Polyhedral combinatorics: An annotated bibliography*

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Polyhedral combinatorics is the study of the integer programming polyhedron

$$P = \mathrm{conv}(X) = \mathrm{conv}(\{x \in X\}),$$

where X is given as a subset of the integer lattice $\mathbb{Z}^n$ and conv denotes the convex hull operator. By Weyl's Theorem

H. Weyl (1935). Elementare Theorie der konvexen Polyeder. *Comentarii Mathematici Helvetici* 7, 290–306,

there exists a matrix $B \in \mathbb{R}^{m \times n}$ and a vector of right hand sides $b \in \mathbb{R}^m$ such that

$$P = \{x \in \mathbb{R}^n : \ Bx \leq b\}.$$

The system $Bx \leq b$ is said to describe P, and each hyperplane $\{x \in \mathbb{R}^n : \ B_{i\cdot}^T x = b_i\}$ is called a cutting plane. One of the central questions in polyhedral combinatorics is to find the cutting planes that describe P. This question is the subject of this chapter.

We start with a section on books and collections of survey articles that treat polyhedral combinatorics in detail. §2 on integer programming by linear programming discusses general schemes by which all cutting planes are generated. We discuss the separation problem, the concepts of total unimodularity and total dual integrality, and give a reference to the computational complexity of deriving an explicit description of conv(X). For NP-hard problems, such as the knapsack, covering, packing, partitioning or mixed integer flow problems, one cannot expect to derive an explicit description of P. Then it is of interest to describe the associated polyhedra partially. Some articles on this issue are listed in §3.

Our policy in selecting references has been as follows. We have chosen books that give a modern account of polyhedral combinatorics. The purpose of §2 is to review the most important theoretical results. When selecting problems for §3 we chose basic combinatorial structures that form substructures of a large collection of combinatorial optimization problems. Some prominent problems of this type are treated in separate chapters of this book, such as the traveling salesman problem, and the maximum cut problem, and are therefore not included here.

*This article will appear as Chapter 3 in the book *Annotated Bibliographies in Combinatorial Optimization*, M. Dell'Amico, F. Maffioli, S. Martello (eds.), Wiley, Chichester.

1 Books

In this section we present a selection of books that are often used as references, and that contain an in-depth treatment of polyhedral combinatorics.

A. Schrijver (1986). *Theory of Linear and Integer Programming*, Wiley, Chichester,
is a broad book directed to researchers. It contains much more than polyhedral combinatorics, and is therefore particularly useful as it puts polyhedral combinatorics in the general context of linear and integer programming.

M. Grötschel, L. Lovász, A. Schrijver (1988). *Geometric Algorithms and Combinatorial Optimization*, Springer-Verlag, Berlin,
derives algorithmic versions of results from geometry and number theory, and links them to combinatorial optimization. One of the outstanding results in polyhedral combinatorics, namely that the separation problem and the optimization problem for a family of polyhedra are polynomially equivalent, is discussed extensively.

G.L. Nemhauser, L.A. Wolsey (1988). *Integer and Combinatorial Optimization*, Wiley, New York,
treats all aspects of polyhedral combinatorics. Next to the general theory, it also gives examples of problem-specific results, both with respect to families of strong valid inequalities, and separation. The book

M. Padberg (1995). *Linear Optimization and Extensions*, Springer, Berlin,
has a comprehensive chapter on the theory of polyhedra. The book discusses all central issues in polyhedral combinatorics, and the links between optimization and separation.

The following books contain selections of survey papers related to polyhedral combinatorics:

W. Cook, P.D. Seymour (eds.) (1990). *Polyhedral Combinatorics.* DIMACS Series in Discrete Mathematics and Theoretical Computer Science 1, AMS, Providence, ACM, New York.
R.L. Graham, M. Grötschel, L. Lovàsz (eds.) (1995). *Handbook of Combinatorics*, Vol II, North-Holland, Amsterdam.

Even though the central theme of the following books is not polyhedral combinatorics, we still want to mention them as they give considerable insight in the study of polyhedra.

E.L. Lawler (1976). *Combinatorial Optimization: Networks and Matroids*, Holt, Rinehart and Winston, New York.
L. Lovász, M.D. Plummer (1986). *Matching Theory*, Akadémiai Kiadó, Budapest.
K. Trümper (1992). *Matroid Decomposition*, Academic Press, San Diego.
R.K. Ahuja, T.L. Magnanti, J.B. Orlin (1993). *Network Flows*, Prentice-Hall, New Jersey.
G.M. Ziegler (1995). *Lectures on Polytopes*, Springer-Verlag, New York.

2 Integer Programming by Linear Programming

If a linear description of $\text{conv}(X)$ is known, then one can solve the problem $\min\{c^T x : x \in X\}$ by linear programming techniques, which is computationally easy.

There is one special case where, for every integral vector $b \in \mathbb{R}^n$, the integrality of the polyhedron $\{x \in \mathbb{R}^n : Ax \leq b\}$ is guaranteed. This situation arises when the matrix A is totally unimodular, i.e., each subdeterminant of A is either -1, 0 or 1. Within the last 40 years a deep theory on totally unimodular matrices has emerged that we cannot discuss here. The interested reader is referred to the books of Schrijver and Trümper listed in §1 for references and surveys on this subject.

If the constraint matrix A is not totally unimodular, then the integrality of the linear programming relaxation is quite rare. However, a linear description of the convex hull of all the feasible integer points of the problem can always be constructed. This topic is discussed next.

For a set $X = \{x \in \mathbb{Z}^n_+ : Ax \leq b\}$, let $\text{conv}(X)$ be the polyhedron defined as the convex hull of all the points in X, and let $\{x \in \mathbb{R}^n_+ : \ Ax \leq b\}$ be its linear programming relaxation. If the constraint matrix A, and the vector of right-hand sides b are integral, and if the set X is bounded, then there exists an implementation of Gomory's cutting plane algorithm, such that for every objective function the procedure terminates after a finite number of iterations with an integral optimum solution.

R.E. Gomory (1958). Outline of an algorithm for integer solutions to linear programs. *Bull. American Math. Soc.* 64, 275–278.

If the coefficients of A and b are real numbers, and if the feasible set is bounded, then Chvátal's rounding scheme will produce $\text{conv}(X)$ after a finite number of iterations.

V. Chvátal (1973). Edmonds polytopes and a hierarchy of combinatorial problems. *Discr. Math.* 4, 185–224¿

An elegant way to formulate, and even generalize Chvátal's result was presented by Schrijver,

A. Schrijver (1980). On cutting planes. M. Deza, I.G. Rosenberg (eds.). *Combinatorics 79 Part II*, Ann. Discr. Math. 9, North-Holland, Amsterdam, 291–296.

Schrijver considered the case where the set of feasible solutions is not necessarily bounded, and where the entries of A and b are rational numbers. In each step of the algorithm he derives a system of linear inequalities $Bx \leq d$ that is totally dual integral (TDI), and where all entries of B are integral, and rounds down the elements of the vector d. A rational system $Bx \leq d$ of linear inequalities is called TDI if for each integral vector c such that $\min\{y^T d : \ y \in \mathbb{R}^n_+, \ y^T B = c\}$ is finite, the minimum is attained by an integral vector. In comparison to Gomory's procedure the step of adding up linear combinations of current inequalities and rounding down the left hand sides becomes redundant if one resorts to a TDI representation of the current polyhedron. The notion of TDI-ness was introduced in

J. Edmonds, R. Giles (1977). A min-max relation for submodular functions on graphs. P.L. Hammer et al. (eds.). *Studies in Integer Programming*, Ann. Discr. Math. 1, North-Holland, Amsterdam, 185–204.

It was proved in

R. Giles, W.R. Pulleyblank (1979). Total dual integrality and integer polyhedra. *Linear Algebra and Appl.* 25, 191–196,

that for every rational polyhedron there exists a TDI-system. This system is unique, if the rational polyhedron is full-dimensional, see

A. Schrijver (1981). On total dual integrality. *Linear Algebra and Appl.* 38, 27–32.

This and many more beautiful results, such as a relation of TDI-ness to Hilbert bases that allows one to derive an integer analogue of Carathéodory's Theorem, can be found in the book of Schrijver, see §1, and, for instance, in the following articles:

R. Chandrasekaran (1981). Polynomial algorithms for totally dual integral systems and extensions. P. Hansen (ed.). *Studies on Graphs and Discrete Programming*, Ann. Discrete Math. 11, 39–51.
W. Cook (1983). Operations that preserve total dual integrality. *Oper. Res. Lett* 2, 31–35.
J. Edmonds, R. Giles (1984). Total dual integrality of linear inequality systems. W.R. Pulleyblank (ed.). *Progress in Combinatorial Optimization*, Academic Press, New York, 117–129.
W. Cook (1986). On box totally dual integral polyhedra. *Math. Program.* 34, 48–61.
W. Cook, J. Fonlupt, A. Schrijver (1986). An integer analogue of Carathéodory's theorem. *J. Combin. Theory B* 40, 63–70.

Related to the question of describing conv(X) by a system of linear inequalities is the study of the so-called *corner polyhedra* that builds a bridge between linear programming and the group problem in integer programming. The reference introducing this topic is

R. Gomory (1969). Some polyhedra related to combinatorial problems. *Linear Algebra and Appl.* 2, 451–558.

If we restrict the variables to take values zero or one only, then there is an alternative procedure for generating the convex hull of all 0/1-vectors satisfying $Ax - b \geq 0$ with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. The basic idea was introduced by Balas, who developed the technique of disjunctive programming. References regarding disjunctive programming techniques are:

E. Balas (1975). Disjunctive programming: Cutting planes from logical conditions. O.L. Mangasarian et al. (eds.) *Nonlinear Programming 2*, Academic Press, 279–312.
C.E. Blair (1976). Two rules for deducing valid inequalities for 0-1 problems. *SIAM J. Appl. Math.* 31, 614–617.
R.G. Jeroslow (1977). Cutting plane theory: Disjunctive methods. *Ann. Discr. Math.* 1, 293–330.
E. Balas (1979). Disjunctive programming. *Ann. Discr. Math.* 5, North-Holland, Amsterdam, 3–51.

The following two references deals with generating the convex hull of a 0-1 integer program by disjunctive programming techniques:

H. Sherali, W. Adams (1990). A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems. *SIAM J. Discr. Math.* 3, 411–430.
L. Lovász, A. Schrijver (1991). Cones of matrices and set-functions and 0-1 optimization. *SIAM J. Optim.* 1, 166–190.

The idea of Lovász and Schrijver can be described as follows. Every constraint $a^T x - \beta \geq 0$ in the current system of inequalities is multiplied by every constraint $(1 - x_j) \geq 0$ and $x_j \geq 0$ for $j = 1, ..., n$, where n is the number of variables in the original formulation. This gives rise to a linear formulation in n^2 variables and $2mn$ constraints under the substitution $x_i x_j := y_{ij}$, $y_{ij} := y_{ji}$ and $x_i^2 := x_i$. The resulting polyhedron is projected down onto the original space of the x-variables. Lovász and Schrijver showed that this process needs to be iterated at most n times before the convex hull of feasible solutions is obtained. In fact, one can also show that it is sufficient to multiply each constraint by one single variable x_j and its complement at a time. This way, the inequality system in the lifted space consists of at most $2n$ variables and $2m$ constraints. This result can be found in the paper by

E. Balas, S. Ceria, G. Cornuéjols (1993). A lift-and-project cutting plane algorithm fo mixed 0-1 programs. *Math. Program.* 58, 295–324.

One of the fundamental results in polyhedral combinatorics is the equivalence between the optimization problem $\min\{c^T x : x \in X\}$ and the separation problem for the polyhedron $\text{conv}(X)$ in terms of computational complexity. The latter problem is to find a hyperplane separating a given point x^* from the polyhedron $\text{conv}(X)$, or to assert that no such hyperplane exists. This result is a theoretical justification for the use of cutting plane algorithms in linear integer programming. It can be found in

M. Grötschel, L. Lovász, A. Schrijver (1981). The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica* 1, 169–197 [Corrigendum: 4, 291–295].

Often one is interested in a solution of the separation problem for a specific family $\mathcal{F}$ of inequalities. This is the problem of finding an inequality in $\mathcal{F}$ that violates x^*, or asserting that no separating hyperplane in this family exists. For certain families of inequalities this problem is sometimes solvable in polynomial time, although the optimization problem for which they are valid is NP-hard, and the number of inequalities in the family is exponential in the encoding length of the optimization problem. This is one of the explanations behind the computational success of polyhedral techniques, see Chapter 4 for further details. An important example of this kind is the separation problem for the family of 2-matching constraints that is valid for the traveling salesman polytope. This family was originally invented by Edmonds for the 2-matching polytope, and gives together with the defining constraints an explicit description of the convex hull of 2-matchings. The fact that the separation problem for the family of 2-matching constraints can be solved in polynomial time was discovered by Padberg and Rao.

J. Edmonds (1965). Maximum matchings and a polyhedron with 0,1-vertices. *Journal of Research of the National Bureau of Standards (B)* 69, 9–14.
M.W. Padberg, M.R. Rao (1982). Odd minimum cut-sets and b-matchings. *Math. Oper. Res.* 7, 67–80.

The Padberg-Rao algorithm can even be used to solve the following more general problem. Let $A \in \{0,1\}^{m \times n}$ be matrix with at most two 1's per column and $b \in \mathbb{Z}^m$. For every point $y \in \mathbb{R}^n$, the $0-1/2$-cut separation problem $\min\{\lfloor \lambda^T b \rfloor - \lambda^T Ay;\ \lambda \in \{0, \frac{1}{2}\}^m\}$ can be solved in polynomial time. In other words, the minimum here is taken over all Chvátal-Gomory cuts with dual multipliers in $\{0, \frac{1}{2}\}$. The result implies that if there exists a Chvátal-Gomory cut with multipliers in $\{0, \frac{1}{2}\}$ that separates y from the polyhedron $\mathrm{conv}(\{x \in \mathbb{Z}^n :\ Ax \leq b\})$, then the most violated inequality in this family can be found in polynomial time. Otherwise it is confirmed by the algorithm that all inequalities in the family are satisfied by y. A result in the same vein was obtained recently by

A. Caprara, M. Fischetti (1993). $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts. *Math. Program.* (to appear).

The paper shows that the $0-1/2$-cut separation problem is polynomially solvable if the constraint matrix modulo 2 is related to the edge-path incidence matrix of a tree. This result is, in particular, applicable to matrices with at most two odd entries for each row, or at most two odd entries for each column.

There is an important technique that can be used to increase the dimension of a face induced by an inequality that is valid for the polyhedron $P_S = \mathrm{conv}(\{x \in \mathbb{Z}^n_+ :\ Ax \leq b\} \cap \{x_j = \xi_j,\ j \in S\})$ where S is a proper subset of $\{1, \ldots, n\}$. For ease of explanation we assume that $\xi_j = 0$ for all $j \in S$. For $a \geq 0$, $\beta > 0$, let $a^T x \leq \beta$ be an inequality that is valid for P_S such that the dimension of the face $\{x \in P_S :\ a^T x = \beta\}$ is equal to t. At each iteration of the so-called *sequential lifting* technique we choose a variable x_k, $k \in S$, set $S := S \setminus \{k\}$, and compute a coefficient $\gamma \geq 0$ such that $a^T x + \gamma x_j \leq \beta$ is valid for P_S. Let γ_0 be the maximum value of γ such that the inequality is valid for P_S. For any choice of $\gamma \leq \gamma_0$ the resulting inequality is valid, and if we choose $\gamma = \gamma_0$, then the face induced by $\{x \in P_S :\ a^T x + \gamma x_j = \beta\}$ has dimension $t+1$. Sequential lifting was first applied by Padberg to the vertex packing problem, see §3.2. As a general procedure it was presented in

M.W. Padberg (1975). A note on zero-one programming. *Oper. Res.* 23, 833–937.

A generalization of this procedure can be found in

L.A. Wolsey (1976). Facets and strong valid inequalities for integer programs. *Oper. Res.* 24, 367–372.

Zemel developed a more general technique called *simultaneous lifting*,

E. Zemel (1978). Lifting the facets of zero-one polytopes. *Math. Program.* 15, 268–277.

Here any subset of the variables in S can be lifted simultaneously, yielding inequalities that in general cannot be obtained by lifting the variables sequentially.

In this chapter we have indicated that it is possible, in principle, to describe the polyhedron conv(X) by means of linear inequalities. The descriptions are, however, implicit and can only be constructed in an iterative fashion. For quite a few polyhedra, such as the matching polyhedron, an explicit description is at hand. A natural question to ask is under which conditions we can expect to derive an explicit description of the convex hull of feasible solutions. The answer was given by

R.M. Karp, C.H. Papadimitriou (1980). On linear characterizations of combinatorial optimization problems. *Proc. 21st Annual IEEE Symp. Found. Comput. Sci.*, IEEE, New York, 1–9,

who proved that if the optimization problem under consideration is NP-hard, then one cannot find an explicit description of the convex hull of feasible solutions, unless NP=co-NP. More precisely, if a certain optimization problem is NP-hard, e.g., the traveling salesman problem, and if the problem to decide whether a valid inequality defines a facet for the associated class of polyhedra is in NP, then this would imply that NP=co-NP. If NP=co-NP, then there exists a compact certificate for the no-answer for all problems in NP, which is unlikely. Despite this negative answer, polyhedral techniques can be effective for NP-hard integer programming problems in the sense that we can find good partial descriptions of the convex hull of feasible solutions. This is the topic of the next section.

3 Selected Combinatorial Problems

Here we study polyhedra associated with special NP-hard combinatorial optimization problems. These problems often appear as substructures in more complex optimization problems. Therefore, it is important to analyze the polyhedra corresponding to such special problems in order to understand the polyhedral structure of more complex problems.

3.1 The Knapsack Problem

The knapsack problem is the basic version of a data dependent problem and is defined as follows. For a capacity $a_0 \in \mathbb{Z}_+$ and a set N of items, where each item j has a weight a_j and a profit c_j, the knapsack problem is the problem of finding a subset of items, with total weight less than or equal to the capacity a_0, that maximizes the total profit. Since a slight change of the weights of the items might drastically change the inequalities that describe the polyhedron, it seems important to understand principles by which valid inequalities are constructed.

Most of the polyhedral studies presented so far involve the basic object of minimal covers, see for instance

L.A. Wolsey (1975). Faces of linear inequalities in 0-1 variables. *Math. Program.* 8, 165–178.

A subset $S \subseteq N$ is a *cover* (or dependent set) if its weight exceeds the capacity. With the cover S one can associate the *cover inequality* $\sum_{j \in S} x_j \leq |S| - 1$ that is valid for the knapsack polyhedron. If the cover S is minimal with respect to inclusion, the associated inequality is called a *minimal cover inequality*. An interesting question is to characterize weight vectors $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ for which the minimal cover inequalities describe the knapsack polyhedron. This question was addressed by

M. Laurent and A. Sassano (1992). A characterization of knapsacks with the max-flow-min-cut property. *Oper. Res. Lett.* 11, 105–110.

They showed that n minimal cover inequalities suffice to describe the knapsack polytope when $a = (a_1, \ldots, a_n)$ is a weakly superincreasing sequence, i.e., $\sum_{j \geq q} a_j \leq a_{q-1}$, for all $q = 2, \ldots, n$.

A slightly more general object than minimal covers are $(1,k)$-configurations that were introduced by

M.W. Padberg (1980). (1,k)-configurations and facets for packing problems. *Math. Program.* 18, 94–99.

A $(1,k)$-*configuration* consists of an independent set S plus one additional item, z say, such that every subset S of cardinality k, together with z, forms a minimal cover. A $(1,k)$-configuration gives rise to the inequality

$$\sum_{j \in S} x_j + (|S| - k + 1)x_z \leq |S|.$$

Padberg showed that if the set N of items defines a $(1,k)$-configuration, then the convex hull of the associated knapsack polyhedron is given by the lower and upper bound constraints and the set of all $(1,l)$-configuration inequalities defined by $T \subseteq S$, where $T \cup \{z\}$ is a $(1,l)$-configuration for some $l \leq k$. This result is generalized by

Y. Pochet and R. Weismantel (1994). The sequential knapsack polytope. Preprint SC 94-30, Konrad-Zuse-Zentrum, Berlin,

to knapsack problems where the weights of the items have the divisibility property, i.e., for every pair of weights, the bigger one is an integer multiple of the smaller one. Inequalities derived from both covers and $(1,k)$-configurations are special cases of extended weight inequalities that have been introduced by

R. Weismantel (1994). On the 0/1 knapsack polytope. *Math. Program.* (to appear),

and that describe the knapsack polyhedron when $a_j = 1$, or $a_j \geq \lfloor a_0/2 \rfloor + 1$, for all $j \in N$. It was also shown in this paper that, independent of the lifting sequence, the lifting coefficient of a variable, x_j, in the extended weight inequality is either equal to the value α_j that this variable would obtain if it was the first one in the sequence, or it equals $\alpha_j - 1$. The correct value of the lifting coefficient for a given sequence can be computed in polynomial time. For cover inequalities these results can be found in the following articles.

E. Balas (1975). Facets of the knapsack polytope. *Math. Program.* 8, 146–164.
E. Zemel (1989). Easily computable facets of the knapsack polytope. *Math. Oper. Res.* 14, 760–764.

For knapsack type problems there are other techniques for lifting lower-dimensional faces of the associated polyhedra. One such techniques is based on an analysis of subadditive functions, see

E.J. Johnson (1980). Subadditive lifting methods for partitioning and knapsack problems. *Journal of Algorithms* 1, 75–96.

We want finisch this subsection by mentioning two successful implementations of generating violated inequalities for the knapsack polytope since they represent major breakthroughs in the use of polyhedral techniques.

H.P. Crowder, E.L. Johnson, M.W. Padberg (1983). Solving large-scale zero-one linear programming problems. *Oper. Res.* 31, 803–834.
T.J. Van Roy, L.A. Wolsey (1987) Solving mixed 0-1 programs by automatic reformulation. *Oper. Res.* 35, 45–57.

For more references on computational aspects and results we refer to Chapter 4.

3.2 Packing, Covering and Partitioning Problems

For a matrix $A \in \mathbb{R}_+^{m \times n}$ and a vector $b \in \mathbb{R}_+^m$ a packing problem is of the form

$$\max\{c^T x : \ Ax \leq b, \ x \in \mathbb{Z}_+^n\},$$

where c is a m-dimensional vector of real coefficients. Replacing the $\leq$-sign by the $\geq$-sign and the max-operation by the min-operation we speak of a covering problem. A partitioning problem is of the form $\min\{c^T x : \ Ax = b, x \in \mathbb{Z}_+^n\}$.

From the theory of linear programming it follows that the dual of the linear programming relaxation of the packing problem is a covering problem. This allows us to derive min-max results for the linear programming relaxations of covering and packing models. In case that b is the m-dimensional vector of all ones, $\mathbf{1}$, we can also use the concept of blocking and anti-blocking polyhedra to derive min-max results for the linear programming relaxations of the packing and covering models. This theory was introduced by

D.R. Fulkerson (1971). Blocking and anti-blocking pairs of polyhedra. *Math. Program.* 1, 168–194.

If $\{x \in \mathbb{R}_+^n : Ax \geq \mathbf{1}\} = \text{conv}\{d^1, \ldots, d^t\} + \mathbb{R}_+^n := \{x + y : x \in \text{conv}\{d^1, \ldots, d^t\}, y \in R_n^+\}$, then the blocking polyhedron is equal to $\{x \in \mathbb{R}_+^n : Dx \geq \mathbf{1}\} = \text{conv}\{a^1, \ldots, a^m\} + \mathbb{R}_+^n$, where A is the matrix with rows $a^1, \ldots a^m$ and D is the matrix with rows $d^1, \ldots, d^t$. One application of the theory of blocking and anti-blocking polyhedra is an elegant proof of the max-flow-min-cut Theorem of Ford and Fulkerson. For further details on blocking and anti-blocking polyhedra we refer to the survey of Schrijver in the Handbook of Combinatorics, see §1. Packing and covering models have been extensively surveyed by

A. Schrijver (1983). Min-max results in combinatorial optimization. A Bachem, M. Grötschel, B. Korte (eds.). *Mathematical Programming: The State of the Art*, Springer, Berlin, 439–500.

Often min-max results for discrete packing and covering models can be derived for totally dual integral systems. Examples are integrality results for crossing families defined on the set of vertices of a digraph and the blocking collection of covers of the crossing family. A comprehensive survey on results in this spirit can be found in

A. Schrijver (1984). Total dual integrality from directed graphs, crossing families, and sub- and supermodular functions. W.R. Pulleyblank (ed.). *Progress in Combinatorial Optimization*, Academic Press, New York, 315–361.

Special, but particularly important cases of packing and covering models arise when the vector b of right hand sides is equal to $\mathbf{1}$, when the coefficients of the matrix A are either 0 or 1, and when the variables are binary. Then one speaks of set packing, set covering, and set partitioning problems, respectively. For the associated polyhedra, explicit descriptions are sometimes known. In particular, if the 0-1 matrix A is *balanced*, i.e., if A does not contain a square submatrix of odd order with two ones per row or columns, then, the set packing as well as the the set covering polyhedra are integral.

C. Berge (1972). Balanced matrices. *Math. Program.* 2, 19–31.

The notion of balancedness was also introduced by Berge.

C. Berge (1970). Sur certains hypergraphes généralisant les graphes bipartites. P. Erdös, A. Rényi, V.T. Sós (eds.). *Combinatorial Theory and Its Applications I*, Colloquia Mathematica Societatis János Bolyai, Vol. 4, North-Holland, Amsterdam, 119–133.

For more details on balanced matrices we refer to Chapter 11, and to the following articles:

M. Conforti, G. Cornuéjols, A. Kapoor, M. R. Rao and K. Vušković (1994). Balanced matrices. J.R. Birge, K.G. Murty (eds.). *Mathematical Programming: State of the Art 1994*, University of Michigan, 1–33.
M. Conforti, G. Cornuéjols, A. Kapoor and K. Vušković (1994). Recognizing balanced 0 ± 1 matrices, *Proc. 5-th Annual ACM-SIAM Symp. Discr. Algorithms*, ACM, New York, SIAM, Philadelphia, 103–111.

Every set packing problem can be interpreted as the problem of finding a maximum stable set in the graph whose nodes correspond to the columns of the matrix and whose edges represent the pairs of columns with intersecting support. The stable set polyhedron has been studied extensively in the literature during the last 20 years starting with the work of Padberg,

M. Padberg (1973). On the facial structure of set packing polyhedra. *Math. Program.* 5, 199–215.

In this paper clique inequalities of the form $\sum_{j \in C} x_j \leq 1$, where C is the node set of a complete subgraph of the given graph, have been introduced. Odd circuit constraints $\sum_{j \in C} x_j \leq \frac{|C|-1}{2}$, with C being a subset of the set of nodes of odd cardinality whose induced subgraph is a cycle without chords, as well as the constraints based on the complements of odd circuits, can be found in this reference, too. Odd circuit constraints can be separated in polynomial time by adapting the odd cycle separation algorithm for the max-cut problem that was introduced by

F. Barahona, A.R. Mahjoub. On the cut polytope. *Math. Program.* 36, 157–173,

see also Grötschel et al. (1988) listed in §1. The graphs for which the set of all odd circuit constraints, the lower bound inequalities, and the $x_i + x_j \leq 1$-constraints for all edges $\{i, j\}$ in the graph suffice to describe the set packing polyhedron, are called t-perfect. This notion was introduced by

V. Chvátal (1975). On certain polytopes associated with graphs. *Journal of Combinatorial Theory B* 18, 305–337.

Similarly, one can ask for the integrality of the polyhedron that is defined by all clique constraints together with the lower bound inequalities and the $x_i + x_j \leq 1$-constraints for all edges $\{i, j\}$ in the graph. Such graphs are called perfect, and are discussed in Chapter 11.

For a thorough treatment of the mathematics that is underlying the constraint generation for the set covering polyhedron we refer the readers to, for instance, the following two papers:

A. Sassano (1989). On the facial structure of the set-covering polytope. *Math. Program.* 44, 181–202.
E. Balas, S.M.Ng (1989). On the set covering polytope I: All the facets with coefficients in $\{0, 1, 2\}$. *Math. Program.* 43, 1–20.

Both the set packing and the set covering polyhedron are equivalent to special cases of independence system polyhedra that have been studied by

R. Euler, M. Jünger, G. Reinelt (1987). Generalizations of odd cycles and anticycles and their relation to independence system polyhedra. *Math. Oper. Res.* 12, 451–462.
M. Laurent (1989). A generalization of antiwebs to independence systems and their canonical facets. *Math. Program.* 45, 97–108.

Polyhedra associated with independence systems are included in the family of transitive packing polyhedra introduced by

A.S. Schulz, R. Müller (1996). Transitive packing. *Springer Lecture Notes in Computer Science*, Proceedings of the 5th International Conference on Integer Programming and Combinatorial Optimization (to appear).

This paper describes a common frame for valid inequalities induced by graphic structures such as cliques, odd cycles, odd anticycles, webs, antiwebs etc. It even generalizes polyhedral results for certain graph partitioning problems. A collection of papers on this subject is given below.

E. Balas, M. Padberg (1976). Set partitioning: A survey. *SIAM Review* 18, 710–760.
M. Grötschel, Y. Wakabayashi (1990). Facets for the clique partitioning polytope. *Math. Program.* 47, 367–388.
S. Chopra, M.R. Rao (1993). The partition problem. *Math. Program.* 59, 87–115.
C.E. Ferreira, A. Martin, C. de Souza, R. Weismantel, L. Wolsey (1994). Formulations and valid inequalities for the node capacitated graph partitioning problem. *Math. Program.* (to appear).

3.3 Mixed Integer Flow Problems

When considering any kind of flow problem in integer programming, there is one fundamental reference.

L.R. Ford, Jr., D.R. Fulkerson (1956). Maximal flow through a network. *Canadian J. Math.* 8, 399–404.

This is the paper containing the well-known max-flow-min-cut Theorem. There are many extensions of this result that have been formulated in the setting of multicommodity flows including duality-results for the problem of packing paths and cuts in graphs under capacity restrictions. In order to cover this and related topics in detail, it would require a book on its own. We refer here to the excellent survey articles contained in

B. Korte, L. Lovász, H.J. Prömel, A. Schrijver (eds.) (1990). *Paths, Flows, and VLSI-Layout*, Springer, Berlin.

Another comprehensive survey can be found in

M.V. Lomonosov (1985). Combinatorial approaches to multiflow problems. *Discr. Appl. Math.* 11, 1 – 94.

Each flow in a graph can be decomposed into paths. Since a subgraph that is induced by a path is node- and edge-connected, a flow can be viewed as a special graph structure that requires connectivity. Besides paths, there are further connectivity structures in graphs that have become important. Consider an undirected graph $G = (V, E)$ and a subset T of V. A Steiner tree in G is a subgraph that spans T and possibly vertices in $V \setminus T$. Polyhedral results regarding various versions of the Steiner tree problem can be found in

M.X. Goemans (1994). The Steiner tree polytope and related polyhedra. *Math. Program.* 63, 157–182.
S. Chopra and M.R. Rao (1994). The Steiner tree problem I: Formulations, compositions and extensions of facets. *Math. Program.* 64, 209–229.
S. Chopra and M.R. Rao (1994). The Steiner tree problem II: Properties and classes of facets. *Math. Program.* 64, 231–246.

In the paper by Goemans a characterization of the convex hull of all incidence vectors of Steiner trees (in the space of the number of nodes plus the number of edges) is given when the underlying graph is series parallel. Chopra and Rao use a directed formulation for the problem of finding a minimum weighted Steiner tree in a graph with weights on the edges. They show that the linear relaxation of the directed formulation is stronger than the linear relaxation of the undirected one. This result is obtained by projecting the polyhedron associated with the directed formulation onto the subspace defined by the variables associated with the undirected formulation.

Similarly as for flows and multicommodity flows, it is interesting to study the problem of packing Steiner trees under capacity restrictions. We do not want to go into details here, but refer the readers to

M. Grötschel, A. Martin, R. Weismantel (1993). Packing steiner trees: Polyhedral investigations. *Math. Program.* 72, 101–123.

In communication network design it is essential to design networks that are reliable in the sense that a failure at any component of the network does not disconnect important clients. This requirement is taken into account by designing networks that are k-node connected or k-edge connected, where the number k has to be specified by the customers. Recent references on this topic are:

M. Stoer (1992). Design of survivability networks. *Lecture Notes Math.* 1531, Springer, Heidelberg.
M. Grötschel, C.L. Monma, M. Stoer (1995). Polyhedral and computational investigations for designing communication networks with high survivability requirements. *Oper. Res.* 43, 1012–1024.

Until now we have briefly sketched results associated with purely integer connectivity type requirements. There is a lot of ongoing research on mixed integer problems that have a flow structure. The basic form of a mixed integer flow structure yields the so-called single-node flow formulation,

$$X_F = \{(x,y) \in \mathbb{R}^n_+ \times \{0,1\}^n : \sum_{j=1}^{n} x_j = b,\ x_j \le u_j y_j,\ \text{for all } j = 1,\dots,n\},$$

where we have a single node with a fixed outflow b, and a set $N = \{1,\dots,n\}$ of arcs with variable upper bounds entering the node. The associated single-node flow polytope is a relaxation of several polyhedra associated with fixed-charge planning and distribution problems, such as lot-sizing and location problems, see further Chapter 15. Let J be a subset of N such that $\sum_{j\in J} u_j = b + \lambda,\ \lambda > 0$. The set J is called a *flow cover*. Let $(m)^+$ denote $\max\{0,m\}$. The flow cover inequalities $\sum_{j\in J} x_j \le b - \sum_{j\in J}(u_j - \lambda)^+(1-y_j)$ were developed by

M.W. Padberg, T.J. Van Roy, L.A. Wolsey (1985). Valid inequalities for fixed charge problems. *Oper. Res.* 33, 842–861.

One way of extending the flow cover inequalities is to include variables $x_j,\ j \in L \subseteq N \setminus J$ in the inequality. This yields an inequality of the form $\sum_{j\in J\cup L} x_j \le b - \sum_{j\in J}(u_j-\lambda)^+(1-y_j) + \sum_{j\in L}(\bar{u}_j - \lambda)y_j$, where $\bar{u}_l = \max\{\max_{j\in J}\{u_j\}, u_l\}$ for all $l \in L$. Padberg et al. (1985) showed that if $u_j = u$ for all $j \in N$, then $\text{conv}(X_F)$ is described by all the constraints in the mixed integer programming formulation and the family of extended flow cover inequalities. It was observed by

K. Aardal, Y. Pochet, L.A. Wolsey (1995). Capacitated facility location: Valid inequalities and facets. *Math. Oper. Res.* 20, 562–582 [Erratum: 21, 253–256],

that the separation problem for the family of extended flow cover inequalities can be solved in polynomial time when $u_j = u$, for all $j \in N$.

A slightly more complicated model arises when arcs can enter and leave the node. All arcs have variable upper bounds. For this model various versions of generalized flow cover inequalities were developed by

T.J. Van Roy, L.A. Wolsey (1986). Valid inequalities for mixed 0-1 programs. *Discr. Appl. Math.* 14, 199–213.

The authors also discuss the separation problems based on these inequalities. Later, Wolsey (1989) generalized several of the families of inequalities mentioned above by introducing the family of submodular inequalities.

L.A. Wolsey (1989). Submodularity and valid inequalities in capacitated fixed charge networks. *Oper. Res. Lett.* 8, 119–124 [Erratum: 8, 295].

For uncapacitated directed fixed-charge networks a general class of inequalities was developed by

T.J. Van Roy, L.A. Wolsey (1986). Valid inequalities and separation for uncapacitated fixed-charge networks. *Oper. Res. Lett.* 4, 105–112.

The inequalities are based on the idea of using the 0-1 variables when bounding the continuous flow that can pass along a subset of the arcs that form a directed cut in the network. Such inequalities have been particularly useful when solving uncapacitated lot-sizing problems, see

Y. Pochet, L.A. Wolsey (1995). Algorithms and reformulations for lot sizing problems. W. Cook, L. Lovász, P. Seymour (eds.). *Combinatorial Optimization*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science 20, AMS, Providence, ACM, New York, 245–293.

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