Pseudo-Linear Scale-Space Theory

Towards the Integration of Linear and Morphological Scale-Space Paradigms

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Abstract

It has been observed that linear, Gaussian scale-space, and nonlinear, morphological erosion and dilation scale-spaces generated by a quadratic structuring function have a lot in common. Indeed, far-reaching analogies have been reported, which seems to suggest the existence of an underlying isomorphism. However, an actual mapping appears to be missing.

In the present work a one-parameter isomorphism is constructed in closed-form, which encompasses linear and both types of morphological scale-spaces as (non-uniform) limiting cases. Apart from establishing such a formal connection, the unfolding of the family provides a means to transfer known results from one domain to the other. Moreover, for any fixed and non-degenerate parameter value one obtains a novel type of "pseudo-linear" multiscale representation that is, in a precise way, "in-between" the familiar ones, and may be of interest in its own right.

1 Introduction

Both morphological as well as linear scale-space representations have received much attention in recent literature. In mathematical morphology a fairly well-understood theory has been developed on erosion and dilation scale-spaces based on a morphological propagator with the shape of a parabola. In this case the reciprocal of the parabola's (constant) curvature serves as the free scale parameter. The two morphological operations, erosion and dilation, induce two distinct one-parameter families of images, each of which represents a given image at various levels of scale. In linear image processing, on the other hand, only one kind of operation is used—at least if homogeneity is required—*viz.* linear correlation or, equivalently, convolution. The notion of a Gaussian scale-space, obtained by linear filtering with a normalised Gaussian of variable width, is particularly familiar. For an overview the reader is referred to the existing books on the subject [6, 8, 14, 18], and to the references therein.

It has been pointed out that the two abovementioned techniques for introducing scale exhibit remarkable analogies. Indeed, in one of their articles on morphological scale-space theory Van den Boomgaard and Smeulders [4] remark:

> "It is our belief that the results presented in this paper barely scratch the surface of the possible integration of differential analysis (including differential geometry) with mathematical morphology. It is our hope that the equivalence between Gaussian scale-space theory and morphological scale-space can be extended even further."

Elsewhere, Boomgaard *et al.* [1, 2, 3] and Dorst *et al.* [5] provide compelling evidence, by a scrutinised analysis of analogies, that seems to warrant this belief. For further details and more references on mathematical morphology, *cf.* Goutsias *et al.* [7], Heijmans [9, 10], Jackway and Deriche [11], and Serra [15, 16].

However, an actual *mapping* between linear and morphological scale-space representations appears to be missing. In this article we construct a one-parameter family of *pseudo-linear scale-space representations*, in which the members are *isomorphically* related by a one-parameter transformation of grey-values. The parameter could be said to express the "degree of nonlinearity[1]". Linear and morphological scale-spaces arise as limiting cases (parameter values 0, respectively $\pm\infty$). Although isomorphism ceases to hold, many results do carry over under the act of taking limits. Thus the unfolding of the family may yield insight into the connection between these familiar multiresolution frameworks beyond the hitherto noticed analogies. Furthermore, the intermediate case of nondegenerate parameter value (± 1, say) is new and may be of interest in its own right. In any case, the closed-form mapping allows one to transfer results from the linear to the morphological domain, *vice versa*, such as the typical scale-space axiom of "causality" (or non-enhancement principle), *cf.* Koenderink [12]. However, emphasis in this article is on the construction of this mapping, not on a scrutinised analysis of potential consequences.

2 Theory: Pseudo-Linear Scale-Spaces and Isomorphisms

Consider the linear isotropic diffusion equation with initial condition:

$$\begin{cases} \partial_s u & = & \Delta u \,, \\ \lim_{s\downarrow 0} u & = & f \,. \end{cases} \tag{1}$$

Subjecting u to an arbitrary transformation

$$u \stackrel{\text{def}}{=} \gamma(v) \quad \text{with} \quad \gamma' > 0 \tag{2}$$

yields the following nonlinear initial value problem for v:

$$\begin{cases} \partial_s v & = & \Delta v + \mu \|\nabla v\|^2 \,, \\ \lim_{s\downarrow 0} v & = & g \,. \end{cases} \tag{3}$$

[1] "Nonlinear" is sloppy jargon for "non-affine".

in which the nonlinearity is defined by

$$\mu \stackrel{\text{def}}{=} (\ln \gamma')' \,, \tag{4}$$

and the initial condition by

$$g \stackrel{\text{def}}{=} \gamma^{-1}(f) \,. \tag{5}$$

We have the following commuting diagram:

$$\begin{array}{ccc} u & \xleftarrow{\gamma} & v \\ {\scriptstyle Eq.\,(1)}\Big\uparrow & & \Big\uparrow{\scriptstyle Eq.\,(3)} \\ f & \xrightarrow{\gamma^{-1}} & g \end{array} \tag{6}$$

Note that if γ tends to an affine transformation, *i.e.* $\mu \downarrow 0$, one reobtains the linear equation. Another case of special interest arises in the limit $\mu \to \pm\infty$. A perturbative approach reveals that one obtains a first order evolution equation, which is the morphological counterpart of Eq. (1):

$$\begin{cases} \partial_t v & = & \pm\|\nabla v\|^2 \,, \\ \lim_{t\downarrow 0} v & = & g \,. \end{cases} \tag{7}$$

See *e.g.* Boomgaard *et al.* [1, 2, 4, 3] and Dorst *et al.* [5].

Affine transformations $\gamma(v) = \alpha + \beta v$ fully exhaust the invariance of Eq. (1) under invertible grey-scale mappings. Consequently it is hard to compare the linear scale-spaces generated in accordance with Eq. (1) for two initial images that differ by a non-affine grey-scale transformation. The general case of Eq. (3) is of interest for its potential role in the development of general multiscale techniques beyond standard linear or morphological methods.

Of all possible non-affine transformations, one class is particularly simple and somewhat special, *viz.* the one for which the coefficient μ of Eq. (4) is a global constant. The corresponding transformation again depends on the choice of two integration constants and is given by the following lemma.

Lemma 1 *Consider the parametrised transformation $u = \gamma_\mu(v)$ given by*

$$\gamma_\mu(v) = \begin{cases} \beta\,\frac{e^{\mu v}-1}{\mu} + \alpha & \textit{if} \quad \mu \neq 0 \,, \\ \beta\, v + \alpha & \textit{if} \quad \mu = 0 \,. \end{cases}$$

This transforms Eq. (1) into Eqs. (3–4) with a constant coefficient μ. Apart from μ there are two degrees of freedom in the transformation, viz. $\alpha \in \mathbb{R}$ *and* $\beta \in \mathbb{R}^+$.

Note that the constants $\alpha = \gamma_\mu(0)$ and $\beta = \gamma'_\mu(0)$ are independent of μ. There is no loss of generality in the discussion that follows if we fix suitable values for α and β. It is convenient to restrict image values to the unit interval, and to maintain this range regardless of the mapping.

Assumption 1 *By means of a suitable affine transformation we henceforth restrict the isomorphism γ_μ of Lemma 1 to the unit interval: $\gamma_\mu(0) \equiv 0$ and $\gamma_\mu(1) \equiv 1$. That is,*

$$\gamma_\mu(v) = \begin{cases} \frac{e^{\mu v}-1}{e^{\mu}-1} & \textit{if} \quad \mu \neq 0 \,, \\ v & \textit{if} \quad \mu = 0 \,. \end{cases}$$

As for dimensional consistency, a convenient interpretation of the mapping in these examples—and of nonlinear transformations in general—will be that the function v and the coefficient μ are dimensionless, while α and β have the same dimension as u. For simplicity we henceforth assume that u is dimensionless[2] as well, as in Assumption 1. We have

$$\gamma_\mu^{-1}(u) = \begin{cases} \frac{1}{\mu} \ln\left(1 + (e^\mu - 1)u\right) & \text{if} \quad \mu \neq 0\,, \\ u & \text{if} \quad \mu = 0\,. \end{cases} \tag{8}$$

Note that both γ_μ and γ_μ^{-1} are continuously differentiable for all $\mu \in \mathbb{R}$, in other words, the isomorphism is even a *diffeomorphism*. This observation may be important in view of techniques or proofs that exploit the commuting diagram of Eq. (6). Furthermore, we have the following limiting cases (χ_I is the indicator function on I, *i.e.* $\chi_I(x) = 1$ if $x \in I$, otherwise $\chi_I(x) = 0$; $[a, b[$ denotes the half-open interval including a but excluding b, etc.):

$$\lim_{\mu\to+\infty} \gamma_\mu(v) = 1 - \chi_{[0,1[}(v)\,, \tag{9}$$

$$\lim_{\mu\to+\infty} \gamma_\mu^{-1}(u) = \chi_{]0,1]}(u)\,, \tag{10}$$

respectively

$$\lim_{\mu\to-\infty} \gamma_\mu(v) = \chi_{]0,1]}(v)\,, \tag{11}$$

$$\lim_{\mu\to-\infty} \gamma_\mu^{-1}(u) = 1 - \chi_{[0,1[}(u)\,. \tag{12}$$

Convergence is pointwise, not uniform. In particular one observes that the limiting pairs are no longer eachother's inverse.

Definition 1 *Recall Eq. (6). Let $\star$ denote correlation,* i.e.

$$f \star \phi(x) \stackrel{\text{def}}{=} \int dz\, f(x+z)\, \phi(z)\,,$$

then we define

$$v \stackrel{\text{def}}{=} \gamma^{-1}\left(\gamma(g) \star \phi\right)\,.$$

In particular, with $\gamma = \gamma_\mu$ as in Assumption 1, we have

$$v_\mu(x) = \frac{1}{\mu} \ln \int dz\, e^{\mu g(x+z)}\, \phi(z)\,.$$

Definition 1 expresses the relationship that holds between the solutions of Eqs. (1) and (3) under Assumption 1, which follows straightforwardly by inspection of Eq. (6). Recall that the Green's function corresponding to Eq. (1) is a normalised Gaussian,

$$\phi(z;\sigma) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi\sigma^2}^{\,n}}\, e^{-\frac{1}{2}\frac{\|z\|^2}{\sigma^2}}\,, \tag{13}$$

in which the *inner scale parameter* σ is related to the evolution parameter s of Eq. (1) by $\sigma = \sqrt{2s}$.

[2] Say we have divided the physical quantity of interest by its maximum value attained over the image domain.

Lemma 2 (Family of Pseudo-Linear Scale-Spaces) *See Definition 1 and Eq. (13). With the mapping as defined in Assumption 1 we have*

$$v_\mu(x;\sigma) = \frac{1}{\mu}\ln\int dz\, e^{\mu g(x+z)}\,\phi(z;\sigma)\,.$$

(The proof is straightforward.) For every $\mu \in \mathbb{R}$ this lemma then gives us the explicit nonlinear filtering procedure for obtaining a particular multiscale representation of the raw image g corresponding to a member of a 1-parameter family of pseudo-linear scale-spaces governed by the control parameter μ. Note that the integral is always well-defined due to renormalisation of grey-values: $0 \leq g(z) \leq 1$ for all z.

Lemma 3 (Linear Scale-Space) *See Lemma 2 and Eq. (13). The limit*

$$v_0 \stackrel{\text{def}}{=} \lim_{\mu\to 0} v_\mu$$

exists and corresponds to linear scale-space filtering:

$$v_0(x;\sigma) = \int dz\, g(x+z)\,\phi(z;\sigma)\,.$$

Proof. Observe that $v_\mu = g \star \phi + \mathcal{O}(\mu)$ as $\mu \to 0$.

The other limiting cases are summarised in the following lemma.

Lemma 4 (Dilation and Erosion Scale-Spaces) *See Lemma 2 and Eq. (13). Define the rescaled parameter $\tau \equiv \sigma\sqrt{|\mu|+1}$, and consider $\overline{v}_\mu(x;\tau) \equiv v_\mu(x;\sigma)$. Keeping τ fixed, the limits*

$$v_\pm \stackrel{\text{def}}{=} \lim_{\mu\to\pm\infty} \overline{v}_\mu$$

exist, and are given by

$$\begin{aligned} v_+(x;\tau) &= \sup_{z\in\mathbb{R}^n}\left[g(x+z)+q_+(z;\tau)\right] \\ v_-(x;\tau) &= \inf_{z\in\mathbb{R}^n}\left[g(x+z)+q_-(z;\tau)\right]\,, \end{aligned}$$

with

$$q_\pm(z;\tau) \stackrel{\text{def}}{=} \mp\frac{1}{2}\frac{\|z\|^2}{\tau^2}\,.$$

In mathematical morphology the functions v_+ and v_- obtained according to this recipe are known as the *dilation*, respectively *erosion of* g *by* $q = q_+$. The function q is known as the *quadratic* or *parabolic structuring function* [2, 4, 5, 13, 17], which, by the recipe of Lemma 4, induces a multiscale representation of g known as the *dilation*, respectively *erosion scale-space*.

Proof. The idea is to keep τ fixed in a physical representation:

$$\overline{v}_\mu(x;\tau) \stackrel{\text{def}}{=} \frac{1}{\mu}\ln\left\{\int dz\, e^{\mu g(x+z)}\,\phi(z;\sigma)\right\}\,.$$

To this end we rewrite the r.h.s. as

$$\frac{1}{\mu}\ln\left\{\sqrt{|\mu|+1}^{\,n}\int dz\, e^{\mu[g(x+z)+q_{\operatorname{sgn}\mu}(z;\tau)]}\,\phi(z;\tau)\right\} = \frac{1}{\mu}\ln\int dz\, e^{\mu[g(x+z)+q_{\operatorname{sgn}\mu}(z;\tau)]}\,\phi(z;\tau)+\mathcal{O}\left(\frac{\ln|\mu|}{\mu}\right)$$

as $\mu \to \pm\infty$. The latter term vanishes in the limit, and the result follows from the standard formulas (using continuity and monotonicity of the logarithm)

$$\begin{aligned}\lim_{\mu\to\infty}\left\{\int dz\, m(z)\,\varphi^{\mu}(z)\right\}^{\frac{1}{\mu}} &= \sup_{z\in\mathbb{R}^n}\varphi(z)\,,\\ \lim_{\mu\to-\infty}\left\{\int dz\, m(z)\,\varphi^{\mu}(z)\right\}^{\frac{1}{\mu}} &= \inf_{z\in\mathbb{R}^n}\varphi(z)\,,\end{aligned}$$

which hold if φ is positive, continuous and bounded, for any measure m for which the integral exists.

The proof makes use of a rescaling $t = s(1+|\mu|)$. If we apply this to Eq. (3) we obtain (setting $2s = \sigma^2$, $2t = \tau^2$):

$$\begin{cases}\partial_t\overline{v}_\mu &= \frac{1}{1+|\mu|}\,\Delta\overline{v}_\mu + \frac{\mu}{1+|\mu|}\,\|\nabla\overline{v}_\mu\|^2\,,\\ \lim_{t\downarrow 0}\overline{v}_\mu &= g\,,\end{cases} \tag{14}$$

which indeed reproduces both Eq. (1) as well as Eq. (7) in the respective limits, but at the same time shows that the associated scale parameters are related in a nontrivial way, *viz.* by an *infinite rescaling*! Although conceptually a bit awkward, such a "renormalisation" is frequently encountered in physics in the context of field theories. The procedure is justified by the argument that "hidden scale parameters" (physical "units") are arbitrary anyway, and that nothing actually depends on their values (*scale invariance*). This implies that the entire morphological scale-space construct, whether based on erosion or on dilation, pertains to the structure of images at infinite resolution[3]. Put differently, we cannot compare linear and morphological scale-spaces on a slice-by-slice basis along the scale axis. This fact explains the sensitivity of morphological scale-space representations with respect to small perturbations of infinite resolution, no matter how small their measure ("noise spikes"). Consequently, "morphological scale"—*i.e.* scale in the sense of the renormalised parameter τ—is of no help in defining well-posed differential structure in the way "linear scale" σ does (recall the quotation in the introduction).

For nonzero, finite values of μ we have a continuous family of intermediate representations. By slick choice of units, starting out from an intermediate value $\mu \neq 0, \pm\infty$, we can always replace any isolated member of this family by one of two canonical forms:

Definition 2 (Pseudo-Linear Scale-Spaces) *See Lemma 2 and Eq. (13). The* E-type pseudo-linear scale-space *of g is defined by*

$$v_{\mathrm{E}}(x;\sigma) \stackrel{\mathrm{def}}{=} v_{-1}(x;\sigma)\,.$$

Likewise, The D-type pseudo-linear scale-space *of g is defined by*

$$v_{\mathrm{D}}(x;\sigma) \stackrel{\mathrm{def}}{=} v_{1}(x;\sigma)\,.$$

[3]Resolution means inverse scale in the sense of linear diffusion.

μ	categorical type
$-\infty$	erosion scale-space
-1	E-type pseudo-linear scale-space
0	linear scale-space
$+1$	D-type pseudo-linear scale-space
$+\infty$	dilation scale-space

Table 1: The μ-family admits five categories of pseudo-linear scale-spaces. The middle three are diffeomorphic (whence the attribute "pseudo").

Put differently, in terms of the physical parameter τ, the (canonical forms of) E- and D-type pseudo-linear scale-spaces correspond to

$$\overline{v}_{\pm 1}(x;\tau) = \pm \ln \int dz\, e^{\pm g(x+z)}\, \phi(z;\tau/\sqrt{2})\,. \tag{15}$$

3 Summary and Conclusion

We have constructed a continuous family of mutually diffeomorphic scale-spaces depending on a single, real-valued parameter μ. We have considered the cases $\mu = 0$ (linear scale-space), $\mu = \pm 1$ (pseudo-linear scale-spaces), and also studied the limits $\mu \to \pm\infty$ (morphological erosion and dilation scale-spaces). Cf. Table 1.

Apart from these special instances, the unfolding of the μ-family is of interest in its own right; by embedding the various scale-spaces into such a family, we have established an explicit connection between important multiresolution frameworks that hitherto seemed only superficially related, *i.e.* by virtue of remarkable analogies, such as the typical scale-space axiom of "causality" (or non-enhancement principle), *cf.* Koenderink [12], and the algebraic similarities between the Gaussian filter in the linear, and the quadratic structuring functions in the morphological framework. The closed-form mapping potentially enables one to transfer results from the linear to the morphological domain, *vice versa*, and thus to understand these analogies.

References

[1] R. van den Boomgaard. *Mathematical Morphology: Extensions towards Computer Vision*. PhD thesis, University of Amsterdam, March 23 1992.

[2] R. van den Boomgaard. The morphological equivalent of the Gauss convolution. *Nieuw Archief voor Wiskunde*, 10(3):219–236, November 1992.

[3] R. van den Boomgaard and L. Dorst. The morphological equivalent of the Gaussian scale-space. In Sporring et al. [18], chapter 15, pages 203–220.

[4] R. van den Boomgaard and A. W. M. Smeulders. The morphological structure of images, the differential equations of morphological scale-space. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 16(11):1101–1113, November 1994.

[5] L. Dorst and R. van den Boomgaard. Morphological signal processing and the slope transform. *Signal Processing*, 38:79–98, 1994.

[6] L. M. J. Florack. *Image Structure*. Computational Imaging and Vision Series. Kluwer Academic Publishers, 1997. To appear.

[7] J. Goutsias, H. J. A. M. Heijmans, and K. Sivakumar. Morphological operators for image sequences. *Computer Vision and Image Understanding*, 62(3):326–346, November 1995.

[8] B. M. ter Haar Romeny, L. M. J. Florack, J. J. Koenderink, and M. A. Viergever, editors. *Scale-Space Theory in Computer Vision: Proceedings of the First International Conference, Scale-Space'97, Utrecht, The Netherlands*, volume 1252 of *Lecture Notes in Computer Science*. Springer-Verlag, Berlin, July 1997.

[9] H. J. A. M. Heijmans. Theoretical aspects of grey-level morphology. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 13(6):568–592, June 1991.

[10] H. J. A. M. Heijmans. Mathematical morphology: a geometrical approach in image processing. *Nieuw Archief voor Wiskunde*, 10(3):237–276, November 1992.

[11] P. T. Jackway and M. Deriche. Scale-space properties of the multiscale morphological dilation-erosion. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 18(1):38–51, January 1996.

[12] J. J. Koenderink. The structure of images. *Biological Cybernetics*, 50:363–370, 1984.

[13] P. D. Lax. *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*. SIAM, 1973.

[14] T. Lindeberg. *Scale-Space Theory in Computer Vision*. The Kluwer International Series in Engineering and Computer Science. Kluwer Academic Publishers, 1994.

[15] J. Serra. *Image Analysis and Mathematical Morphology*. Academic Press, London, 1982.

[16] J. Serra and P. Soille, editors. *Mathematical Morphology and its Applications to Image Processing*, volume 2 of *Computational Imaging and Vision Series*. Kluwer Academic Publishers, Dordrecht, 1994.

[17] J. Smoller. *Shock-Waves and Reaction Diffusion Equations*. Springer-Verlag, New York, 1983.

[18] J. Sporring, M. Nielsen, L. M. J. Florack, and P. Johansen, editors. *Gaussian Scale-Space Theory*, volume 8 of *Computational Imaging and Vision Series*. Kluwer Academic Publishers, Dordrecht, 1997.