Computing Fence Designs for Orienting Parts *

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Abstract

A common task in automated manufacturing processes is to orient parts prior to assembly. We consider sensorless orientation of a polygonal part by a sequence of fences. We show that any polygonal part can be oriented by a sequence of fences placed along a conveyor belt, thereby settling a conjecture by Wiegley *et al.* [17], and present the first polynomial-time algorithm to compute the shortest such sequence. The algorithm is easy to implement and runs in time $O(n^3 \log n)$, where n is the number of vertices of the part.

1 Introduction

Many automated manufacturing processes require parts to be oriented prior to assembly. A part feeder takes in a stream of identical parts in arbitrary orientations and outputs them in a uniform orientation. Part feeders often use data obtained from some kind of sensing device to accomplish their task. We consider the problem of *sensorless orientation* of parts, in which the initial pose of the part is assumed to be unknown. In sensorless manipulation, parts are positioned and/or oriented using passive mechanical compliance. The input is a description of the part shape and the output is a sequence of open-loop actions that moves a part from an unknown initial pose into a unique final pose. Among the sensorless part feeders considered in literature are the parallel-jaw gripper [6, 9], a single pushing jaw [2, 10, 11, 13], a conveyor belt with a sequence of (stationary) fences placed along its sides [5, 14, 17], a conveyor belt with a single rotational fence (1JOC) [1], a tilting tray [8, 12], and vibratory plates and programmable vector fields [3, 4].

The pushing jaw [2, 10, 11, 13] orients a part by an alternating sequence of pushes and jaw reorientations. The problem of sensorless orientation by a pushing jaw is to find a sequence of push directions that will move the part from an arbitrary initial orientation into a single known final orientation. Such a sequence is referred to as a *push plan*. Goldberg [9] showed that any polygonal part can be oriented by a sequence of pushes. Chen and Ierardi [6] proved that any polygonal part with n vertices can be oriented by $O(n)$ pushes. They showed that this bound is tight by constructing (pathological) n-gons that require $\Omega(n)$ pushes to be oriented. Goldberg gave an algorithm for computing the shortest push plan for a polygon. His algorithm runs in $O(n^2)$ time.

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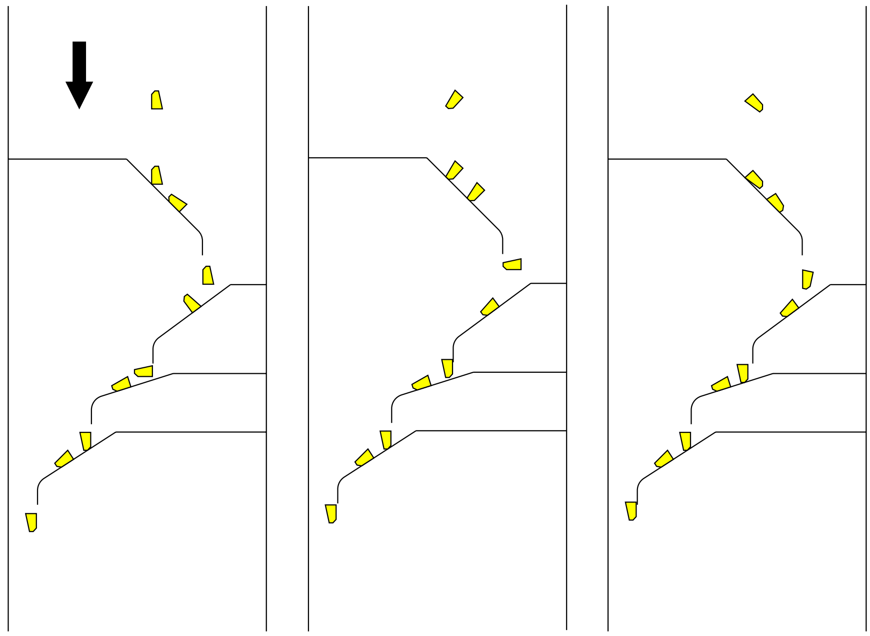


Figure 1: Three overhead views of the same conveyor belt and fence design. The traversals for three different initial orientations of the same part are displayed. The traversals show that the part ends up in the same orientation in each of the three cases.

The problem of *fence design* is to determine a sequence of fence orientations (see Figure 1) such that the fences with these orientations align the part as it moves down a conveyor belt and slides along these fences [5, 14, 17]. The motion of the belt effectively turns each slide into a push action by the fence in the direction normal to the fence. The fact that the direction of the push, i.e., the normal at the fence, must have a non-zero component in the direction opposite to the motion of the belt imposes a restriction on successive push directions. Fence design can be regarded as finding a constrained sequence of push directions (see Subsection 2.2 for the actual constraints). The additional constraints make fence design considerably more difficult than sensorless orientation by a pushing jaw. Wiegley *et al.* [17] conjectured that a fence design exists for any polygonal part. They gave an exponential algorithm for computing the shortest sequence of fences for a given part.

In this paper, we prove the conjecture that a fence design exists for any polygonal part. In addition, we give an $O(n^3 \log n)$ algorithm for computing a fence design of minimal length (in terms of the number of fences used). We show that fence designs of length $O(n)$ exist for a large class of parts. The algorithm is easy to implement and the resulting program returns fence designs for input parts within a fraction of a second. The program can be tuned to take into account certain quality requirements on the fence design, like minimum and maximum (successive) fence angles to prevent impractical steep and shallow fences and a long series of fences on a single side of the belt, which would require an impractically wide conveyor belt. The cost of the incorporation of quality measures is an increase of the algorithm's running time to $O(n^4)$.

Throughout the paper, we assume zero friction between the part and the fences. Since any push action acts on the convex hull of the part rather than on the part itself, we assume without loss of generality that the part under consideration is convex. Furthermore, we assume that the parts do not have meta-stable edges, i.e. the perpendicular projection of the center-of-mass on an edge does not intersect a vertex of the edge.

This paper is organized as follows. In Section 2, we first review the key notion of a push plan, and identify the constraints on the relative push angles for fence design. Section 3.1 discusses an $O(n^3 \log n)$ algorithm for computing the shortest fence design for a part. In Section 4, we focus on a large class of asymmetric polygonal parts and show that these parts can be oriented by fence designs of linear length (in the number of vertices). Section 5 generalizes the results of Section 4 to parts with some kind of symmetry.

2 Push plans and fence designs

2.1 The push function

In this section we focus on the push function of a part. Let P a convex polygonal part with n vertices and center-of-mass c. We assume that a fixed coordinate frame is attached to P. Directions are expressed relative to this frame. The contact direction of a supporting line l of P is uniquely defined as the direction of the normal of l pointing into P. The radius function $r : [0, 2\pi) \to \mathbb{R}^+$ maps a direction ϕ onto the distance from c to the supporting line of P with contact direction ϕ (see Figure 2). For a polygonal part, the radius function is a continuous piecewise sinusoidal function, and can be computed in $O(n)$ time by checking each vertex [11]. The final orientation of a part that is being pushed can be determined from its radius function.

In most cases, parts will start to rotate when pushed. If pushing in a certain direction does *not* cause the part to rotate, then the contact normal at one of its points of contact with the jaw passes through the center-of-mass [11]. We refer to the corresponding direction of the contact normal as an *equilibrium* push direction or orientation. If pushing does change the orientation, then this rotation changes the orientation of the pushing device relative to the part. We assume that pushing continues until the part stops rotating and settles in a stable equilibrium pose, which is an equilibrium with a preimage of non-zero length.

The *push function* $p : [0, 2\pi) \to [0, 2\pi)$ links every orientation ϕ to the orientation $p(\phi)$ in which the part P settles after being pushed by a jaw with initial contact direction ϕ (relative to the frame attached to P). The rotation of the part due to pushing causes the contact direction of the jaw to change. The final orientation $p(\phi)$ of the part is the contact direction of the jaw after the part has settled. The equilibrium push directions are the fixed points of p.

The push function p of a polygonal part consists of steps, which are intervals $I \subset [0, 2\pi)$ for which $p(\phi) = C$ for all $\phi \in I$ and some constant $C \in I$. The steps of the push function are easily constructed from the radius function r. If the part is pushed in a direction corresponding to a point of non-horizontal tangency of the radius function then the part will rotate in the direction in which the radius decreases. The part finally settles in an orientation corresponding to a local minimum of the radius function. As a result, all points in the open interval I bounded by two consecutive local maxima of the radius function r map onto the orientation $\phi \in I$ corresponding to the unique local minimum of r on I. (Note that ϕ itself maps onto ϕ because it is a point of horizontal tangency.) This results in the steps of the push function. In preparation for the next sections, we define two open intervals $l(v) = \{\phi < v | p(\phi) = v\}$ and $r(v) = \{\phi > v | p(\phi) = v\}$ for each fixed point – or equilibrium orientation – v of the

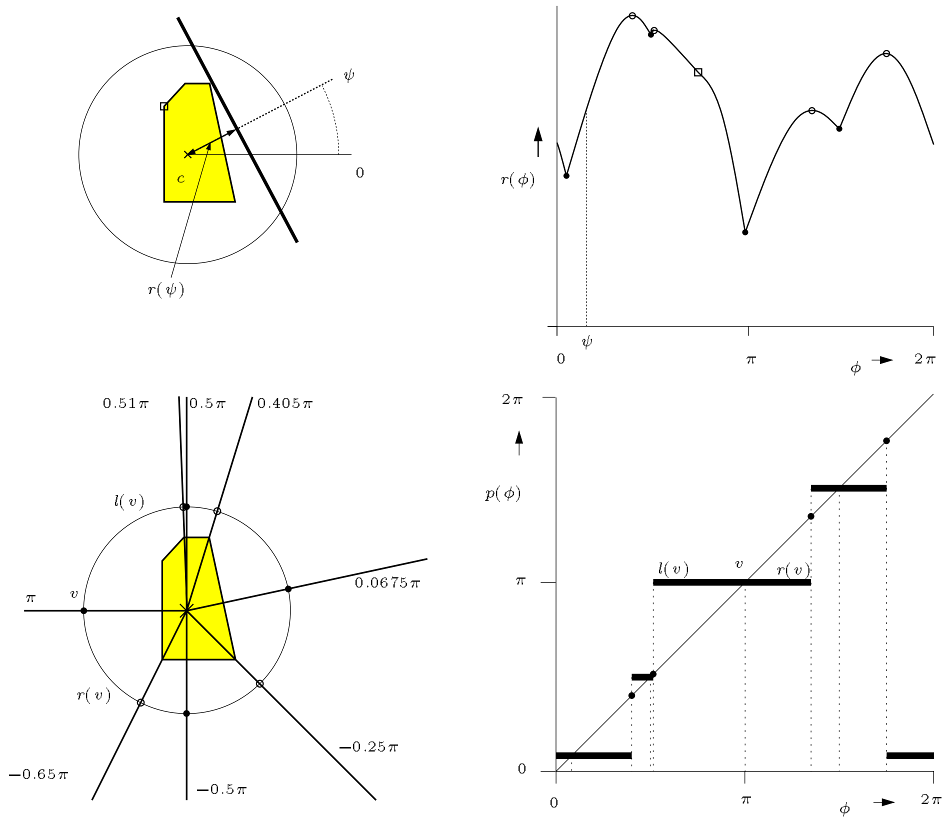


Figure 2: At the top, a polygonal part and its radius function $r : [0, 2\pi) \rightarrow \mathbb{R}^+$. The boxed kink in the radius corresponds to the boxed vertex of the part, which is not a maximum of the radius function. Below, the same part and its push function, in a circular and regular representation.

push function. We refer to these intervals as v's left and right environment respectively. The interval $l(v)$ corresponds to the half-step left of $v = f(v)$ and $r(v)$ corresponds to the half-step right of $v = f(v)$ (see Figure 2). Note that an equilibrium v is stable if $l(v) \cup r(v) \neq \emptyset$. Besides the steps, there are isolated points satisfying $p(\phi) = \phi$ in the push function, corresponding to local maxima of the radius function. Figure 2 shows a polygonal part and its push function. The behavior of the part while being pushed is fully described by the push function. In the appendix we show that any step function having only non-zero-length half-steps represents a polygonal part, and is therefore a valid push function.

A push function p is said to have period d if $p(\phi) = p((\phi + d) \bmod 2\pi)$ for all $\phi \in [0, 2\pi)$. Any part can only be oriented up to (periodic) symmetry in its push function.

We use the abbreviation p_α to denote the (shifted) push function defined by

$$p_\alpha(\phi) = p((\phi + \alpha) \bmod 2\pi),$$

for all $\phi \in [0, 2\pi)$. Note that $p_\alpha(\phi)$ is the final orientation of a part in initial orientation ϕ after a reorientation by α followed by a push. We can now define a *push plan*.

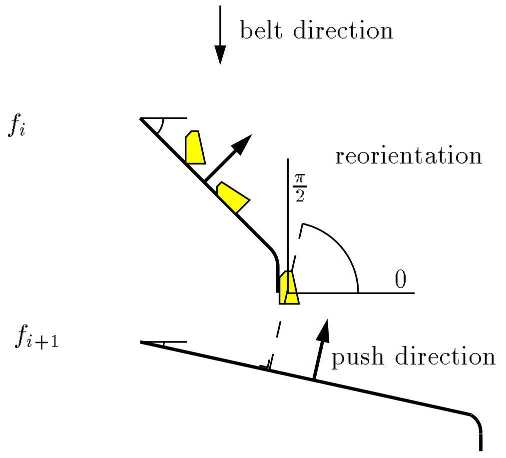
Definition 2.1 *A* push plan *is a sequence* $\alpha_1, \ldots, \alpha_m$ *such that* $p_{\alpha_m} \circ \ldots \circ p_{\alpha_1}(\phi) = \Phi$ *for all* $\phi \in [0, 2\pi)$ *and some fixed* $\Phi \in [0, 2\pi)$.

An important property of the push function is monotonicity, or the order preserving property [12]. A sequence $s_1, \ldots, s_n$ of elements of a set S is *ordered* if $s_1, \ldots, s_n$ are encountered in order when the generating cycle of S is traced *once*, starting from s_1. A function $p : S \to S$ is *monotonic* if for any ordered sequence $s_1, \ldots, s_n$ the sequence $p(s_1), \ldots, p(s_n)$ is also ordered.

2.2 Fence design

In this section we address the problem of designing a series of fences $f_1, \ldots, f_m$ that will orient P when it moves down a conveyor belt and slides along these fences $f_1, \ldots, f_m$. Let us assume that the conveyor belt moves horizontally from top to bottom, as indicated in the overhead view in Figure 4. We distinguish between left fences, which are placed along the left belt side, and right fences, which are placed along the right side. The fence angle β_i of a fence f_i denotes the angle between the upward pointing vector opposing the motion of the belt and the normal to the fence with a positive component in upward direction. The motion of the belt turns the sliding of the part along a fence into a push by the fence. The direction of the push is – by the zero friction assumption – orthogonal to the fence with a positive component in the direction opposing the motion of the belt. Thus, the motion of the belt causes any push direction to have a positive component in the direction opposing the belt motion. We now transform this constraint on the push direction relative to the belt into a constraint on successive push directions relative to the part.

Sliding along a fence f_i causes one of P's edges e to align with the fence. The curved tip of the fence [5] guarantees that e is aligned with the belt sides as P leaves the fence. If f_i is a left fence then e faces the left belt side (see Figure 3). If f_i is a right fence, it faces the right side. Assume f_i is a left fence. At the moment of leaving f_i, hence, after the push, the contact direction of f_i is perpendicular to the belt direction and towards the right belt side. So, the reorientation of the push is expressed relative to this direction. Figure 3(a) shows that the reorientation α_{i+1} is in the range $(0, \pi/2)$ if we choose f_{i+1} to be a left fence. If we take a right fence f_{i+1} then the reorientation is in the range $(\pi/2, \pi)$. A similar analysis can



(a)

f_i	α_{i+1}	f_{i+1}
left	$(0, \pi/2)$	left
left	$(\pi/2, \pi)$	right
right	$(-\pi, -\pi/2)$	left
right	$(-\pi/2, 0)$	right

(b)

Figure 3: (a) For two successive left fences, the reorientation of the push direction lies in the range $(0, \pi/2)$. (b) The ranges op possible reorientations of the push direction for all pairs of fence types.

be done when P leaves a right fence and the edge e faces the left belt side. The result is given in the table of Figure 3(b).

The table shows that the type of fence f_i imposes a bound on the reorientation α_{i+1}. Application of the same analysis to fences f_{i-1} and f_i and reorientation α_i leads to the following definition of a valid fence design [17].

Definition 2.2 *A* fence design *is a push plan* $\alpha_1, \ldots, \alpha_m$ *satisfying for all* $1 \leq i < m$*:*

- $\alpha_i \in (0, \frac{\pi}{2}) \cup (-\pi, -\frac{\pi}{2}) \Rightarrow \alpha_{i+1} \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$
- $\alpha_i \in (-\frac{\pi}{2}, 0) \cup (\frac{\pi}{2}, \pi) \Rightarrow \alpha_{i+1} \in (-\frac{\pi}{2}, 0) \cup (-\pi, -\frac{\pi}{2})$.

The push plan on the left in Figure 4 satisfies the constraints of Definition 2.2, and is therefore also a fence design.

3 Computing fence designs

3.1 A graph based approach

As every fence puts the part in a *stable* equilibrium orientation, the part is in one of these $m_s = O(n)$ orientations as it travels from one fence to another. Let us label these stable equilibria $a_0, \ldots, a_{m_s-1}$. After a first fence, the part can be in any of the stable equilibria $a_0, \ldots, a_{m_s-1}$. The problem is to reduce the set of possible orientations of P to one stable equilibrium a_i by a sequence of fences. We build a directed graph on all possible *states* of the part as it travels from one fence to a next fence. A state consists of a set of possible orientations of the part plus the type (left or right) of the last fence, as the latter imposes a restriction on the reorientation of the push direction. Although there are $O(2^{m_s})$ subsets of $\{a_0, \ldots, a_{m_s-1}\}$, we shall see that we can restrict ourselves to subsets consisting of sequences of adjacent stable equilibria. Any such sequence can be represented by the closed interval s

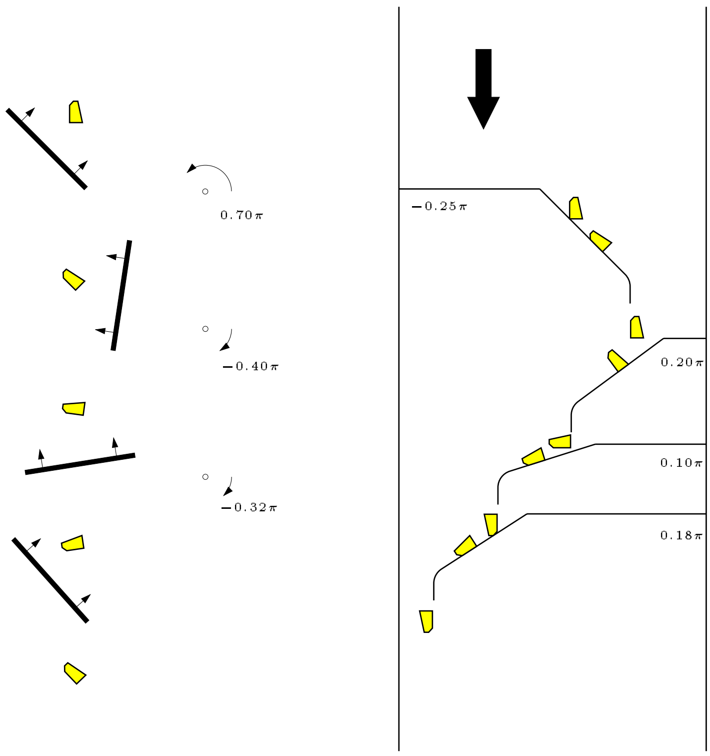


Figure 4: The left picture shows a plan for a pushing jaw. The jaw reorientations are (from top to bottom) 0.70π, -0.40π, and -0.32π. The conveyor belt on the right orients the same part. The fence angles, which are measured relative to the upward vector opposing the direction of belt motion, are -0.25π, 0.20π, 0.10π, and 0.18π. (Note that the sequence of fence orientations is not the same as the sequence of orientations of the pushing jaw because the curved fence tip rotates the stable edge to become aligned with the belt direction.)

defined by the first and last stable orientation a_i and a_j of the sequence. The resulting graph has $O(m_s^2)$ nodes.

Consider two graph nodes (s,t) and (s',t'), where $s = [a_i, a_j]$ and s' are intervals of stable equilibria and t and t' are fence types. Let $I_{t,t'}$ be the open interval (of length $\pi/2$) of reorientations admitted by the successive fences of types t and t' according to the table in Subsection 2.2. There is a directed edge from (s,t) to (s',t') if there is an angle $\alpha \in I_{t,t'}$ such that a reorientation of the push direction by α followed by a push moves any stable equilibrium in s into a stable orientation in s'. To check this condition, we determine the preimage $(u,w) \supseteq s'$ of s' under the (monotonic) push function. Observe that if $|s| = a_j - a_i < w - u$, any reorientation in the open interval $r = (u - a_i, w - a_j)$ followed by a push will map s into s'. We add an edge from (s,t) to (s',t') if the intersection of r and the interval $I_{t,t'}$ of admitted reorientations is non-empty, and label this edge with $r \cap I_{t,t'}$. For convenience, we add a source and a sink to the graph. We connect the source to every node (s,t) in the graph for which s contains all m_s stable equilibria, and we connect every node (s,t) with $s = [a_i, a_i]$ to the sink. Every path from the source to the sink now represents a fence design; a fence design of minimum length corresponds to a shortest such path.

Theorem 3.1 *The shortest path in the graph corresponds to the shortest fence design, if a fence design exists.*

Proof: We prove that each path corresponds to a fence design and vice versa. The theorem follows immediately if we realize that every edge in the path corresponds to a fence in the design.

($\Rightarrow$) If there is a path, we must prove that there is a corresponding fence design. Since there is an edge from (s,t) to (s',t') if and only if the successive fences t and t' allow for a reorientation that maps s into s', this follows immediately from the construction of the graph.

($\Leftarrow$) If there is a fence design, we prove there is a path in the graph that represents this fence design. Let $f_1, \ldots, f_N$ be a fence design. We track the possible orientations of the fence design, and prove by induction that for every set of possible orientations, there is a node in the graph, and furthermore, there is a path from the source to such a node. Let A_i denote the set of all possible orientations of P leaving f_i. It is easy to see that for each A_i there are multiple nodes that include the set of possible orientations.

After the first fence f_1, all stable equilibria are possible. Since we added edges from the source to all nodes containing all stable orientations, these nodes are reachable.

We now assume that for fence f_i having type t in our fence design the nodes (s,t) with $s \supseteq A_i$ are reachable from the source. Let t' be the fence type of f_{i+1}. Let (s',t') be a node such that $s' \supseteq A_{i+1}$. Let (u,w) denote the preimage of s'. Since the push function is monotonic and by existence of the fence design which maps A_i onto A_j, there is an admitted reorientation α_{i+1} by f_{i+1} such that $(u - \alpha_{i+1}, w - \alpha_{i+1}) \supset A_i$. Therefore, let (s,t) be a node such that $(u - \alpha_{i+1}, w - \alpha_{i+1}) \supset s \supseteq A_i$. There is an edge from (s,t) to (s',t'), and there is a path from the source to (s,t). Since $s' \supseteq A_{i+1}$ is arbitrary, all (s',t') with $s' \supseteq A_{i+1}$ are reachable from the source. □

An important observation is that some graph edges are redundant if we are just interested in a fence design of minimum length. Consider a node (s,t) and all its outgoing edges to nodes $(s' = [a_i, a_j], t')$ for a fixed left endpoint a_i and a fixed fence type t'. We show that only one of these outgoing edges is sufficient. The following lemma is the key to this result.

Lemma 3.2 *Let* (s,t), (s',t'), *and* (s'',t') *be nodes with* $s' = [a_i, a_j]$ *and* $s'' = [a_i, a_k]$ *and let* $s' \supset s''$. *If there are edges from* (s,t) *to both* (s',t') *and* (s'',t'), *then the edge from* (s,t) *to* (s',t') *can be deleted without affecting the length of the shortest path in the graph.*

Proof: Assume that τ is a path from source to sink containing the edge $((s,t),(s',t'))$ and assume $((s',t'),(s''',t'''))$ succeeds this edge in τ. Because $s' \supset s''$, there must also be an edge $((s'',t'),(s''',t'''))$ in the graph. Hence, we can replace the edges $((s,t),(s',t'))$ and $((s',t'),(s''',t'''))$ in τ by $((s,t),(s'',t'))$ and $((s'',t'),(s''',t'''))$ without affecting the length of τ. □

The repeated application of Lemma 3.2 to the graph (until no more edges can be deleted) leads to a reduced graph in which every node has just one outgoing edge per set of nodes with intervals with a common left endpoint and with a common fence type. The single edge from the initial graph that remains after the repeated application of Lemma 3.2 is the one to the node corresponding to the shortest interval. Since there are $O(m_s) = O(n)$ possible left endpoints and just two fence types, the number of outgoing edges from one node is reduced to $O(n)$. The total number of edges of the reduced graph is therefore $O(n^3)$.

Lemma 3.3 *The reduced graph contains* $O(n^2)$ *nodes and* $O(n^3)$ *edges.*

The (reduced) graph can be constructed in the following way. First we compute the push function and store it in such a way that preimages can be found in $O(1)$ time. For each node (s,t), left endpoint a_i, and fence type t', we must determine the shortest interval $s' = [a_i, a_j]$ such that an edge exists between (s,t) and (s',t'). We can do this by a binary search on j. Since checking whether an edge exists between a pair of nodes corresponds to computing the preimage of an interval (which can be done in constant time), this binary search takes $O(\log n)$ time. As a similar binary search must be performed for each combination of a node (s,t), a left endpoint a_i, and a fence type t', the total time required to compute the graph edges is $O(n^3 \log n)$. A breadth-first search of the graph takes $O(n^3)$ time (see e.g. [7]). This yields the following theorem.

Theorem 3.4 *The computation of the optimal fence design for a polygonal part* P *with* n *vertices takes* $O(n^3 \log n)$ *time.*

Let τ be a path in the graph from the source to the sink. Every edge of τ corresponds to a non-empty angular interval of possible reorientations of the push direction. We simply pick the midpoint of every such interval as the reorientation, and get a push plan which is a fence design. We can easily compute the fence angles from the reorientation angles on the path, using the properties of the fence framework [17].

3.2 Implementation

We implemented the described algorithm to test its behavior in practice. This turned out to be rather easy, using only some basic geometric computations for the push function, and some standard graph algorithms. The resulting code is very fast; it returned fence designs within a fraction of a second for all parts we tried. All fence designs shown in this paper were generated by the program. Our implementation offers the user the additional possibility of adding costs to graph edges. By doing so, the user can prevent the algorithm from outputting certain types of fence designs. Assigning high costs to edges between any pair of nodes of the same fence type t, for example, will cause the algorithm to output a sequence of alternating (left and right)

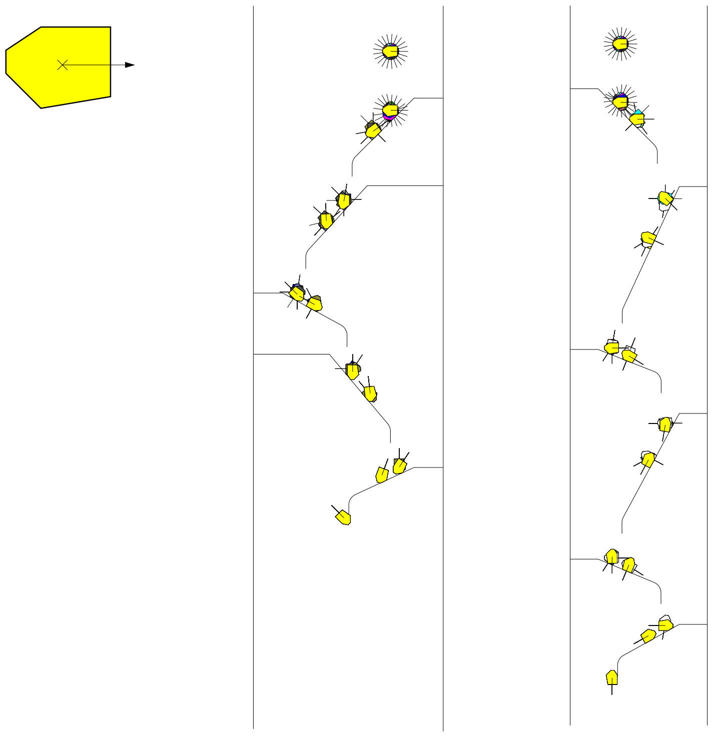


Figure 5: An optimal design of five fences, and a design of six alternating fences allowing for a narrower belt. Every line segment emanating from the part represents a possible orientation of the part.

fences if such a sequence exists. Alternating sequences are often preferred over sequences containing cascades of left (or right fences), as they generally allow for narrower conveyor belts (see Figure 5). Different cost assignments can be found to prevent e.g. unwanted steep and shallow fences. The costs make it impossible to apply Lemma 3.2 to reduce the graph size, as this may cause the removal of equally long or longer paths with lower cost from the graph. The size of the resulting graph is therefore $O(n^4)$. Dijkstra's algorithm (see e.g. [7]) has been used to find the minimal cost path through the graph in time $O(n^4)$.

4 The existence of fence designs for asymmetric parts

In this section we concentrate on parts with push functions with a unique longest left environment $l(a)$ and a unique longest right environment $r(b)$. We prove that for these *asymmetric* parts a fence design always exists and has length $O(n)$. In Section 5 we deal with parts that are not asymmetric.

Chen and Ierardi [6] use a sequence of equivalent basic actions to orient a polygonal part P with a unique longest right environment $r(a)$ of length α. Each basic action consists of a reorientation of the jaw by an angle of $\alpha - \mu$, with $\mu > 0$ such that $\alpha - \mu > |r(a')|$ for any equilibrium orientation $a' \neq a$, and a subsequent push. Note that a reorientation of the jaw by $\alpha - \mu$ corresponds to a change of the orientation of the part by $\alpha - \mu$. Every basic action puts the part into an equilibrium orientation. After each basic action, the part is therefore in one of a finite number of equilibrium orientations. Let us label the m equilibrium orientations $a_0, \ldots, a_{m-1}$ in order of *decreasing* angle starting from $a_0 = a$. After the first push, the part P can be in any of the equilibrium orientations $a_0, \ldots, a_{m-1}$. Chen and Ierardi show that every next basic action eliminates the last orientation in the sequence as possible orientation of the part. Assume that P is in one of the orientations $a_0, \ldots, a_k$, for some $k \geq 1$. The key idea behind the proof is that a next basic action will cause P, when in orientation a_0, to stay in orientation a_0 because $\alpha - \mu < |r(a_0)| = |r(a)|$, and, when in orientation a_i for some $1 \leq i \leq k$, to move into some orientation a_j with $0 \leq j \leq i-1$ because $\alpha - \mu > |r(a_i)|$. Upon completion of the basic action, the part will therefore be in one of the orientations $a_0, \ldots, a_{k-1}$. As a consequence, a total of $m+1$ basic actions suffices to put P into orientation $a_0 = a$. In other words, the sequence $(\alpha - \mu)^{m+1}$ is a valid push plan for P. In a similar manner, we could use reorientations by $-(\beta - \mu)$, and obtain the sequence $(-\beta + \mu)^{m+1}$, where β is the length of the longest left environment. The observation that $m = O(n)$ leads to the result of Lemma 4.1.

Lemma 4.1 [6] *A polygonal part P with n vertices can be oriented by $N = O(n)$ pushes.*

In order for the push plan $(\alpha - \mu)^{m+1}$ or $(-\beta + \mu)^{m+1}$ to be a valid fence design, we have to show that it satisfies the constraints formulated in Definition 2.2. We observe first of all that there can be no more than three environments of length at least $\pi/2$, because the longest two left environments have different lengths and the longest two right environments have different lengths. As a result, there is at most one left environment of size at least $\pi/2$ or at most one right environment of size at least $\pi/2$. Assume without loss of generality that there is at most one right environment of size at least $\pi/2$. Although the length α of the longest right environment $r(a)$ can be at least $\pi/2$, the length α' of the second largest right environment $r(a')$ must be smaller than $\pi/2$. If we now choose μ such that $\alpha' < \alpha - \mu < \min\{\alpha, \pi/2\}$, then we get that $\alpha - \mu > |r(v)|$ for all equilibrium orientations $v \neq a$. In addition, we clearly have that $\alpha - \mu < \pi/2$, which makes it easy to verify that $(\alpha - \mu)^{m+1}$ is a fence design.

Theorem 4.2 *An asymmetric polygonal part P with n vertices can be oriented by a fence design of length $N = O(n)$.*

5 Arbitrary parts

The considerations in Section 4 show that we can orient a part by a linear length fence design if its push function has a unique longest left or right environment for which the second largest interval has a length smaller than $\pi/2$. For asymmetric parts, there always exists such an environment. If we deal with arbitrary parts, there can be several environments with the same size α, even with size greater than $\pi/2$. In this section we show that for every polygonal part there is a fence design that orients the part up to symmetry in its push function. We recall that m_s denotes the number of stable (equilibrium) orientations of P. We first show that we can orient P if the period of the push function is 2π. The plans can actually be used to orient any part up to the period in its push function. The method we use is similar to the method Chen and Ierardi introduced to generate push plans [6]. Recall that a fence design is a push plan satisfying constraints on the reorientations of the jaw. We will try to produce push plans that only use reorientations in either $(0, \pi/2)$ or $(-\pi/2, 0)$, as such plans clearly satisfy Definition 2.2. There are two problems with the implementation of the push plans of

Chen and Ierardi as a fence design. If there are several right environments with size greater than $\pi/2$, we cannot orient the part. If there is no unique largest right environment, Chen and Ierardi apply their so called 'stretching lemma' which in general uses any reorientation of the pushing jaw. The stretching lemma shows that we can shift two possible orientations which both have a maximal left or right environment closer to each other with one single push; in other words: we can break the symmetry if there are multiple orientations with equally long maximal left or right environments. For a push plan Lemma 4.1 remains valid [6, 16]. For fences this is not necessarily the case. We can, however, reduce the possible orientations of the part to those with maximal right environments or with right environments of length greater than $\pi/2$ by an alternating sequence of jaw applications and jaw reorientations by an angle $\min\{\alpha-\mu, \pi/2-\mu\}$ (α and μ as in Section 4; no right environment shorter than $\pi/2$ is longer than $\pi/2-\mu$). (We can similarly reduce the possible orientations to orientations with such left environments.) As observed, for arbitrary parts, the number of possible orientations can be larger than one. We have to use more sophisticated fence designs to further reduce the number of possible orientations.

We start with some definitions which we use throughout the rest of this section. Let α again be the length of the longest right environment and let β be the length of the longest left environment. Furthermore, let $\mathcal{A} = a_0 \dots a_{m_s-1}$ be the cyclic sequence of stable equilibria, cut at some arbitrary orientation and ordered by *increasing* angle. The sequence $\mathcal{A}$ is said to have right cycle d if and only if $|r(a_i)| = |r(a_{i+d})|$ for all $0 \leq i < m_s$. (Indexing is modulo m_s.) Similarly, the sequence $\mathcal{A}$ has left cycle d if and only if $|l(a_i)| = |l(a_{i+d})|$ for all $0 \leq i < m_s$. We denote by $\mathcal{R}$ the set of orientations in $\mathcal{A}$ with right environments of length α, or length greater than or equal to $\frac{\pi}{2}$. We define the right measure $M_{\mathcal{R}}$ of an interval $[v, w]$ of orientations by $M_{\mathcal{R}}([v,w]) = |\{a \in \mathcal{R} | v \leq a < w\}|$. In a similar manner, we can define a set $\mathcal{L}$ of orientations with left environments of length β, and a left measure $M_{\mathcal{L}}$. We let $\mu > 0$ be a small constant such that $\alpha - \mu$ and $\alpha + \mu$ are both smaller than $\pi/2$ but larger than any environment of length less than $\pi/2$. In addition, the constant μ is smaller than the length of any environment. We recall that polygonal parts without meta-stable edges have push functions without left and right environments of zero length.

Our push plans for arbitrary parts consist of three types of basic building blocks. These building blocks are referred to as MOVE, SHIFT, and REDUCE.

- Suppose that $[v, w]$ with $v, w \in \mathcal{A}$ is the current interval of possible orientations. Then, the push plan $(\min(\alpha, \frac{\pi}{2}) - \mu)^{m_s}$ MOVEs the interval $[v, w]$ into an interval $[v', w']$ with $v', w' \in \mathcal{R}$ of equal right measure.

- Suppose that $[v = a_i, w = a_j]$, $v, w \in \mathcal{A}$ is the current interval of possible orientations. Then, the push plan $(|r(v)| + \mu)$ SHIFTs the interval $[v, w]$ into an interval $[v', w']$ with $v' = a_{i+1}, w' \in \mathcal{A}$. If $v, w \in \mathcal{R}$ then $w' = a_{j+1}$ of and the right measure of $[v', w']$ equals the right measure of $[v, w]$. Notice that in that case a SHIFT followed by a MOVE maps $[v, w]$ into an interval $[v'', w'']$ with $v'', w'' \in \mathcal{R}$ and $v' \neq v, w' \neq w$ (provided that $\mathcal{R}$ has at least two elements) of equal right measure.

- Suppose that $[v, w]$ with $v, w \in \mathcal{R}$ is the current interval of possible orientations. We want to define an operation which REDUCEs an interval $[v, w]$ to some interval $[v', w']$ with $v', w' \in \mathcal{A}$ of smaller measure. The REDUCE exploits specific asymmetries that are present in the push function to achieve the reduction of the measure. Different classes of push functions lead to different REDUCEs.

The objective is to use these building blocks in a push plan that is guaranteed to result in an interval of possible orientations of measure zero.

According to the 'stretching lemma' of Chen and Ierardi [6], any interval $[v, w], v \neq w$ of possible orientations can be mapped onto a shorter interval of possible orientations $[v', w']$ by a single push. An additional push will then map $[v', w']$ onto an interval $[v'', w'']$ satisfying $M_{\mathcal{R}}([v'', w'']) < M_{\mathcal{R}}([v, w])$. The problem in applying these ideas in fence design is that the two required reorientations of the push direction may not be achievable by a sequence of fences. We will use the SHIFT and MOVE plans to overcome this problem. We classify the push functions, based on the left and right cycle of the stable equilibria, and the sizes of the environments. The implementation of the REDUCE is the main difference between the distinct classes of push functions. We distinguish the following classes of push functions.

1. Either the left or the right cycle is m_s, and no left or right environment is longer than $\pi/2$ in that cycle.
2. Both the left and the right cycle are smaller than m_s.
3. Both the left and the right cycle are m_s and there is more than one environment of length greater than $\pi/2$.

Let us first concentrate on push functions in the first and second class. Suppose that $[v, w]$ with $v, w \in \mathcal{R}$ is the current interval of possible orientations. Let v' and w' be the smallest orientation in $\mathcal{R}$ larger than v and w respectively. If the sequence of right environments of orientations between v and v' differs from the the sequence of right environments between w and w', then there exists a simple strategy that maps $[v, w]$ either onto $[v', u]$ with $u \in \mathcal{A} \setminus \mathcal{R}$ and $w < u < w'$ or onto $[u, w']$ with $u \in \mathcal{A} \setminus \mathcal{R}$ and $v < u < v'$. Moreover, it can be shown that any interval $[v, w]$ with $v, w \in \mathcal{R}$ can be transformed—by means of SHIFT and MOVE plans—into a different interval $[v'', w'']$ with $v'', w'' \in \mathcal{R}$ of equal measure, such that the simple strategy is guaranteed to map $[v'', w'']$ onto $[v', u]$, with $u \in \mathcal{A} \setminus \mathcal{R}$ and $w < u < w'$, and where v' and w' are the smallest orientations in $\mathcal{R}$ larger than v and w respectively. A subsequent reorientation by $\alpha + \mu$ followed by a push will eliminate v' as a possible orientation of the part without adding new possible orientations from $\mathcal{R}$. As a consequence, the measure of the resulting interval of possible orientations is smaller than $M_{\mathcal{R}}([v', u])$. For push functions from the first class, this strategy (or its equivalent using left environments) suffices to reduce the interval of possible orientations to an interval of measure zero. The reorientations of the push direction in the entire scheme are restricted to $(0, \pi/2)$ or $(-\pi/2, 0)$, which makes the sequence of pushes a valid fence design. If the left and right cycle are both smaller than m_s, then we must eventually switch our attention from the right environments to left environments to break the rotational symmetry of the right environments (or vice versa). It turns out that such a switch can be accomplished without violating the reorientation constraints for fence designs.

The third class of push functions requires some modifications to the MOVE, SHIFT, and REDUCE framework. There are, however, at most three environments of length greater than $\pi/2$, which makes it possible to treat the different cases one by one, and provide dedicated push plans which use both left and right fences. These dedicated push plans satisfy the reorientation constraints and are therefore valid fence designs.

The three classes of push functions will be dealt with in the next three subsections. Theorem 5.1 summarizes the result of this section.

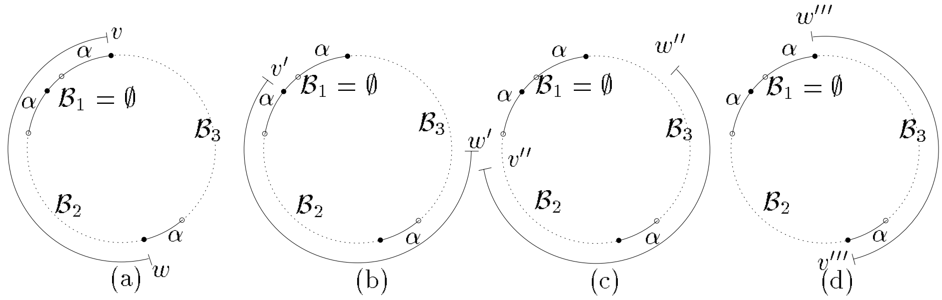


Figure 6: The general reduce. (a) The interval $[v, w]$ of possible orientations consists of three orientations in $\mathcal{R}$. The measure of $[v, w]$ is two. (b) After augmenting the plan with $\alpha + \mu$ (SHIFT), the orientations are appropriately reoriented. (c) The measure of $[v, w]$ is shrunk to one, by appending $\alpha + \mu$ (SHIFT) to the plan. (d) A MOVE moves v, w back onto orientations in $\mathcal{R}$.

Theorem 5.1 *Any polygonal part can be oriented up to rotational symmetry by a fence design.*

A consequence of Theorem 5.1 is that the algorithm for computing fence designs presented in Section 3.1 always outputs a fence design. Since the graph used in the algorithm has $O(n^2)$ nodes, the length of the shortest path is $O(n^2)$.

Theorem 5.2 *Any polygonal part with n vertices can be oriented up to rotational symmetry by a fence design of length $O(n^2)$. The optimal fence design can be computed in $O(n^3 \log n)$ time.*

5.1 Parts with aperiodic push functions with short environments

In this section, we show how to use the MOVE, SHIFT and REDUCE actions to orient a part for which either the left or the right cycle of the stable equilibria is m_s, and no left or right environment is longer than $\frac{\pi}{2}$ in that cycle. Without loss of generality, we may assume that the right cycle is m_s, and the right environments are shorter than $\frac{\pi}{2}$. In this section, we only use reorientations of the jaw in the angular interval $(0, \frac{\pi}{2})$, which automatically leads to valid fence designs. Since $\alpha < \frac{\pi}{2}$, the SHIFT and MOVE push plans only use reorientations in the angular interval $(0, \frac{\pi}{2})$. The techniques of this section are also applicable if the left environments are shorter than $\frac{\pi}{2}$. In that case the reorientations of the jaw are in the angular interval $(-\frac{\pi}{2}, 0)$.

We start with a MOVE. This assures that all possible orientations are in $\mathcal{R}$. Now, if $M_{\mathcal{R}}[v, w] = 0$, there is only one possible orientation, and we are done. If $M_{\mathcal{R}}[v, w] > 0$ we have to reduce $[v, w]$. The goal of the reduce operation is to first develop a plan which maps v onto an orientation v' in $\mathcal{R}$ and w onto an orientation w' not in $\mathcal{R}$, with $M_R[v', w'] = M_R[v, w]$. We extend this plan with a SHIFT. The SHIFT maps $[v', w']$ onto an interval with smaller measure, and therefore completes a REDUCE. After another MOVE operation, the interval $[v'', w'']$ of possible orientations has $v'', w'' \in \mathcal{R}$. We repeat until there is just one possible orientation left. Figure 6 gives an example of application of one step of the general framework.

The main body of the reduce is based on the difference of subsequences $\mathcal{B}_{a_i} = a_{i+1}, \ldots a_{i+k-1}$, such that $[a_i, a_{i+k}]$ is a subsequence of $\mathcal{A}$, $a_i, a_{i+k} \in R_\alpha$, and $\mathcal{B}_{a_i} \cap \mathcal{R} = \emptyset$. The sequence $\mathcal{B}_{a_i}$

is so to say an ordered set of rightintervals *between* two orientations with rightintervals of length α.

Let $\mathcal{B}_{a_i}$ and $\mathcal{B}_{a_j}$, subsequences of $\mathcal{A}$, be given. We define an order on the sequences $\mathcal{B}_{a_i}$ and $\mathcal{B}_{a_j}$. Let $init(\mathcal{B}_{a_i})$ and $init(\mathcal{B}_{a_j})$ be the subsequences of $\mathcal{B}_{a_i}$ and $\mathcal{B}_{a_j}$ obtained by repeatedly removing the last elements a and a' if $|r(a)| = |r(a')|$. As a result, the right environments of the last orientations in $init(\mathcal{B}_{a_i})$ and $init(\mathcal{B}_{a_j})$ have different lengths. The order on the sequences $\mathcal{B}_{a_i}$ and $\mathcal{B}_{a_j}$ is as follows:

Definition 5.3 *Let $\mathcal{B}_{a_j} \neq \mathcal{B}_{a_j}$. Let γ_i and γ_j be the length of the longest right environment of any stable equilibrium in $init(\mathcal{B}_{a_i})$ and $init(\mathcal{B}_{a_j})$ respectively. Let k_i be the number of orientations $a \in init(\mathcal{B}_{a_i})$ with $r(a) = \gamma_i$ and let k_j be the number orientations $a \in init(\mathcal{B}_{a_j})$ with $r(a) = \gamma_j$. Then $\mathcal{B}_{a_i} \prec \mathcal{B}_{a_j}$ if*

- *$\mathcal{B}_{a_j} = \emptyset$, or*
- *$\mathcal{B}_{a_j} \neq \emptyset$ and $\gamma_i > \gamma_j$, or*
- *$\mathcal{B}_{a_j} \neq \emptyset$ and $\gamma_i = \gamma_j$ and $k_i > k_j$,*
- *$\mathcal{B}_{a_j} \neq \emptyset$ and $\gamma_i = \gamma_j$ and $k_i = k_j$ and $\mathcal{C}_{a_i} \prec \mathcal{C}_{a_j}$, where $\mathcal{C}_{a_i}$ and $\mathcal{C}_{a_j}$ are the trailing subsequences of $init(B_{a_i})$ and $init(\mathcal{B}_{a_j})$ starting right after the last orientations in $init(\mathcal{B}_{a_i})$ and $init(\mathcal{B}_{a_j})$ with right environments of length $\gamma_i = \gamma_j$.*

It follows that $\mathcal{B}_{a_i} \neq \mathcal{B}_{a_j}$ iff $\mathcal{B}_{a_i} \prec \mathcal{B}_{a_j}$ or $\mathcal{B}_{a_j} \prec \mathcal{B}_{a_i}$. Furthermore, $\prec$ is transitive and antisymmetric. Suppose now that for an interval of possible orientations, $[v, w]$, $v, w \in \mathcal{R}$, we have $\mathcal{B}_w \prec \mathcal{B}_v$. Lemma 5.4 gives us that we can now REDUCE the right measure of $[v, w]$.

Lemma 5.4 *Let $[v, w]$ with $v, w \in \mathcal{R}$ and $\mathcal{B}_w \prec \mathcal{B}_v$. There exists a push plan that maps the interval of possible orientations onto another interval with measure smaller than the measure of $[v, w]$.*

Proof: The fence design to reduce the measure of $[v, w]$ is quite straightforward. Given that $\mathcal{B}_w \prec \mathcal{B}_v$. We know that that $|r(v)| = |r(v)|$. We first apply a shift. This causes both v and w to map onto a different orientation. The rest of the plan: reorient the jaw by $(\min\{r(a_1), r(a_k)\} + \mu)$ and push, until $|r(a_k)| = \alpha$. Finish with a reorientation of the jaw is $\alpha + \mu$, followed by a push. This fairly simple fence design indeed maps v onto the next orientation in $\mathcal{R}$, while w is mapped onto an orientation in $\mathcal{B}_w$. We verify this for each item of Definition 5.3. If $\mathcal{B}_v = \emptyset$, the proof is trivial. If $B_v \neq \emptyset$ and $\gamma_w > \gamma_v$, then $(\min\{r(a_1), r(a_k)\}) \neq \gamma$ until v coincides with an orientation in $\mathcal{R}$. If $\mathcal{B}_v \neq \emptyset$ and $\gamma_w = \gamma_v$, but $k_w > k_v$, then w stays on the last orientation with right environments of length γ_w, until v is on a orientation in $\mathcal{R}$. If $B_v \neq \emptyset$ and $\gamma_w = \gamma_v$ and $k_w = k_v$, then both v and w are at the $k_w = k_v$'th and the last orientations with right environments of length $\gamma_w = \gamma_v$ after a number of pushes. Since $\mathcal{B}_w \prec \mathcal{B}_v$, the proof follows inductively from the order of the trailing sequences. □

Using (SHIFT, MOVE)$^\kappa$, we can map an interval of possible orientations $[v, w]$, $v, w \in \mathcal{R}$ onto another interval $[v', w']$, $v', w' \in \mathcal{R}$. We want to determine κ which leads to $[v', w']$, $v', w' \in \mathcal{R}$ having $\mathcal{B}_{w'} \prec \mathcal{B}_{v'}$. In order to prove that such κ exist we observe the following: The measure $M_{\mathcal{R}}$ of $[v, w]$ divides the set of $\{\mathcal{B}_z | z \in \mathcal{R}\}$ into equivalence classes. An example of these classes

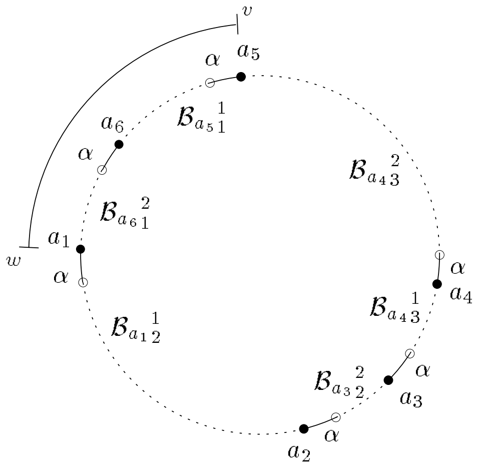


Figure 7: The measure of $[v, w]$ is two. The number of right intervals with length α is 6. $[v, w]$ generates two equivalence classes in the set of $\mathcal{B}$'s. These equivalence classes are denoted with the superscript 1 and 2.

is depicted in Figure 7. The elements of one equivalence class lie $M_{\mathcal{R}}([v, w])$ apart. We shall show that at least one of the equivalent classes contains sequences $\mathcal{B}_i$ and $\mathcal{B}_{i+M_{\mathcal{R}}([v,w])}$, with $\mathcal{B}_{i+M_{\mathcal{R}}([v,w])} \prec \mathcal{B}_i$.

We formally define the equivalence classes. Let $M_{\mathcal{R}}([v, w])$ denote the measure of the possible orientations. Let I be the principal ideal generated by $M_{\mathcal{R}}([v, w])$ in $\mathbb{Z}_{|\mathcal{R}|}$. The cosets of $\mathbb{Z}_{|\mathcal{R}|}/I$ correspond to the equivalence classes of the orientations in $\mathcal{R}$, generated by the measure of $M_{\mathcal{R}}([v, w])$. The cosets are sets of orientations in $\mathcal{R}$ such that the measure of an interval $[a_i, a_j]$ a_i, a_j of the same coset is $\kappa \cdot M_{\mathcal{R}}([v, w])$, for an integer κ.

Lemma 5.5 *If a coset $s + I$ of $\mathbb{Z}_{|\mathcal{R}|}/I$ does not have complete identical elements, then there is a configuration of the interval $[v, w]$ of possible orientations, such that $\mathcal{B}_w \prec \mathcal{B}_v$.*

Proof: Since $\mathbb{Z}_{|\mathcal{R}|}$ has a finite number of elements, so does each coset of $\mathbb{Z}_{|\mathcal{R}|}/I$. The elements of a coset $s + I$ by assumption, are not identical, but $s + I$ is cyclic since **for all** $i \in \mathbb{Z} : s = s \cdot |\mathcal{R}|$ for s in $\mathbb{Z}_{|\mathcal{R}|}$.

The rest of the proof follows from the transitivy, and the antisymmetry of $\prec$. □

The following lemma states the existence of a coset that has a pair of non-identical elements.

Lemma 5.6 *There is a coset $s + I$ of $\mathbb{Z}_{|\mathcal{R}|}/I$ that does not have identical elements.*

Proof: If all cosets of $\mathbb{Z}_{|\mathcal{R}|}/I$ have identical elements, then the right cycle of $\mathcal{A}$ is $M_{\mathcal{R}}([v, w])$. Since the right cycle of $\mathcal{A}$ is not determined by $\mathcal{R}$, and the intervals $\mathcal{B}_z (z \in \mathcal{R})$ are divided into identical equivalence classes. There is no difference between any pair of right-environments that are $M_{\mathcal{R}}([v, w])$ apart. However, the right cycle is m_s. This is a contradiction and there has to be a coset of $\mathbb{Z}_{|\mathcal{R}|}/I$ that has a pair of non-identical elements. □

To use this assymetric property, we have to be able to shift the interval of possible orientations such that $\mathcal{B}_w \prec \mathcal{B}_v$.

Lemma 5.7 *Given a interval* $[v, w]$ *of possible orientations. There is an fence design which maps* $[v, w]$ *onto* $[v', w']$*,* $M_{\mathcal{R}}[v', w'] = M_R[v, w]$ *such that* $\mathcal{B}_{w'} \prec \mathcal{B}_{v'}$*.*

Proof: Lemma 5.6 gives us that there is a pair $v', w', M_{\mathcal{R}}[v', w'] = M_{\mathcal{R}}[v, w]$, with $\mathcal{B}'_w \prec \mathcal{B}'_v$. The push plan (SHIFT, MOVE)$^\kappa$, with $\kappa = M_{\mathcal{R}}[v, v']$ maps v onto v' and w onto w' and is a valid fence design. This completes the proof. □

We now put the building blocks together. The following theorem gives us fence designs to orient parts without large right environments and right cycle m_s.

Theorem 5.8 *Given a part with possible (stable) orientations* $\mathcal{A}$*. Let the right cycle of* $\mathcal{A}$ *be* $m_s = |\mathcal{A}|$*. If for all* $v \in \mathcal{A} : |r(v)| < \frac{\pi}{2}$*, then we can orient the part.*

Proof: The following fence design orients the part.

1. MOVE
2. *let* $[v, w]$ *denote an interval of possible orientations.*
3. **while** $M_{\mathcal{R}}[v, w] > 0$ **do**
   - 3.1 (SHIFT, MOVE)$^\kappa$, such that $\mathcal{B}_w \prec \mathcal{B}_v$
   - 3.2 REDUCE

□

This approach also works for parts for which the left environments have the same properties, the reorientations are then in the angular interval $(-\frac{\pi}{2}, 0)$.

5.2 Parts with left and right cyclic push functions

In this section we treat parts for which both the left and the right cycle of the possible orientations $\mathcal{A}$ are strictly smaller than $m_s = |\mathcal{A}|$. For these parts, the symmetry of the push function is neither broken by the left environments nor by the right environments, but by the combination of the left and right environments.

5.2.1 Preliminaries

Let a denote the left cycle of $\mathcal{A}$, and b the right cycle of $\mathcal{A}$. Let us assume that the period of the push function is 2π, which implies that the least common multiple of a and b is m_s.

We already noticed that the existence of left or right evironments which are longer than $\frac{\pi}{2}$ makes it less straightforward to give push plans which are fence designs as well. Long environments raise the need of large reorientations of the jaw. Such reorientations have to be carefully embedded into a push plan in order to satisfy the fence design constraints. We first identify a number of properties of the push function for parts with cyclic left and right environments, and analyze the possible existence of large environments. The right cycle is less than m_s. This implies for each $a_i \in \mathcal{A}$, there is an a_j, such that $i \neq j$ and $|r(a_j)| = |r(a_i)|$. More specific, this implies that if there is an a_i with $|r(a_i)| \geq \frac{\pi}{2}$, there is an a_j, such that $i \neq j$ and $|r(a_j)| = |r(a_i)|$. Since the period of the push function is 2π, this means that every other orientation a_k, $k \notin \{i, j\}$, for which $|r(a_k)| \geq \frac{\pi}{2}$, satisfies $|r(a_k)| = |r(a_i)|$. The following lemma gives a result on existence of left or right environments longer than equal $\frac{\pi}{2}$.

Lemma 5.9 *Let P be a part. Let $\mathcal{A} = a_0 \ldots a_{m_s-1}$ be the set of m_s possible stable orientations of P. Let a denote the left cycle of $\mathcal{A}$, and b denote the right cycle of $\mathcal{A}$. If $a, b < m_s$ and there is an stable equilibrium v for which $|l(v)| \geq \frac{\pi}{2}$, then the set of stable equilibria $\{v' \in \mathcal{A} \,||\, l(v')| = |l(v)| \geq \frac{\pi}{2}\}$ has cardinality two or three. Furthermore, there is no stable equilibrium v' for which $|r(v')| \geq \frac{\pi}{2}$.*

Proof: We know that $|l(a_i)| = |l(a_{i+a})|$, for all i. Since the left cycle a of $\mathcal{A}$ is less than m_s, each left environment length occurs at least twice. Since the period of the push function is 2π, and each left and right environment has non-zero length, there are at most three (left or right) environments with length greater than or equal $\frac{\pi}{2}$. Suppose now, that there is a right environment with length greater than or equal $\frac{\pi}{2}$. Since the right cycle of $\mathcal{A}$ is less than m_s as well, there must be a second right environment with length greater than or equal $\frac{\pi}{2}$, which together with the at least two large left environments not fit the period of the push function. This contradicts the assumption that there is a right environment with length greater than or equal $\frac{\pi}{2}$, and completes the proof. □

The lemma also holds for a right environment longer than or equal $\frac{\pi}{2}$, thus either the longest left environments, or the longest right environments are longer than $\frac{\pi}{2}$, or none of the left or right environments is longer than $\frac{\pi}{2}$. Without loss of generality, throughout this section, we assume that no right environment of the part is longer than $\frac{\pi}{2}$.

Assumption 5.10 *Throughout this section we assume, without loss of generality, that there is no orientation a_i such that $|r(a_i)| \geq \frac{\pi}{2}$.*

5.2.2 The reduce

The push plan for a part with a left and right cyclic push function is an extension of the techniques to orient parts with acyclic push functions and short environments, which we discussed in Section 5.1. Recall that we developed a plan to orient a part with right cycle m_s. In this section, however, the cycle of the right environments unequals m_s. The scheme of Section 5.1 might look useless at first glance; if we apply the scheme, we can only reduce the measure $M_R[v, w]$ of the interval of possible orientations up to the cycle of the right environments. At this point, $|r(a_{i+k})| = |r(a_{j+k})|$, for any k, but the part is not oriented up till symmetry yet.

The reduce we will develop, REDUCE$_{dc}$, further orients the part. Since solely the right environments of the parts push function do not capture the part's asymmetry, REDUCE$_{dc}$ exploits the difference of the left environments as well. To understand REDUCE$_{dc}$ we define a second order, $\prec_{dc}$, on intervals of possible orientations.

Definition 5.11 *Let $[a_i, a_j]$ denote the interval of possible orientations. Let $|l(a_{i+1})| \neq |l(a_{j+1})|$. We define $a_i \prec_{dc} a_j$ if $|l(a_{i+1})| < |l(a_{j+1})|$.*

Consider the interval of possible orientations. We shall show that we can define a reduce operation which first decreases the number of orientations in the interval of possible orientations tand then reduces the right measure of interval of possible orientations of the part.

Our first goal is to shift interval of possible orientations. We use the plan (SHIFT)$^{\kappa}$, with κ such that, at the end, the interval $[v, w]$ of possible orientations satisfies $v \prec_{dc} w$. Such κ indeed exists. The proof is similar to the equivalence class discussion in Section 5.1 and is not repeated here.

Lemma 5.12 *Given an interval* $[v, w]$ *of possible orientations. There is a fence design which maps* $[v, w]$ *onto* $[v', w']$*, with the same number of stable equilibria, but* $v' \prec_{dc} w'$.

Next, we will define a reduce for double cyclic push functions, REDUCE$_{dc}$, which actually decreases the right measure of the interval of possible orientations.

Let $[a_i, a_j], a_i \prec_{dc} a_j$ denote the current interval of possible orientations. Firstly, we map a_i onto a_{i+2}, and a_i onto a_{j+1}. The required reorientation of the jaw to accomplish this is $|r(a_i)| + |l(a_{i+1})| + |r(a_{i+1})| + \mu$.

Small reorientations of the jaw

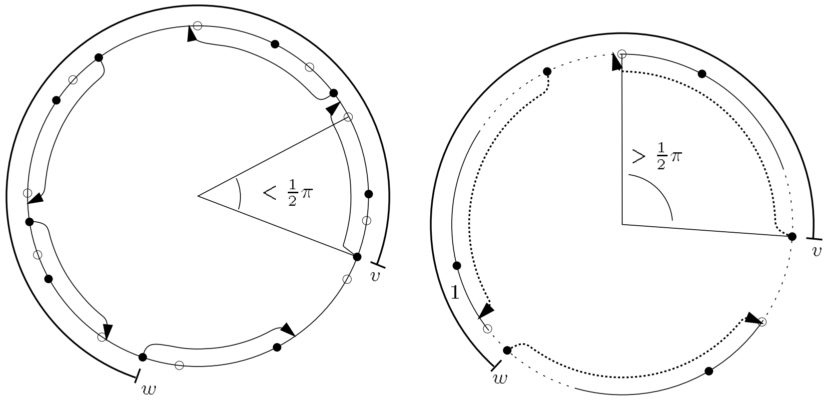
Let us first assume that $|r(a_i)| + |l(a_{i+1})| + |r(a_{i+1})| < \frac{\pi}{2}$ (The other case, which uses a large roerentation of the jaw, is treated later). Under this assumption, REDUCE$_{dc}$, is built up as follows: a reorientation of the jaw of $|r(a_i)| + |l(a_{i+1})| + |r(a_{i+1})| + \mu$ maps a_i onto a_{i+2}, and a_j onto a_{j+1}. Secondly, we continue with $(\text{SHIFT})^{\kappa'}$, until a_i is mapped onto an orientation in $\mathcal{R}$. We finish with another SHIFT, to reduce the right measure of the interval of possible orientations. The next lemma gives that at the end of REDUCE$_{dc}$, $[v'w']$, the image of $[a_i, a_j]$, is such that the measure $M_{\mathcal{R}}([v', w']) < M_{\mathcal{R}}([v, w])$

Lemma 5.13 *Let* $[a_i, a_j]$ *denote the interval of possible orientations. Let* $|l(a_{i+1}| < |l(a_{j+1})|$. *After* REDUCE$_{dc}$*, the new interval* $[v', w']$ *of possible orientations has* $M_{\mathcal{R}}[v', w'] < M_{\mathcal{R}}[a_i, a_j]$.

Proof: Before the reorientation of $|r(a_i)| + |l(a_{i+1})| + |r(a_{i+1})| + \mu$, we have $[a_i, a_j]$, with $i = j \pmod{m_s/b}$, but not $i = j \pmod{m_s}$. The number of stable equilibria in the interval of possible orientations is $j - i + 1 (\operatorname{mod} m_s)$. After application of the jaw, the number of stable equilibria in the interval of possible orientations is less than $j - i + 1 (\operatorname{mod} m_s)$. If $a_{i+1} \in \mathcal{R}$, and the number of stable equilibria in the interval of possible orientations is less than $j - i + 1 (\operatorname{mod} m_s)$, then we reduced the right measure. Otherwise, we have to show that the plan $(\text{SHIFT})^{\kappa'}$, such that $a_{i+1+\kappa'} \in \mathcal{R}$ maintains the property that the number of orientations in the interval of possible orientations is less than $j - i + 1 (\operatorname{mod} m_s)$. Suppose now that at a certain point during the execution of $(\text{SHIFT})^{\kappa}$, the number of orientations in the interval of possible orientations becomes larger than or equal to $j - i + 1$. Before the last push, the number of orientations in the interval of possible orientations was less than $j - i + 1$. Let $[a_{i'}, a_{j'}]$ denote this interval of possible orientations, thus $j' - j < (i' - i)$. SHIFT $(= |r(a_{i'})| + \mu)$ maps $[a_{i'}, a_{j'}]$ onto $[a_{i'+1}, a_{j''}]$, with $j'' - j' \geq (i' - i) + 1$, and $j' - j < (i' - i)$, thus $j'' > j' + 1$. We know that for any l, $|r(a_{i+l})| = |r(a_{j+l})|$, which implies that there is an orientation $j''' \in (j', \ldots, j'']$ such that $r(j''') = r(i')$. This means that the last SHIFT, which mapped $a_{j'}$ onto $a_{j''}$, involves a reorientations of the jaw of at least $|r(a_{i'})| + |r(a_{j'})| > |r(a_i')| + \mu$. □

Large reorientations of the jaw

If we drop the assumption that there is a configuration of the interval of possible orientations $[a_i, a_j]$, such that $|r(a_i)| + |l(a_{i+1})| + |r(a_{i+1})| < \frac{\pi}{2}$, the REDUCE$_{dc}$ plan from the previous paragraph does not correspond to a valid fence design. We will slightly alter REDUCE$_{dc}$ to overcome this problem. The following theorem gives us that in this special case the length of the longest left environment, $\beta < \frac{\pi}{2}$. This property will turn out to be useful. Recall that we assumed that $\alpha < \frac{\pi}{2}$ as well (Assumption 5.10).



(a) (b)

Figure 8: (a) The situation where we can skip one special large right-interval (1). (b) The situation where we can use a change of fence side, and continue with the mirrored strategy.

Lemma 5.14 *Let P be a part. Let $\mathcal{A} = a_0 \ldots a_{m_s-1}$ be the set of m_s possible stable orientations of P. Let a denote the left cycle of $\mathcal{A}$, and b denote the right cycle of $\mathcal{A}$. Let $a, b < m_s$. Let $\beta = max_{i \in 0 \ldots m_s-1}|l(a_i)|$. If there is a k such that $|r(a_i)| + |l(a_{i+1})| + |r(a_{i+1})| > \frac{\pi}{2}$ for all i, for which $|r(a_i)| + |l(a_{i+1})| + |r(a_{i+1})| < |r(a_{i+(k\frac{m_s}{b})})| + |l(a_{i+1+(k\frac{m_s}{b})})| + |r(a_{i+1+(k\frac{m_s}{b})})|$, then $\beta < \frac{\pi}{2}$.*

Proof: Let us assume, on the contrary that $\beta > \frac{\pi}{2}$. According to Lemma 5.9, there are at least two left environments which have length β, say $l(v_1)$ and $l(v_2)$ with $v_1, v_2 \in \mathcal{A}$. We will show that for each $k > 0$, there is an i, such that $|r(a_i)| + |l(a_{i+1})| + |r(a_{i+1})| < |r(a_{i+(k\frac{m_s}{b})})| + |l(a_{i+1+(k\frac{m_s}{b})})| + |r(a_{i+1+(k\frac{m_s}{b})})|$, and $|r(a_i)| + |l(a_{i+1})| + |r(a_{i+1})| < \frac{\pi}{2}$. Let $k > 0$ be given. Let now $a_{j-1+(k\frac{m_s}{b})}$ coincide with v_1, and $a_{j'+1+(k\frac{m_s}{b})}$ coincide with v_2. Since $a \neq b$, $\frac{m_s}{b} \neq \frac{m_s}{a}$, and clearly $|l(a_{j+1})|, |l(a_{j'+1})| < |l(a_{j+1+(k\frac{m_s}{b})})|$. If $|r(a_j)| + |l(a_{j+1})| + |r(a_{j+1})| < \frac{\pi}{2}$, we let $i = j$. Otherwise, we get $|r(a'_j)| + |l(a_{j'+1})| + |r(a_{j'+1})| < \frac{\pi}{2}$. In order to have $a, b < m_s$, there have to be more stable equilibria than four. Therefore, if $\{a_{j'}, a_{j'+1}\} \cap \{a_j, a_{j+1}\} = \emptyset$, then $|r(a'_j)| + |l(a_{j'+1})| + |r(a_{j'+1})| \leq 2\pi - 2\beta - |r(a_j)| - |l(a_{j+1})| - |r(a_{j+1})| - |l(a_q)| - |r(a_q)| < \frac{\pi}{2}$, with a_q a fifth stable equilibrium. It remains to prove that $\{a_{j'}, a_{j'+1}\} \cap \{a_j, a_{j+1}\} = \emptyset$. Firstly, $a_j \neq a_{j'}$, and $a_{j+1} \neq a_{j'+1}$, because otherwise $v_1 = v_2$. Secondly, $a_j \neq a_{j'+1}$ and $a_{j+1} \neq a_{j'}$, because otherwise a_j neigbors a'_j, and therefore v_1 neighbors v_2 which implies that $a = 1$ and $b = m_s$. This contradicts the assumption. So, $i = j'$ in this case, which completes the proof. □

We know that $|r(a_i)| + |l(a_{i+1})| + |r(a_{i+1})| \geq \frac{\pi}{2}$, so $\beta < \frac{\pi}{2}$. Furthermore $|r(a_i)| + |l(a_{i+1})| + |r(a_{i+1})| < \pi$, since otherwise the period of the push function is greater than 2π. If we now reorient the jaw by $|r(a_i)| + |l(a_{i+1})| + |r(a_{i+1})| + \mu$, and apply the jaw, we reduced the length (but not necessarily the right measure) of the interval of possible orientations. At this point, we have to use reorientations in the angular intervals $(-\frac{\pi}{2}, 0)$ or $(\pi, \frac{\pi}{2})$. The part of the reduce that follows is a *mirrored* version of the reduce presented up till now. We can therefore also continue to orient the mirrored part, and keep in mind that we have to replace the angles by which we reorient the jaw by their negated values. Also, a mirror swaps the values of a and b, α and β, and the sets $\mathcal{R}$ and $\mathcal{L}$. Figure 9 depicts a part and its mirror image. We now

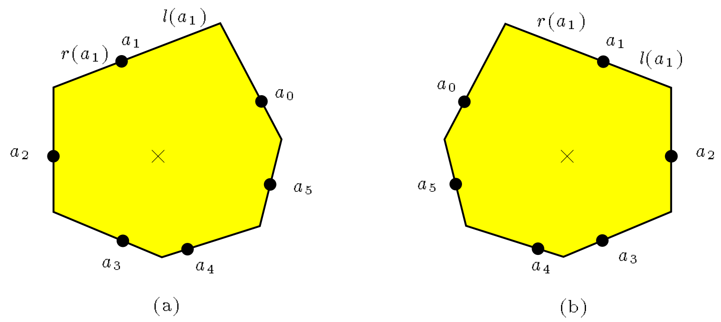


Figure 9: (a) A part, and (b) its mirror image.

continue to reduce the inverted right measure of the interval $[w, v]$ of possible orientations.

If $i \neq j (\mathrm{mod}\, m_s/b)$ we continue REDUCE$_{dc}$ by *mirror*(SHIFT$^{\kappa}$), until $w \in \mathcal{R}$, a final SHIFT reduces the measure, and is valid since (the new) $\alpha < \frac{\pi}{2}$.

We now know $i = j (\mathrm{mod}\, m_s/b)$ and try to (further) reduce the mirrored right measure of the interval of possible orientations, using mirrored SHIFT's and REDUCE$_{dc}$'s. Note that if it happens to be that case that the REDUCE$_{dc}$ is again forced to be mirrored, then, after two subsequent mirrors of the reduce, the interval of possible orientations $[v', w']$ is shorter than the interval of possible orientations $[v, w]$ before the former mirror. Since two mirrors of the part cancel out, we continue REDUCE$_{dc}$ as if we did not use the two mirrors until the part is oriented.

5.2.3 The final reduce

In this section we derived the following:

Theorem 5.15 *Given a part P, if the parts push functions stable equilibria $\mathcal{A}$ have a left cycle $a < |\mathcal{A}|$ and a right cycle $b < |\mathcal{A}|$, then there is a fence design that orients the part up to symmetry.*

Proof: The discussion in this section leads to the following REDUCE$_{dc}$.

1. MOVE
2. *let $[v, w]$ denote an interval of possible orientations.*
3. **while** $v \neq w$ **do**
    - 3.1 **if** there is a pair of orientations $v', w' \in \mathcal{R}$,
      with $M_R[v', w'] = M_R[v, w]$
      and $\mathcal{B}'_w \prec \mathcal{B}'_v$ **do**
        - 3.1.1. (SHIFT, MOVE)$^{\kappa}$, κ such that $[v', w']$ is the current interval
        - 3.1.2. REDUCE
    - 3.2. **else**
        - 3.2.1 (SHIFT)$^{\kappa}$, κ such that $[v', w']$ is the current interval
          and REDUCE$_{dc}$ is applicable
        - 3.2.2 REDUCE$_{dc}$; take *mirror* into account.

□

5.3 Parts with push functions with long environments

We now treat the special cases where some environment (left or right) is longer than $\frac{\pi}{2}$. Without loss of generality we may assume that some right environment has size $\geq \frac{\pi}{2}$. Unfortunately, the MOVE, SHIFT and REDUCE, as defined in the introduction of this section, no longer only use reorientations of the jaw in the angular interval $(0, \frac{\pi}{2})$. In this section, we use the same general idea of MOVE, SHIFT and REDUCE, but we have to modify them into feasible fence designs.

We use the following push plans as building blocks:

- Suppose that $[v, w]$ $(v, w \in \mathcal{R})$ is the current interval of possible orientations. We want a SHIFT$_l$ operation which maps $[v, w]$ onto $[v', w']$ with equal measure. This is a useful operation when $|\mathcal{R}| > 2$. The SHIFT$_l$ will be presented in Section 5.3.1.

- We need an operation REDUCE$_l$ which reduces the measure of $[v, w]$.

Note that the maximum number of right environments longer than $\frac{\pi}{2}$ is three (otherwise, the period of the push function is larger than 2π). If there is one right environment that is larger than $\frac{\pi}{2}$, we simply use a fence design $(\frac{\pi}{2} - \mu)^{m_s}$. It remains to prove that we can give a fence design for parts with two or three large right environments. Throughout this section, we denote the stable equilibria in $\mathcal{R}$ by v_1, v_2, and if there is a third large right environment, v_3. We call the possible orientations p_1, p_2, and if appropriate, p_3. Initially, p_1 coincides with v_1, p_2 with v_2, and p_3 with v_1. We denote $\mathcal{B}_{v_1}$ by $\mathcal{B}_1$, $\mathcal{B}_{v_2}$ by $\mathcal{B}_2$ and $\mathcal{B}_{v_3}$ by $\mathcal{B}_3$.

We will clarify the presented push plans by giving pictures which symbolicly show the reorientations of the jaw. The part is represented by the circular representation of its push function. The stable equilibria correspond to discs, the unstable equilibria correspond to circles. Recall that reorientations of the jaw in the angular intervals $(0, \frac{\pi}{2})$ and $(-\pi, -\frac{\pi}{2})$ are implemented by a left fence. Reorientations of the jaw in the angular intervals $(-\frac{\pi}{2}, 0)$ and $(\frac{\pi}{2}, \pi)$ are implemented by a right fence. In the pictures, reorientations of the jaw implemented by a left fence correspond to solid arrows. Reorientations of the jaw implemented by of a right fence correspond to dashed arrows. A push plan $\alpha_1, \alpha_2, \alpha_3$ adds three arrows per possible orientation to the figure. The first reorientation is marked by **1**, the second by **2**, and the third by **3**.

We start by moving the set of possible orientations to the long right environments. In Figure 10 the MOVE is depicted for a part with long environments. In the next subsections we treat the several cases which can occur if there are long right environments.

5.3.1 Push functions with three long environments

In this section we treat parts of which three orientations have a right environment longer than $\frac{\pi}{2}$. In the three large-right-interval case, we now apply a three phase approach

1. REDUCE$_l$ the three possible orientations to two orientations with a large right environment. This phase thus reduces two large orientations to one. We design the reduce plan such that the last reorientation we use is in the angular interval $(0, \frac{\pi}{2})$, or $(-\pi, -\frac{\pi}{2})$, i.e. the last fence was a left fence.

2. SHIFT$_l$ the possible orientations. Depending on the REDUCE$_l$, either one or two shifts are necessary.

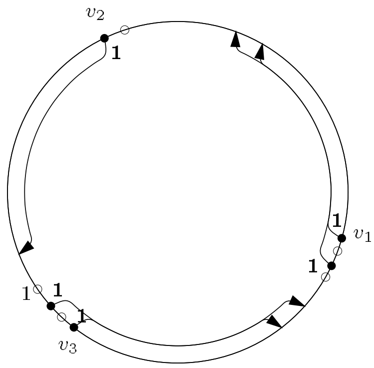


Figure 10: The MOVE for a part with three long right environments $r(v_1), r(v_2)$, and $r(v_2)$. One reorientation of the jaw suffices to MOVE the possible orientations onto $\{v_1, v_2, v_3\}$.

3. Apply the REDUCE$_l$ again. Note that in phase one, there are two orientations that are mapped onto the same destination orientation, so phase two and three are always applicable.

We first present SHIFT$_l$, which is the same in all cases in this section. The last reorientation of the jaw was in the angular interval $(0, \frac{\pi}{2})$, and the possible orientations are in $\mathcal{R}$ (the right environments are longer than $\frac{\pi}{2}$). This means that any reorientation in $(0, \frac{\pi}{2})$ is useless, it does not change the possible orientations. So, in order to give a valid fence design, we have to use at least one reorientation in the angular interval $(\frac{\pi}{2}, \pi)$. The first reorientation we use is $\max_{a \in \mathcal{R}}(|r(a)|) + \mu$. After application of the jaw, clearly, $p_1 \neq v_1$, $p_2 \neq v_2$ and $p_3 \neq v_3$. The second reorientation we use is $(-(\frac{\pi}{2} - \mu))$. Since there are three right environments longer than $\frac{\pi}{2}$, and at least three left environments, the possible orientations of the part are now again $p_1 = v_1$, $p_2 = v_2$, and $p_3 = v_3$. However, we can now use reorientations in the angular intervals $(-\pi, -\frac{\pi}{2})$ and $(-\frac{\pi}{2}, 0)$. The first two reorientations of the shift are implemented by right fences. The third and last reorientation of the jaw in SHIFT$_l$ is $-(\frac{\pi}{2} + \mu)$, we skip the angular intervals between $r(v_1)$, $r(v_2)$ and $r(v_3)$, resulting in a configuration of the possible orientations in which $p_1 = v_3$, $p_2 = v_1$ and $p_3 = v_2$. This orientation is implemented by a left fence. We have now shifted the possible orientations and can use the same reorientations of the jaw as just after the MOVE. The shift is depicted in Figure 11. Summarizing the shift, suppose that $[v, w]$ $(v, w \in \mathcal{R})$ is the current interval of possible orientations. Then, the push plan $(\frac{\pi}{2} + \mu), (-(\frac{\pi}{2} - \mu)), (-(\frac{\pi}{2} + \mu))$ maps the interval $[v, w]$ into an interval $[v', w'] \neq [v, w]$, $(v, v', w, w' \in \mathcal{R})$ of equal (right) measure.

In the next subsection we present reduce plans which are applicable for parts with three long right intervals. Since there are three long right environments, there cannot be a left environment longer than $\frac{\pi}{2}$. If the left cycle of the part is less than m_s, then we can use the techniques of Section 5.1. Since we already showed how to orient a part of which both the left and right cycle are less than m_s, we may assume that the right cycle is m_s.

5.3.2 Three long intervals of equal length

In this section $|r(v_1)| = |r(v_2)| = |r(v_3)|$. Recall that the right cycle is m_s. We have two cases.

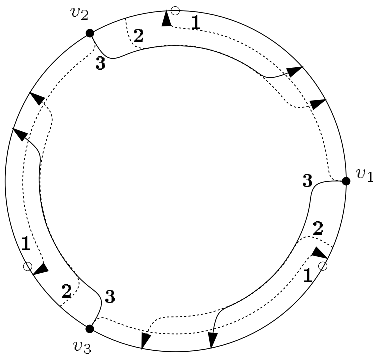


Figure 11: The SHIFT$_l$ plan for three long right environments is $(\frac{\pi}{2}+\mu), (-(\frac{\pi}{2}-\mu)), (-(\frac{\pi}{2}+\mu))$.

- The angular intervals between $r(v_1)$, $r(v_2)$ and $r(v_3)$, $\mathcal{B}_1, \mathcal{B}_2$, and $\mathcal{B}_3$ have exactly the same size. Clearly for $i \in 1,2,3$: $|\mathcal{B}_i|+|r(v_i)| < \pi$, and therefore $B_i \cap (v_i+\frac{\pi}{2}, v_i+\pi) = B_i$. In words, with a valid reorientation of the jaw, we can map any possible orientation onto any angle in the angular intervals between the long right environments. Chen and Ierardi show in Lemma 2.4 of their paper [6] that there is a reorientation of the jaw for which one orientation, say p_v is mapped onto a left environment, and another orientation p_w is mapped onto a right environment. The third orientation p_z is mapped either on a left or a right environment. Thus, after the application of the jaw, which is a right fence, we know that *not* $(p_1 - (v_1 + |r(v_1)|)) = (p_2 - (v_2 + |r(v_2)|)) = (p_3 - (v_3 + |r(v_3)|))$. Furthermore, we know that $(p_i - (v_i + |r(v_i)|)) < \frac{\pi}{2}$. We can now reorient the jaw by $-(p_w - (v_w + |r(v_w)| + \mu))$ and apply the jaw, which maps p_w onto v_w, but p_v onto an orientation in $\mathcal{B}_v$. The third possible orientation p_z is either mapped onto an orientation in $\mathcal{B}_z$ or onto v_z. The reorientation is in the angular interval $(-\frac{\pi}{2}, 0)$; this is accomplished by a right fence as well. We can now reorient the jaw by $-(\frac{\pi}{2}+\mu)$. An application of the jaw shifts the possible orientation(s) in $\mathcal{R}$ to the preceding orientation in $\mathcal{R}$, e.g. p_w is mapped onto v_{w-1}. The possible orientations in orientations not in $\mathcal{R}$ are also shifted to the preceding orientation in $\mathcal{R}$, e.g. $p_v \in B_v$ is mapped onto v_v. Since there is at least one orientation in $\mathcal{R}$ and one orientation not in $\mathcal{R}$, and the orientations are a cyclic sequence, there this application of the jaw collapses two orientations. This last application is realized by a left fence. We have a feasible reduce. In Figure 12, the cases for p_z mapped onto B_z and p_z mapped onto v_z are displayed).
- If not $|\mathcal{B}_1| = |\mathcal{B}_2| = |\mathcal{B}_3|$, we break the symmetry using the difference in length of the $\mathcal{B}_i$'s. We start with reorientation $|r(v_1)| + \mu$, this maps a possible orientation $p_j = a_i$ onto the next possible orientation a_{i+1}. We can now reorient the jaw by $-(\frac{\pi}{2} - \mu)$, after applying the jaw, $p_1 = v_1$, $p_2 = v_2$, and $p_3 = v_3$. The third reorientation breaks the symmetry, we can now reorient in the angular interval $(\frac{\pi}{2}, 0)$, and we choose the reorientation such that at least one orientation is mapped onto an angle in $B_i (i \in 1,2,3)$, and at least one orientation is mapped onto an angle in $r(v_i)(i \in 1,2,3)$. We apply the jaw. We can now with one left fence implement a reorientation of the jaw of $-(\frac{\pi}{2}+\mu)$. This fence maps two possible orientations to one. There are again two cases. See Figure 13(a) and (b).

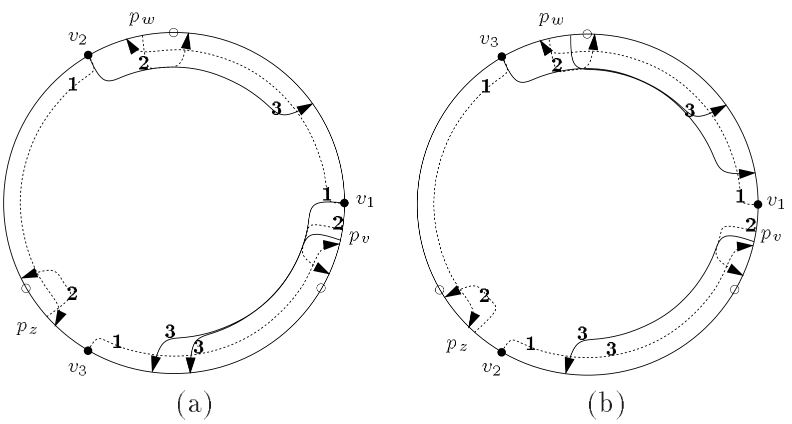


Figure 12: The REDUCE$_l$ for parts with three equally long right environments and $|\mathcal{B}_1| = |\mathcal{B}_2| = |\mathcal{B}_3|$ first break the symmetry by a reorientation in $(\frac{\pi}{2}, \pi)$ the rest of the plan is $-(p_w - (v_w + |r(v_w)| + \mu)), -(\frac{\pi}{2} + \mu)$.

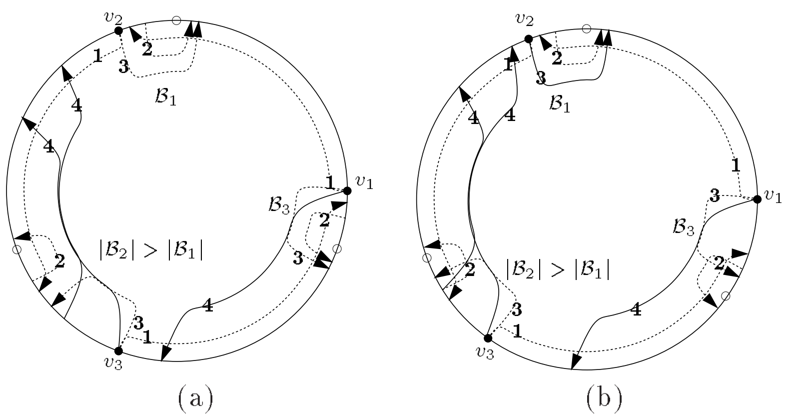


Figure 13: The REDUCE$_l$ for parts with three equally long right environments and not $|\mathcal{B}_1| = |\mathcal{B}_2| = |\mathcal{B}_3|$ start with $(|r(v_1)| + \mu), (-(\frac{\pi}{2} - \mu)$, then break the symmetry by a reorientation in $(-\frac{\pi}{2}, 0)$. The rest of the plan is $-(\frac{\pi}{2} + \mu)$.

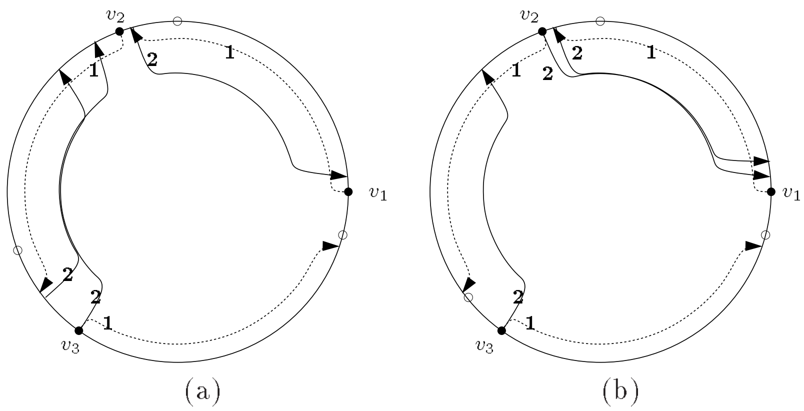


Figure 14: The REDUCE$_l$ for parts with three not equally long, long right environments consists of $(\min_{i\in(1,2,3)} r(v_1) + \mu), -(\frac{\pi}{2} - \mu)$.

5.3.3 Three long right environments of different length

If not $|r(v_1)| = |r(v_2)| = |r(v_3)|$, then we can reduce number of possible orientations to one using the reduce plan starting with a right fence which maps at least one possible orientation onto itself, and at least one possible orientation onto another orientation. This is accomplished by a reorientation of the jaw by $(\min_{i\in\{1,2,3\}} r(v_i) + \mu)$. After pushing, we reorient the jaw by $-(\frac{\pi}{2} - \mu)$. This is accomplished by a left fence which reduces the number of possible orientations two. In Figure 14(a), and (b) the REDUCE$_l$ for a unique largest right interval is displayed.

5.3.4 Two long right intervals

If there are two orientations with long right intervals, v_1, v_2. There is no longer a reason to SHIFT$_l$. After a MOVE, there are two possible orientations, and after a reduce, there is only one possible orientation left. In the case of three long environments, the intervals between the long environments, $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ were shorter than $\frac{\pi}{2}$. If there are only two long right environments, this clearly is not true. Given this, the reduce for two long right environments is more complicated. We devote the following subsections to present the reduce for the different cases.

5.3.5 Two long intervals of equal length

We first give the reduce if both large right intervals have the same size, i.e. $|r(v_1)| = |r(v_2)|$. The size of domain of the push-function minus $(r(v_1) \cup r(v_2))$ is smaller than π. If the left-intervals have cycle m_s, we can have orient the part using the techniques of Section 5.1, using reorientations in the angular interval $(-\frac{\pi}{2}, 0)$, since in that case there is at most one left environment with size greater than $\frac{\pi}{2}$. Now suppose that the left cycle is less than m_s, say a. Since we already treated the case for which the right cycle is less than m_s as well, we may now assume the right cycle m_s. We now distinguish two cases.

1. The lengths of the angular intervals between $r(v_1)$ and $r(v_2)$ differ. With one reorien-

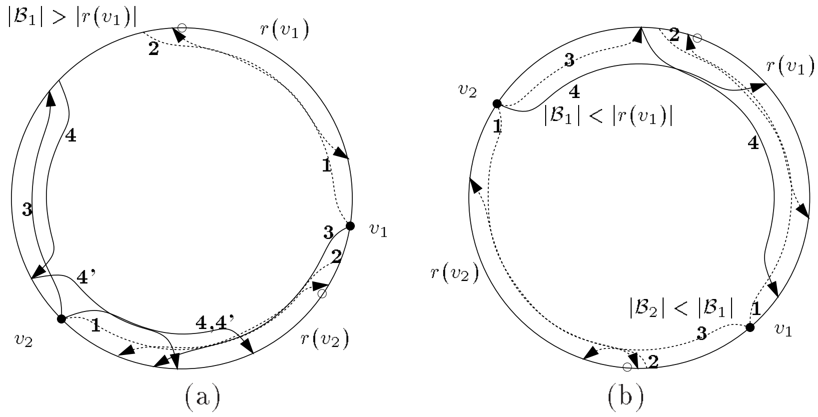


Figure 15: (a) The REDUCE$_l$ for two equally long right environments and $|\mathcal{B}_1| > \frac{\pi}{2}$ is $(r(v_1) + \mu), -(\frac{\pi}{2} - \mu), -(\frac{\pi}{2} + \mu)$ followed by a MOVE $(= 4)$. (b) If $|\mathcal{B}_1| < |r(v_1)|$, REDUCE$_l$ is as follows: $(r(v_1) + \mu), -(\frac{\pi}{2} - \mu), -(|\mathcal{B}_2| + \mu), (-|r(v_1)|)$.

tation of the jaw of $(|r(v_1)| + \mu)$ followed by a push, implemented by a right fence, we shift both p_1 and p_2 one orientation to the next stable equilibrium of the part. With another right fence we shift p_1 back to v_1 and p_2 back to v_2, using reorientation of the jaw of $\frac{\pi}{2} - \mu$. The largest angular interval between $r(v_1)$ and $r(v_2)$ is either larger than $|r(v_1)| = |r(v_2)|$, or this interval is as long as or shorter than $|r(v_1)| = |r(v_2)|$.

Let us first assume that the largest angular interval between $r(v_1)$ and $r(v_2)$ is longer than $|r(v_1)| = |r(v_2)|$, and thus larger than $\frac{\pi}{2}$ as well. Let us assume that this angular interval is $\mathcal{B}_1$, and thus $\{v_1, \mathcal{B}_1, v_2,\}$ is counterclockwisely ordered. The last reorientation of the jaw was in the interval $(-\frac{\pi}{2}, 0)$ which gives us that using a reorientation of the jaw of $-(\frac{\pi}{2} + \mu)$ and a push by a left fence, we shift the orientation p_1 to v_2, and p_2 to an orientation in $\mathcal{B}_1$. In counterclockwise order, this orientation is positioned before v_2, so with a MOVE this orientation maps to v_2 as well, while the possible orientation p_1 remains fixed, and hereby complete a valid and feasible fence design. A picture of this reduce is given in Figure 15(a).

If, on the other hard, the largest angular interval between $r(v_1)$ and $r(v_2)$, $\mathcal{B}_1$, is as long as or shorter than $|r(v_1)| = |r(v_2)|$. We know that $|\mathcal{B}_2| < \frac{\pi}{2}$. Again, $\{v_1, \mathcal{B}_1, v_2, \mathcal{B}_2\}$ is counterclockwisely ordered. With a reorientation of the jaw of $-(|\mathcal{B}'| + \mu)$ and push by a right fence, we shift the orientation p_1 past $\mathcal{B}_2$, while p_2 is mapped onto an orientation in $\mathcal{B}_1$. With a reorientation of the jaw by $(-|r(v_1)|)$ and a push by a left fence, we map both possible orientations to v_1. A picture of this REDUCE$_l$ is given in Figure 15(b).

2. The intervals between the large right intervals, $\mathcal{B}_1$ and $\mathcal{B}_2$, have exactly the same size we construct a fence design as follows: Since the push function has period 2π, Lemma 2.4 of Chen and Ierardi's paper [6] gives us that there has to be a pair of orientations (at least one between the orientations with the large right environments), reachable with one push in angular interval $(\frac{\pi}{2}, \pi)$ by a right fence, such that the distance between p_1 and v_1 is not equal to the distance between p_2 and v_2. Apply this push and we assume w.l.o.g. that $(p_1 - v_1) < (p_2 - v_2)$.

   With another right fence, we reorient the jaw by $-(p_1 - v_1)$ and map with a push p_1

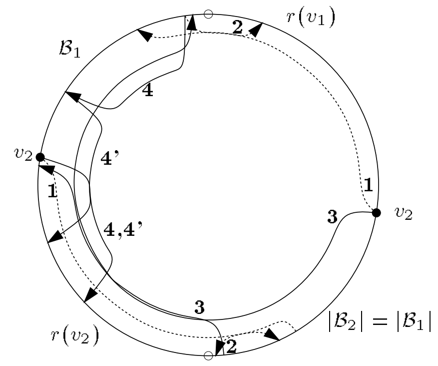


Figure 16: The REDUCE$_l$ for a part with two long right environments, $|r(v_1)| = |r(v_2)|$, $|\mathcal{B}_1| = |\mathcal{B}_2|$. The plan starts with a symmetry breaking push after a reorientation fo the jaw in $(\frac{\pi}{2}, \pi)$. The plan continues with $-(p_1 - v_1), -(\pi - \mu)$ and ends with a MOVE.

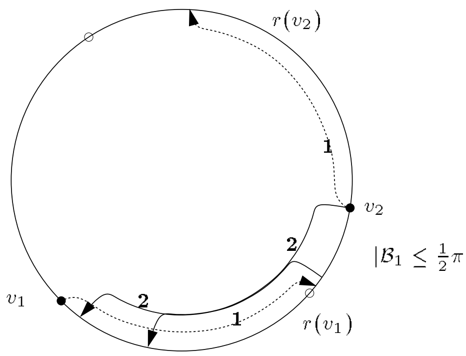


Figure 17: The REDUCE$_l$ for a part with two large right environment, which are not equally large. The plan is $(|r(v_1) + \mu), -(\frac{\pi}{2} + \mu)$.

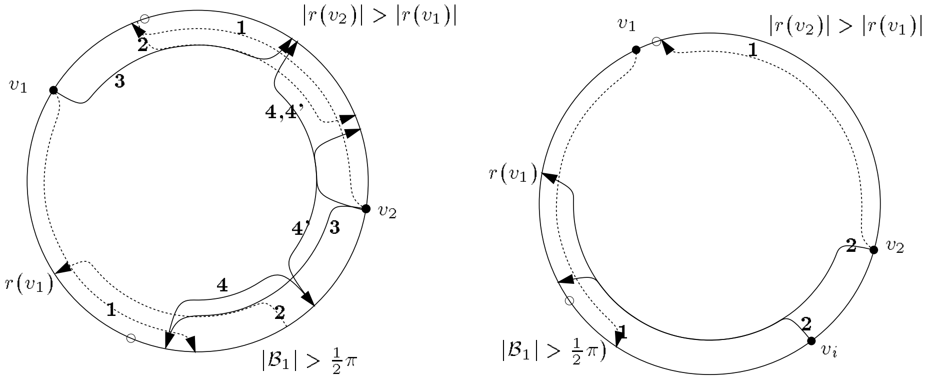
onto v_1 and p_2 onto $\mathcal{B}_2$. We now reorient the jaw by $-(\pi - \mu)$ and push with a left fence. This shifts p_1, which coincides with v_1, to v_2. Simultaneously, p_2 is shifted to an orientation in $\mathcal{B}_1$. With a MOVE, we complete the plan and we have a feasible fence design to orient this type of parts. A picture of the reduce is given in Figure 16.

5.3.6 Two long intervals of different length

Let v_1 be the orientation with the shortest right interval greater than $\frac{\pi}{2}$; the other orientation is called v_2, the angular interval between $r(v_1)$ and $r(v_2)$ (in counterclockwise order) is therefore called $\mathcal{B}_1$. We distinguish between two cases that differ in the length of $\mathcal{B}_1$.

1. The length of $\mathcal{B}_1$ is less than or equal to $\frac{\pi}{2}$. With a reorientation of the jaw by $(|r(v_1)+\mu)$ and a push right fence, we map p_1 onto the orientation next to v_1, leaving p_2 unchanged. With a left fence we construct a reorientation of the jaw of $-(\frac{\pi}{2}+\mu)$, and shift p_2 to the left, such that it is mapped onto the orientation v_1 (skipping $\mathcal{B}_1$). The other possible orientation, p_1, is also mapped onto v_1. See Figure 17 for a picture of this reduce.

2. The length of $\mathcal{B}_1$is larger than $\frac{\pi}{2}$. We now know that both large right intervals are smaller



(a) (b)

Figure 18: The REDUCE_l plans for parts with two long right environments, and the angular interval between the shortest and the longest intervals longer than $\frac{\pi}{2}$. (a) $(|r(v_2)| + \mu), -(\frac{\pi}{2} - \mu), -(\frac{\pi}{2} + \mu)$. The plan end with MOVE (4). (b) If $-(\frac{\pi}{2} - \mu)$ of the reduce in (a) does not map p_1 back onto v_1. The plan is different: $(|r(v_2)| - \mu), -(|\mathcal{B}_1| + \mu)$.

than π. We use a right fence to construct a reorientation of the jaw by $(|r(v_2)| + \mu)$, followed by a push, and skip both right intervals. With again a right fence, we construct push angle $-(\frac{\pi}{2} - \mu)$. Now p_1 again coincides with v_1, and p_2 is on v_2. In this case, we can use a left fence to push $-(\frac{\pi}{2} + \mu)$ to map a_2 to v_1, and p_1 orientation is mapped to an orientation in $\mathcal{B}$. With a MOVE, we move p_1 onto v_1 as well.

If it is not possible to map p_1 back to v_1, we use a different plan. Firstly, we use a push of $|r(v_2)| - \mu$, by a right fence. p_1 is mapped onto the same orientation as in the previous plan; p_2 remains stable now, though. We now use a reorientation of the jaw of $-(|\mathcal{B}_1| + \mu)$, by a left fence to map both p_1 and p_2 onto v_2. This is possible, since $p_1 - p_2 < \frac{\pi}{2}$. See Figures 18(a) and (b) for a picture of the corresponding push plans.

In this section we derived the following

Theorem 5.16 *Given a part P, if the parts push functions stable equilibria $\mathcal{A}$ have right cycle $|\mathcal{A}|$ and there are right environments longer than or as long as $\frac{\pi}{2}$ then there is there is a fence design that orients the part up to symmetry. Similarly, if the stable equilibria $\mathcal{A}$ have left cycle $|\mathcal{A}|$ and there are left environments longer than or as long as $\frac{\pi}{2}$ then there is there is a fence design that orients the part up to symmetry.*

6 Conclusion

In this paper we investigated the problem of sensorless part orientation by sequences of pushes. We showed that any polygonal part can be oriented by a sequence of fences placed along a conveyor belt. We presented the first polynomial-time algorithm for computing the shortest fence design for any given polygonal part. The algorithm is easy to implement and runs in time $O(n^3 \log n)$. The structure of the algorithm yields an $O(n^2)$ bound on the length of the shortest fence design. We showed that for asymmetric parts the length is actually bounded by $O(n)$. It remains an open problem whether an $O(n)$ bound exists for parts that are not asymmetric.

Although pathological polygons can be constructed that lead to push plans and fence designs of length $\Omega(n)$, it turns out that the length of most plans remains far below the worst-case length. In [16], Van der Stappen *et al.* have shown that only $O(1)$ actions are required for parts with non-zero eccentricity, i.e., with non-square minimum-width bounding box. The analysis also applies to curved parts, providing the first complexity bound for non-polygonal parts. The results generalize to fence designs for parts with acyclic left and right environments. It remains an open question whether this bound can be transferred to arbitrary parts.

7 Acknowledgements

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A From a Push Function to a Part

In this appendix we show that, given a push function, there is a polygonal part which has this push function. This shows that the study of push functions is equivalent to the study of pushing polygonal parts. In [15], Rao and Goldberg gave related results for diameter functions, which predict the rotation of a part, when squeezed by a parallel jaw gripper.

Let p be a push function. Let γ be the minimum length of all left and right environments. If $\gamma > 0$, then we can construct a part that fits p. It is easily verified that this constraint is satisfied by any push function of a polygonal part without meta-stable edges, i.e. the perpendicular projection of the center-of-mass on an edge does not intersect a vertex of the edge.

We construct a convex polygon for which every maximum has radius 1 and every minimum has radius $\max\{0.5, \cos\gamma\}$. Each stable equilibrium orientation in the push function must correspond to an edge of the polygon such that the perpendicular projection of the center-of-mass lies inbetween the endpoints of the edge, and furthermore, the direction of the projection equals the equilibrium orientation. The isolated equilibrium points in the push function must correspond to vertices of the polygon that lie at a distance 1 from the center-of-mass such that there is a tangent to the vertex whose normal passes through the center-of-mass. So, if the push function has m_s stable equilibria, we obtain an alternating collection of m_s vertices at distance 1 from the center-of-mass and m_s edges at distance $\max\{0.5, \cos\gamma\}$.

It remains to connect the vertices to the edges. These connecting chains should create no additional minima or maxima in the radius function. So, we must make sure that the radius is strictly decreasing from the vertex to the adjoining edges. To this end we start in each minimum a polyline that spirals out to the maximum to the left and to the right. We make

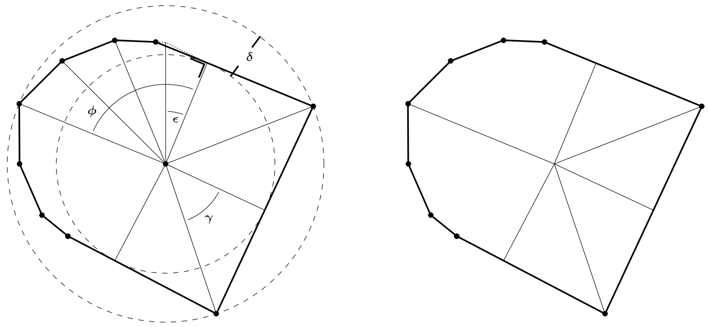

Figure 19: The construction of a polygonal part with a given push function. For one environment of the part, $\mu = \phi/\epsilon$ regions bounded by rays with angular distance ϵ are depicted, together with the resulting polyline.

sure that no segment of the polyline connecting a minimum to a maximum has a orthogonal line crossing the center of the part.

We construct a polyline from a minimum orientation a^- to an adjoining maximum orientation a^+ as follows. Without loss of generality, we assume that the interval between a^- and a^+ is the right environment of a^-. We divide the angular interval (a^-, a^+) into μ equal sized intervals. Let $\phi = |(a^-, a^+)|$, then $\epsilon = \phi/\mu$ is the size the intervals between a^- and a^+. The vertices of the polyline are placed on rays emanating from c with angle $a^-, a^- + \epsilon, \ldots, a^+$. Let $\delta > 0$ denote the difference in radius of the maxima and the minima of the part. The i-th vertex of the polyline has distance $1 - (1 - \frac{i}{\mu}) \cdot \delta$ to c. (See Figure 19 for an example.)

In order for the part to have no minimum or maximum in the angular interval (a^-, a^+), ϵ has to be small enough to satisfy the following constraint: for every pair of successive vertices (a, a') on the polyline, with distance x resp. $x + \delta/\mu$ to the center-of-mass of the part, the vertical projection of c on the supporting line of (a, a') lies not on the segment. Thus, a' is more distant from c than the intersection of the line through a, which is orthogonal to the ray of a, and the ray of a'. Using trigonometry this constraint is

$$\frac{x}{x + \frac{\delta}{\mu}} = \frac{x}{x + \epsilon\frac{\delta}{\phi}} < \cos\epsilon.$$

Using Taylor gives us a lower bound for $\cos\epsilon$:

$$\frac{x}{x + \epsilon\frac{\delta}{\phi}} < 1 - \frac{1}{2}\epsilon^2,$$

and after some manipulation, using $x < 1$,

$$0 < \frac{\delta}{\phi}\epsilon - \frac{1}{2}\epsilon^2 - \frac{\delta}{2\phi}\epsilon^3.$$

Since $\frac{\delta}{\phi}$ is a constant, we can choose ϵ small enough to satisfy this constraint, and then there is no contact normal of the part in the angular interval (a^-, a^+), intersecting c. To make a^- a minimum, we extend the normal of the circle with radius $1 - \delta$ from a^- to the intersection with the constructed polyline, and cut away the area of the part outside the extension. We conclude that we can connect each minimum to its successive maximum, and thus construct a convex polygonal part for every push function with non-zero minimum length of left and right environments.