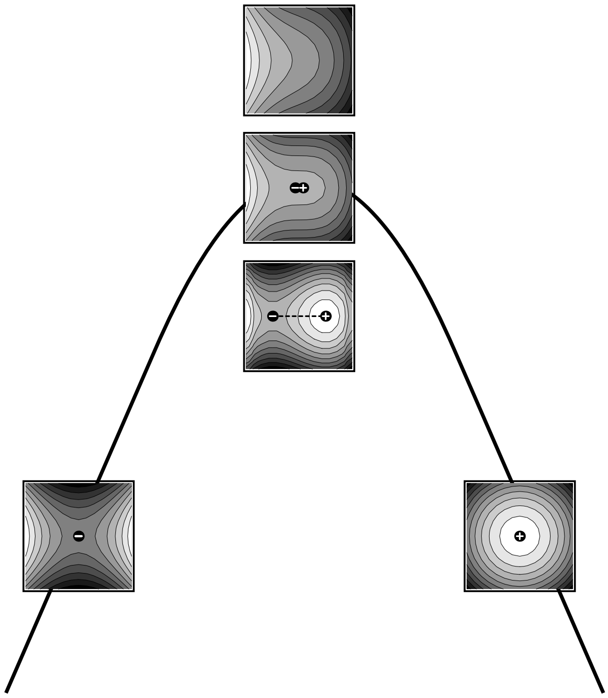

Figure 2: In 2D, positive Hessian determinant signifies an extremum, while a negative one indicates a saddle. The morsification is visualised here for the annihilation event, Eq. (1), showing five typical, fixed-scale local pictures at different points on or near the critical curve.

properties of critical curves, unless one has a recipe that links them to a well-known coordinate system. This limitation led us to develop the covariant formalism.

The covariant formalism declines from the explicit construction of canonical coordinates altogether. It allows us (i) to carry out computations in any *user-defined, global coordinate system*, requiring only a few image convolutions per level of scale, and (ii) to compute *metrical properties* of topological events, such as angles, directions, velocities (or tangents), accelerations (or curvatures), *etc.*

It is easy to see that to first order approximation the location of a degenerate critical point in $(n+1)$-dimensional scale-space is given by the linear system

$$\begin{bmatrix} \mathbf{H} & \mathbf{w} \\ \mathbf{z}^{\mathrm{T}} & \mathrm{c} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ t \end{bmatrix} = - \begin{bmatrix} \mathbf{g} \\ \det \mathbf{H} \end{bmatrix} , \tag{6}$$

in which the coefficients are given by first order derivatives of the image's gradient $\mathbf{g}$ and Hessian determinant $\det \mathbf{H}$, evaluated at the point of expansion near the critical point of interest, as follows:

$$\mathbf{H} = \nabla \mathbf{g} \, , \mathbf{w} = \partial_t \mathbf{g} \, , \mathbf{z} = \nabla \det \mathbf{H} \, , \mathrm{c} = \partial_t \det \mathbf{H} \, , \tag{7}$$

or, if we restrict our attention to 2D images (generalisation to arbitrary spatial dimensions is straightforward [6]),

$$\mathbf{H} = \begin{bmatrix} L_{xx} & L_{xy} \\ L_{xy} & L_{yy} \end{bmatrix} \tag{8}$$

$$\mathbf{w} = \begin{bmatrix} \Delta L_x \\ \Delta L_y \end{bmatrix} , \tag{9}$$

$$\mathbf{z} = \begin{bmatrix} L_{xxx}L_{yy} + L_{xx}L_{xyy} - 2L_{xy}L_{xxy} \\ L_{yyy}L_{xx} + L_{yy}L_{xxy} - 2L_{xy}L_{xyy} \end{bmatrix}, \tag{10}$$

and

$$\mathrm{c} = L_{xx}\Delta L_{yy} - 2L_{xy}\Delta L_{xy} + L_{yy}\Delta L_{xx}\,. \tag{11}$$

Here Δ denotes the Laplacean operator. Apparently the first order scheme requires spatial derivatives up to *fourth* order. It is important to note that Eqs. (6–11) hold *in any Cartesian coordinate system* (in fact, in any other coordinate system as well if one replaces partial derivatives by so-called covariant derivatives [27]). This property of form invariance is known as *covariance*.

Our next goal is to invert the system of Eqs. (6–11) *while maintaining covariance*. This obviates the need for numerical inversions or the construction of canonical frames. The inversion differs qualitatively for Morse and non-Morse critical points and so we consider the two cases separately.

If we discard the degeneracy condition of vanishing Hessian determinant we obtain a linear approximation of the critical curve through the Morse critical point of interest:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{L_{xx}L_{yy} - L_{xy}^2} \begin{bmatrix} -L_x L_{yy} + L_{xy}L_y \\ L_x L_{xy} - L_{xx}L_y \end{bmatrix} + \frac{1}{L_{xx}L_{yy} - L_{xy}^2} \begin{bmatrix} -L_{yy}\Delta L_x + L_{xy}\Delta L_y \\ L_{xy}\Delta L_x - L_{xx}\Delta L_y \end{bmatrix} t\,. \tag{12}$$

For non-Morse critical points this clearly makes no sense. In that case we must consider the full system so as to obtain 3 equations in 3 unknowns (x, y and t), and we must expect to find isolated points in scale-space only. With the established results it is now possible to invert the linear system, Eq. (6).

$$\begin{bmatrix} \mathbf{x} \\ t \end{bmatrix} = -\begin{bmatrix} \mathbf{H} & \mathbf{w} \\ \mathbf{z}^{\mathrm{T}} & \mathrm{c} \end{bmatrix}^{\mathrm{inv}} \begin{bmatrix} \mathbf{g} \\ \det \mathbf{H} \end{bmatrix}. \tag{13}$$

Note that the extra scale dimension permits us to invert the coefficient matrix regardless of regularity of the Hessian. The simplest way to do this is to use the definition of the transposed cofactor matrix $\widetilde{\mathrm{M}}$ corresponding to a matrix M, which in the regular case satisfies

$$\mathrm{M}^{\mathrm{inv}} = \frac{1}{\det \mathrm{M}}\widetilde{\mathrm{M}}, \tag{14}$$

but is also defined if M is singular. Identifying M with the coefficient matrix of Eq. (6) we then find

$$\widetilde{\mathrm{M}} \stackrel{\mathrm{def}}{=} \begin{bmatrix} \overline{\mathbf{H}} & \overline{\mathbf{w}} \\ \overline{\mathbf{z}}^{\mathrm{T}} & \overline{\mathrm{c}} \end{bmatrix}, \tag{15}$$

with (in 2D) $\mathbf{H} = \mathbf{z} \otimes \mathbf{w} - \mathbf{z} \cdot \mathbf{w}\, \mathbf{I}$, or

$$\overline{\mathbf{H}} = \begin{bmatrix} -(L_{yyy}L_{xx} - 2L_{xy}L_{xyy} + L_{yy}L_{xxy})\Delta L_y & (L_{yyy}L_{xx} - 2L_{xy}L_{xyy} + L_{yy}L_{xxy})\Delta L_x \\ (L_{xxx}L_{yy} - 2L_{xy}L_{xxy} + L_{xx}L_{xyy})\Delta L_y & -(L_{xxx}L_{yy} - 2L_{xy}L_{xxy} + L_{xx}L_{xyy})\Delta L_x \end{bmatrix}, \tag{16}$$

$\overline{\mathbf{w}} = -\widetilde{\mathbf{H}}\mathbf{w}$, or

$$\overline{\mathbf{w}} = \begin{bmatrix} L_{xy}\Delta L_y - L_{yy}\Delta L_x \\ L_{xy}\Delta L_x - L_{xx}\Delta L_y \end{bmatrix}, \tag{17}$$

$\overline{\mathbf{z}} = -\widetilde{\mathbf{H}}\mathbf{z}$, or

$$\overline{\mathbf{z}} = \begin{bmatrix} -L_{yy}(L_{xxx}L_{yy} - 2L_{xy}L_{xxy} + L_{xx}L_{xyy}) - L_{xy}(L_{yyy}L_{xx} - 2L_{xy}L_{xyy} + L_{yy}L_{xxy}) \\ L_{xy}(L_{xxx}L_{yy} - 2L_{xy}L_{xxy} + L_{xx}L_{xyy}) - L_{xx}(L_{yyy}L_{xx} - 2L_{xy}L_{xyy} + L_{yy}L_{xxy}) \end{bmatrix}, \tag{18}$$

and $\overline{\mathrm{c}} = \det \mathbf{H}$, *i.e.*

$$\overline{\mathrm{c}} = L_{xx}L_{yy} - L_{xy}^2 \,. \tag{19}$$

Furthermore we have

$$\begin{aligned}
&\det \mathbf{M} = \\
&\quad ([L_{xxyy} + L_{yyyy}]L_{xx} + [L_{xxxx} + L_{xxyy}]L_{yy} - 2[L_{xxxy} + L_{xyyy}]L_{xy})(L_{xx}L_{yy} - L_{xy}^2) + \\
&\quad -\{L_{xx}[L_{xxy} + L_{yyy}][L_{yyy}L_{xx} + L_{yy}L_{xxy} - 2L_{xy}L_{xyy}] + \\
&\quad +L_{yy}[L_{xxx} + L_{xyy}][L_{xxx}L_{yy} + L_{xx}L_{xyy} - 2L_{xy}L_{xxy}] + \\
&\quad -L_{xy}([L_{xxx} + L_{xyy}][L_{yyy}L_{xx} + L_{yy}L_{xxy} - 2L_{xy}L_{xyy}] + \\
&\quad [L_{xxy} + L_{yyy}][L_{xxx}L_{yy} + L_{xx}L_{xyy} - 2L_{xy}L_{xxy}])\} \,.
\end{aligned} \tag{20}$$

Note that this expression contains a term proportional to the Hessian determinant, which therefore vanishes at catastrophes, but odds are that the additional terms will remain nonzero, so that $\mathbf{M}$ can indeed be inverted in the generic case. Again, all these expressions are valid in any coordinate system as required. Eq. (13) can now be evaluated for the 2D case at hand using Eqs. (14–20) so as to produce the scale-space location of the catastrophe.

The sign of $\det \mathbf{M}$ subdivides the image domain into regions to which all generic catastrophes are confined. In fact, we have $\det \mathbf{M} < 0$ at annihilations, and $\det \mathbf{M} > 0$ at creations. One way to see this is to note that it holds for the canonical forms $f^{\mathrm{A}}(x, y; t)$ and $f^{\mathrm{C}}(x, y; t)$ of Eqs. (1–4). If we now transform these under an arbitrary coordinate transformation that leaves the diffusion equation invariant, it is easily verified that the sign of $\det \mathbf{M}$ is preserved. In fact, the following, more general result, holds at the location of a generic catastrophe:

$$t = \frac{1}{2}\frac{1}{\det \mathbf{M}}\left(\mathbf{z}^{\mathrm{T}}\mathbf{x}\right)^2 + \mathcal{O}(\|\mathbf{x}\|^3, \|\mathbf{x}\|\,t, t^2) \,. \tag{21}$$

The curvature of the critical path at the catastrophe is given by $(\overline{\mathbf{w}}^{\mathrm{T}}\nabla)^2\, t_{\text{catastrophe}} = \det \mathbf{M}$. The proof is based on a second order scale-space consistent Taylor expansion [5] evaluated at the catastrophe, and is given elsewhere [6].

A remarkable property of $\det \mathbf{M}$ is that in a "typical" case it is more likely to have a negative than a positive value. In view of the above claim this suggests that annihilations are expected to occur more frequently than creations, which indeed turns out to be the case in practice. After all, although their number does not necessarily decrease in a monotonic fashion, critical points are bound to disappear after a sufficient amount of blurring. See Fig. 3. Having established covariant expressions we have drawn several geometric conclusions that do not follow from the canonical formalism. Below we give a few more examples, using explicit Cartesian coordinates.

At any point on the critical curve—including the catastrophe—the scale-space tangent vector is proportional to $(\overline{\mathbf{w}}; \overline{\mathrm{c}})$, *cf.* Eqs. (17,19):

$$\begin{bmatrix} \mathbf{v}'_x \\ \mathbf{v}'_y \\ \mathrm{c}' \end{bmatrix} = \begin{bmatrix} \Delta L_y L_{xy} - \Delta L_x L_{yy} \\ \Delta L_x L_{xy} - \Delta L_y L_{xx} \\ L_{xx}L_{yy} - L_{xy}^2 \end{bmatrix} .$$

One can interpret this as a "velocity" in the sense of a displacement of the critical point per unit of "time", if one identifies the latter with $\tau = t/(L_{xx}L_{yy} - L_{xy}^2)$. This modified evolution parameter varies monotonically along the critical path through the catastrophe, in contrast with t itself.

Finally, the tangent plane to the Hessian zero-crossing in scale-space is given by the following equation in any Cartesian coordinate system:

$$\begin{aligned}
(L_{xxx}L_{yy} + L_{xx}L_{xyy} - 2L_{xy}L_{xxy})\,x \;&+\; (L_{yyy}L_{xx} + L_{yy}L_{xxy} - 2L_{xy}L_{xyy})\,y\, + \\
&+\; (\Delta L_{yy}L_{xx} + \Delta L_{xx}L_{yy} - 2\Delta L_{xy}L_{xy})\,t = 0 \,,
\end{aligned} \tag{22}$$

Figure 3: Left: 2D 256×256 slice from a brain MR scan. Middle: 64×64 subimage. Right: sign $\det \mathbf{M}$, evaluated for this subimage at a scale of $\sigma = 4.0$ pixels; light (dark) means positive (negative). In this case positive values are found at approximately 9% of the pixels.

in which one recognises the normal vector to be identical to $(\mathbf{z}; \mathrm{c})$ of Eqs. (10–11).

3 Conclusion and Discussion

We have presented an operational scheme to characterise critical curves, both in the vicinity of Morse as well as (generic) non-Morse critical points. These curves completely characterise the topology of a scale-space image, at least in the generic case. Lowest order metrical properties of these curves, notably tangent vectors and curvatures, can be straightforwardly computed by combining the output of a small number of linear filterings for computing image derivatives up to fourth order, as indicated.

The possibility to establish links between successive levels of scale is probably the most important feature provided by the Gaussian scale-space paradigm. Because bifurcations in a link-tree ideally reflect the morsifications of the catastrophes in scale-space (as these determine the unfolding of topological structure over scale), the results of this study can be used to establish a rigorous mathematical underpinning of various multiresolution techniques that are used in image analysis. The theory can be extended to arbitrary dimensions [6].

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