Capacitated Facility Location: Separation Algorithms and Computational Experience

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Dedicated to the memory of Eugene L. Lawler

Abstract

We consider the polyhedral approach to solving the capacitated facility location problem. The valid inequalities considered are the knapsack, flow cover, effective capacity, single depot, and combinatorial inequalities. The flow cover, effective capacity, and single depot inequalities form subfamilies of the general family of submodular inequalities.

The separation problem based on the family of submodular inequalities is NP-hard in general. For the well-known subclass of flow cover inequalities, however, we show that if the client set is fixed, and if all capacities are equal, then the separation problem can be solved in polynomial time.

For the flow cover inequalities based on an arbitrary client set, and for the effective capacity and single depot inequalities we develop separation heuristics. An important part of all these heuristic is based on constructive proofs that two specific conditions are necessary for the effective capacity inequalities to be facet defining. The proofs show precisely how structures that violate the two conditions can be modified to produce stronger inequalities.

The family of combinatorial inequalities was originally developed for the uncapacitated facility location problem, but is also valid for the capacitated problem. No computational experience using the combinatorial inequalities has been reported so far. Here we suggest how partial output from the heuristic identifying violated submodular inequalities can be used as input to a heuristic identifying violated combinatorial inequalities.

We report on computational results from solving 60 small and medium size problems.

Key words: Cutting planes; Facets; Location problems.

We study the cutting plane approach to solving the capacitated facility location (CFL) problem. The polyhedral structure of CFL has been studied by Aardal (1992) and Aardal, Pochet and Wolsey (1993), by Leung and Magnanti (1989) for the equal capacity case, and by Deng and Simchi-Levi (1993) for the case of equal capacities and unsplit demands. Cornuéjols, Sridharan and Thizy (1991) compared the strength of various Lagrangean relaxations of CFL, and relaxations obtained by completely dropping different sets of the defining constraints. They also did an extensive computational study using these relaxations and several heuristics proposed in the literature. Very little experience, however, in using strong valid inequalities to solving CFL has been reported on so far, except for some preliminary work by Aardal (1992). For related capacitated network design problems encouraging computational results are obtained by e.g. Bienstock and Günlük (1994), and Magnanti, Mirchandani and Vachani (1995).

In Section 1 we first introduce necessary notation, and a formulation of CFL that is non-standard as it contains redundant aggregate variables and constraints. The purpose of adding aggregate information is to provide the building blocks for two relaxations of CFL. Facet defining inequalities for these relaxations are well-known, and will be generated automatically, given our formulation, by the mixed-integer optimizer used in the computational study.

In Section 2 we describe the family of submodular inequalities for CFL as introduced by Aardal et al., and state conditions for special cases of the submodular inequalities, namely flow cover, effective capacity, and single depot inequalities, to be facet defining. Here we also give proofs that two conditions are necessary for the effective capacity inequalities to be facet defining. The proofs, different from the proof in Aardal et al., are constructive and provides precise ways of strengthening inequalities that violate the specific necessary conditions. The family of combinatorial inequalities, developed by Cornuéjols and Thizy (1922), and Cho et al. (1983a) for the uncapacitated facility location problem, are described briefly in Section 3.

In Section 4 we first show that the separation problem based on the flow cover inequalities for a fixed client set can be solved in polynomial time if all capacities are equal. Next we suggest a general heuristic for finding appropriate subsets of depots and clients, which are then used as input to specialized heuristics for finding violated flow cover, effective capacity and single depot inequalities. An important part of the heuristics for identifying violated effective capacity and single depot inequalities builds on the proofs of the necessary conditions presented in Section 2. We also suggest how the subsets generated by the general heuristic can be used to identify violated combinatorial inequalities.

The separation heuristics developed in Section 4 have been implemented as subroutines of MINTO (Savelsbergh et al. (1994)) and are used, together with the system constraints of MINTO, to solve CFL. In Section 5 we report on computational experience from solving 60 small and medium size test problems. The computational results indicate that the knapsack cover inequalities, which are automatically generated by MINTO from our formulation of CFL, are the single most important ones as they close a substantial part of the duality gap for many instances. In most cases, however, more problem specific inequalities are needed in order to further close the gap.

The heuristics we propose seem effective in generating violated effective capacity and single depot inequalities, and we are able to solve seven problems to optimality at the root node of the branch-and-bound tree, by generating these inequalities in addition to the MINTO system inequalities. For twenty-one of the problems the percentage time reduction was more than 75%, compared to the time needed if we solved the problems by pure branch-and-bound.

1 Problem Formulation

Let $M = \{1,\ldots,m\}$ be the set of facilities (depots) and $N = \{1,\ldots,n\}$ the set of clients. Client k has demand d_k and the capacity of depot j is m_j. The total demand of the clients in the subset $S \subseteq N$ is denoted by $d(S)$. The fixed cost of opening facility j is f_j and the per unit cost of flow between depot j and client k is c_{jk}.

Let $y_j = 1$ if depot j is open and $y_j = 0$ otherwise. The flow from depot j to client k on the arc (j,k) is denoted by v_{jk}. The total flow from depot j to all clients is denoted by v_j. We want to determine which depots should be opened and how the flow should be distributed in order to satisfy all clients demand, as well as the capacity restrictions, at minimum cost. Consider the following formulation of CFL:

$$\min\{\sum_{j\in M}\sum_{k\in N} c_{jk}v_{jk} + \sum_{j\in M} f_j y_j : (v,y) \in X^{CFL}\}$$

where

$$\begin{aligned}
X^{CFL} = \{(v,y) \in \mathbb{R}^{m\times n} \times \mathbb{Z}^m : \sum_{j\in M} v_{jk} &= d_k, \quad k \in N, &(1)\\
v_j &= \textstyle\sum_{k\in N} v_{jk}, \quad j \in M, &(2)\\
v_j &\leq m_j y_j, \quad j \in M, &(3)\\
0 \leq v_{jk} &\leq d_k y_j, \quad j \in M,\ k \in N, &(4)\\
\sum_{j\in M} v_j &= d(N), &(5)\\
y_j &\leq 1, \quad j \in M\}. &(6)
\end{aligned}$$

The standard formulation of CFL does not include the aggregate constraints (2) and (5), and has capacity constraints $\sum_{k\in N} v_{jk} \leq m_j y_j,\ j \in M$ instead of constraints (3). The reason why we include the aggregate information is that by combining appropriate aggregate constraints with constraints (6), we obtain the surrogate knapsack polytope X^K, and the single-node flow polytope X^{SNF},

$$X^K = \{y \in \mathbb{Z}_+^m : \sum_{j\in M} m_j y_j \geq d(N),\ y_j \leq 1,\ j \in M\}, \tag{7}$$

$$X^{SNF} = \{(v,y) \in \mathbb{R}_+^m \times \mathbb{Z}_+^m : \sum_{j\in M} v_j = d(N),\ v_j \leq m_j y_j,\ y_j \leq 1,\ j \in M\}. \tag{8}$$

The knapsack cover inequalities and flow cover inequalities, developed for X^K and X^{SNF} respectively, see Wolsey (1975) and Padberg et al. (1985), are shown to be facet defining for conv(X^{CFL}) by Aardal (1992). Separation heuristics based on these families of inequalities are incorporated in the system routines of the mixed-integer optimizer MINTO, and can therefore be applied directly to formulation (1)–(6). Hence, formulation (1)–(6) is useful for computational purposes, even though it contains redundant information.

2 The Submodular Inequalities and Important Special Cases

Submodular inequalities were first introduced by Wolsey (1989) for general fixed-charge network flow problems, and adapted to CFL by Aardal (1992), and Aardal et al. (1993). Choose a subset

$K \subseteq N$ of clients, and let $J \subseteq M$ be a subset of depots. For each depot $j \in J$ choose a subset $K_j \subseteq K$. Once the sets K, J and K_j for $j \in J$ are known, we can define the *effective capacity* of depot j as $\bar{m}_j = \min(m_j, d(K_j))$.

Definition 1 A set function f defined on $N = \{1, ..., n\}$ is *submodular* on N if

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

for all $A, B \subseteq N$.

Proposition 1 (Aardal et al. (1993).) *The function*

$$\begin{aligned} f(J) = \max\{\sum_{j \in J} \sum_{k \in K_j} v_{jk} : & & & \\ \sum_{k \in K_j} v_{jk} \leq & \quad \bar{m}_j y_j, & & j \in J, \\ \sum_{\{j \in J : K_j \ni k\}} v_{jk} \leq & \quad d_k, & & k \in K, \\ v_{jk} \geq & \quad 0, & & j \in J,\ k \in K, \\ y_j = & \quad 1, & & j \in J, \} \end{aligned} \tag{9}$$

is submodular on M.

The value of $f(J)$ is the maximum flow from the depots in J to the clients in K given the arc set $\{(j,k) : j \in J,\ k \in K_j\}$. The difference in maximum flow when all depots are open, and when all depots except depot j are open, is called the *increment function* and is defined as $\rho_j(J \setminus \{j\}) = f(J) - f(J \setminus \{j\})$.

Proposition 2 (Aardal et al. (1993).) *Let $K \subseteq N$, $J \subseteq M$, and choose for each $j \in J$ a subset $K_j \subseteq K$. The submodular inequality*

$$\sum_{j \in J} \sum_{k \in K_j} v_{jk} + \sum_{j \in J} \rho_j(J \setminus \{j\})(1 - y_j) \leq f(J) \tag{10}$$

is valid for conv (X^{CFL}).

Definition 2 $J \subseteq M$ is a *cover* with respect to N if $\sum_{j \in J} \bar{m}_j = d(N) + \lambda$ with $\lambda > 0$.

For the submodular inequalities to be valid we do not require the set J to be a cover. In all special cases that are studied here, we assume, however, that J is a cover.

The first special case we shall consider is the well-known family of *flow cover* inequalities developed by Padberg et al. (1985) for X^{SNF}. Let $(x)^+ = \max(0, x)$ and let, for a given set $K \subseteq N$, $v'_j = \sum_{k \in K} v_{jk}$, $j \in M$.

Lemma 3 *Let $K \subseteq N$. If J is a cover with respect to K, and if $K_j = K$ for all $j \in J$, then $f(J) = d(K)$ and $\rho_j(J \setminus \{j\}) = (m_j - \lambda)^+$, where $\lambda = \sum_{j \in J} m_j - d(K) > 0$.*

In the flow cover case, the submodular inequalities (10) take the following form:

$$\sum_{j\in J} v'_j + \sum_{j\in J}(m_j - \lambda)^+(1-y_j) \le d(K). \tag{11}$$

Given our formulation of CFL, MINTO will automatically recognize the single-node flow structure (8) and generate flow cover inequalities (11) for $K = N$.

In the general case, where $K_j \subset K$ for at least one $j \in J$, there exists no closed-form expression for $\rho_j(J \setminus \{j\})$, which makes it difficult to characterize facet defining submodular inequalities. By using maximum flow arguments, it is straightforward to show that $\rho_j(J\setminus\{j\}) \ge (\bar{m}_j - \lambda)^+$ for all $j \in J$. Aardal et al. (1993) completely characterized the subclass of facet defining submodular inequalities having $\rho_j(J\setminus\{j\}) = (\bar{m}_j - \lambda)^+$, which are referred to as *effective capacity* inequalities, and two subclasses of submodular facets with $\rho_j(J \setminus \{j\}) > (\bar{m}_j - \lambda)^+$ for at least one $j \in J$, called single depot and multi depot inequalities. We begin by describing the effective capacity inequalities.

Theorem 4 (Aardal et al. (1993).) *Let $J \subset M$ be a cover with respect to K, and let $Q \subset J$ be the subset of depots for which $\bar{m}_q < m_q$. Assume that $\sum_{j\in M} m_j > d(N) + m_r$ for all $r \in J$. The effective capacity inequality*

$$\sum_{j\in J}\sum_{k\in K_j} v_{jk} + (\bar{m}_j - \lambda)^+(1-y_j) \le d(K) \tag{12}$$

defines a facet of $\text{conv}(X^{CFL})$ *if and only if*

- **(1)** *for each pair of depots $q_1,\ q_2 \in Q,\ K_{q_1} \cap K_{q_2} = \emptyset$,*
- **(2)** *$K_j = K$ for all $j \in J \setminus Q$,*
- **(3)** *$(\cup_{q\in Q} K_q) \subset K$,*
- **(4)** *$\bar{m}_q > \lambda$ for all $q \in Q$,*
- **(5)** *if $|Q| \le 1$, then $\exists\ j \in J$ with $\bar{m}_j = m_j > \lambda$.*

Remark: If $Q = \emptyset$ in Theorem 4, we obtain the facet defining *flow cover* inequalities (11).

From the point of view of separation, it is natural to ask what to do in case we generate a structure that violates one or more of conditions (1)-(5) of Theorem 4. Below we give proofs, different from the proofs by Aardal et al., that Conditions (1) and (4) are necessary for the effective capacity inequalities (12) to be facet defining, by showing that an inequality that violates any of these conditions can be strengthened. The way the proof is designed gives direct indications on how the strengthening can be done algorithmically. The strengthening procedure forms an important part of the separation heuristics presented in Section 4.2. To make the first proof more clear, we give an example of how we can tighten an inequality when Condition (1) is violated, i.e., when $K_{q_1} \cap K_{q_2} \neq \emptyset$ for $q_1, q_2 \in Q$.

Example 1 Consider the the CFL structure shown in Figure 1. Let $K = \{1,2,3,4\}$, and let $J = \{1,2,3\}$. Let furthermore $K_1 = \{1,2\}$, $K_2 = \{2,3\}$, and $K_3 = K$. Since $\bar{m}_j < m_j$ for $j = 1,2$ we have $Q = \{1,2\}$. Note that $K_1 \cap K_2 = \{2\}$. The set J defines a cover with respect to K, and $\lambda = 5$. The effective capacity inequality given this structure is

$$v_{11} + v_{12} + v_{22} + v_{23} + v_{31} + v_{32} + v_{33} + v_{34} + 4(1-y_1) + 6(1-y_2) + 4(1-y_3) \le 24. \tag{13}$$

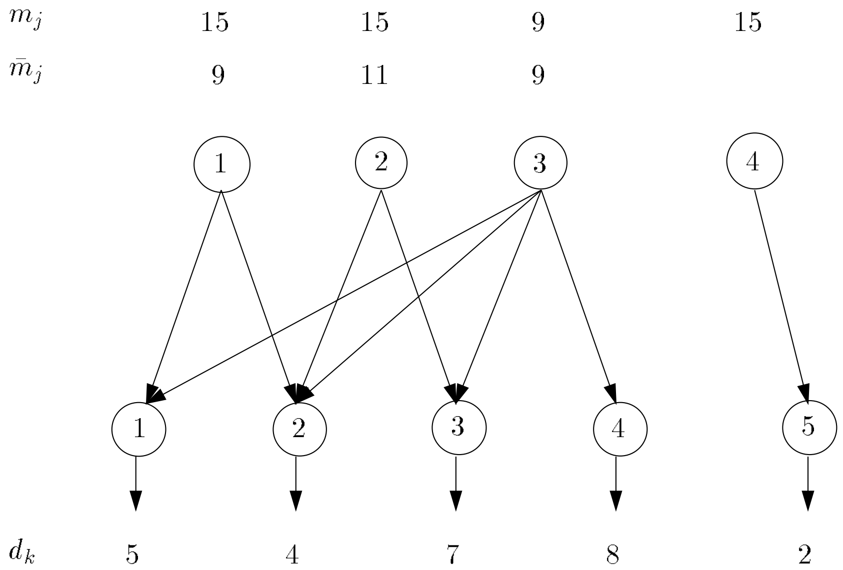

Figure 1: An example of how to strengthen an effective capacity inequality if $K_{q_1} \cap K_{q_1} \neq \emptyset$.

If client 2 is removed from K we get a new client set $K' = \{1,3,4\}$, and $K'_1 = \{1\}$, $K'_2 = \{3\}$, and $K'_3 = K'$. The new excess $\lambda' = 1$. The effective capacity inequality based on sets J, K', and K'_j for all $j \in J$ is

$$v_{11} + v_{23} + v_{31} + v_{33} + v_{34} + 4(1-y_1) + 6(1-y_2) + 8(1-y_3) \leq 20. \tag{14}$$

Adding inequality $\sum_{j \in M} v_{j2} \leq 4$ to inequality (14) gives

$$v_{11} + v_{23} + v_{31} + v_{33} + v_{34} + v_{12} + v_{22} + v_{32} + v_{42} + 4(1-y_1) + 6(1-y_2) + 8(1-y_3) \leq 24, \tag{15}$$

which is stronger than inequality (13) since variable v_{42} has been added and since the coefficient of $(1-y_3)$ has increased from four to eight. ■

Proof of necessity of Condition (1).

We will show that if all clients belonging to any intersection of sets K_q are removed, and the simple inequality $\sum_{j \in M} \sum_{k \in K^\cap} v_{jk} \leq d(K^\cap)$ is added, where $K^\cap$ is the set of removed clients, then a stronger inequality is obtained. Let

$$\begin{aligned} K^\cap &= \bigcup_{q \in Q} (\cup_{j \in Q \setminus \{q\}} (K_j \cap K_q)), \\ Q^\cap &= \{j \in Q : K_j \cap K^\cap \neq \emptyset\}, \\ K' &= K \setminus K^\cap, \\ K'_j &= K_j \setminus K^\cap, \; j \in J. \end{aligned}$$

The values of the effective capacities, given the new sets K', and K'_j for all $j \in J$, are:

$$\bar{m}'_j = \min(m_j, d(K')), \; j \in J \setminus Q,$$

$$\bar{m}'_j = \bar{m}_j, \; j \in Q \setminus Q^\cap,$$
$$\bar{m}'_j = \bar{m}_j - \sum_{k \in K_j \cap K^\cap} d_k, \; j \in Q^\cap,$$

giving a new value of the excess capacity, $\lambda' = \sum_{j \in J} \bar{m}'_j - d(K')$.

Assume that $\sum_{j \in J \setminus Q} m_j > d(K \setminus (\cup_{j \in Q} K_j))$, in which case $\lambda' > 0$. If $\sum_{j \in J \setminus Q} m_j \leq d(K \setminus (\cup_{j \in Q} K_j))$, no tight point with $y_j = 1$ for all $j \in \{j \in J : \bar{m}_j > \lambda\}$ exists.

We now show that the inequality

$$\sum_{j \in J} \sum_{k \in K'_j} v_{jk} + \sum_{j \in M} \sum_{k \in K^\cap} v_{jk} + \sum_{j \in J} (\bar{m}'_j - \lambda')^+ (1 - y_j) \leq d(K') + d(K^\cap) \tag{16}$$

is stronger than the effective capacity inequality based on the sets J, K and K_j.

Since $\lambda' > 0$, inequality (16) is valid. (In general, if the excess λ is negative, then the coefficient $(\bar{m}_j - \lambda)^+$ will overestimate the amount of flow that is lost if depot j is closed, and hence the inequality will not be valid.) The right-hand sides of both inequality (16) and the effective capacity inequality based on sets J, K and K_j are equal, since $d(K') + d(K^\cap) = d(K)$. All coefficients of the variables v_{jk} are equal in the two inequalities except the coefficients of v_{jk}, $j \in Q \setminus Q^\cap$, $k \in K^\cap$ and $j \in M \setminus J$, $k \in K^\cap$, which have value one in inequality (16) and zero in the effective capacity inequality based on sets J, K and K_j, $j \in J$. Hence, what remains to show is that $(\bar{m}'_j - \lambda')^+ \geq (\bar{m}_j - \lambda)^+$ for all $j \in J$.

Case 1: $j \in J \setminus Q$

$\bar{m}'_j - \lambda' = \min(m_j, d(K')) - \sum_{l \in J} \bar{m}'_l + d(K') = \min(m_j, d(K')) -$
$\sum_{l \in J \setminus Q} \min(m_l, d(K')) - \sum_{l \in Q \setminus Q^\cap} \bar{m}_l - \sum_{l \in Q^\cap} (\bar{m}_l - \sum_{k \in K_l \cap K^\cap} d_k) + d(K) - d(K^\cap) =$
$- \sum_{l \in (J \setminus Q) \setminus \{j\}} \min(m_l, d(K')) - \sum_{l \in Q \setminus Q^\cap} \bar{m}_l - \sum_{l \in Q^\cap} (\bar{m}_l - \sum_{k \in K_l \cap K^\cap} d_k) + \sum_{l \in J} \bar{m}_l - \lambda - d(K^\cap) =$
$- \sum_{l \in (J \setminus Q) \setminus \{j\}} \min(m_l, d(K')) + \sum_{l \in J \setminus Q} \bar{m}_l + \sum_{l \in Q^\cap} \sum_{k \in K_l \cap K^\cap} d_k - \lambda - d(K^\cap) =$
$\bar{m}_j - \lambda + \sum_{l \in (J \setminus Q) \setminus \{j\}} (\bar{m}_l - \min(m_l, d(K'))) + \sum_{l \in Q^\cap} \sum_{k \in K_l \cap K^\cap} d_k - d(K^\cap) \geq \bar{m}_j - \lambda,$

where the inequality follows from the term $\sum_{l \in (J \setminus Q) \setminus \{j\}} (\bar{m}_l - \min(m_l, d(K')))$ being nonnegative, and from $\sum_{l \in Q^\cap} \sum_{k \in K_l \cap K^\cap} d_k \geq d(K^\cap)$.

Case 2: $j \in Q \setminus Q^\cap$

$\bar{m}'_j - \lambda' = \bar{m}_j - \sum_{l \in J} \bar{m}'_l + d(K') =$
$\bar{m}_j - \sum_{l \in J \setminus Q} \min(m_l, d(K')) - \sum_{l \in Q \setminus Q^\cap} \bar{m}_l - \sum_{l \in Q^\cap} (\bar{m}_l - \sum_{k \in K_l \cap K^\cap} d_k) + d(K) - d(K^\cap) =$
$\bar{m}_j - \sum_{l \in J \setminus Q} \min(m_l, d(K')) - \sum_{l \in Q \setminus Q^\cap} \bar{m}_l - \sum_{l \in Q^\cap} (\bar{m}_l - \sum_{k \in K_l \cap K^\cap} d_k) + \sum_{l \in J} \bar{m}_l -$
$\lambda - d(K^\cap) =$
$\bar{m}_j - \lambda + \sum_{l \in J \setminus Q} (\bar{m}_l - \min(m_l, d(K'))) + \sum_{l \in Q^\cap} \sum_{k \in K_l \cap K^\cap} d_k - d(K^\cap) \geq \bar{m}_j - \lambda.$

Case 3: $j \in Q^\cap$

$\bar{m}'_j - \lambda' = \bar{m}_j - \sum_{k \in K_j \cap K^\cap} d_k - \sum_{l \in J} \bar{m}'_l + d(K') =$
$\bar{m}_j - \sum_{k \in K_j \cap K^\cap} d_k - \sum_{l \in J \setminus Q} \min(m_l, d(K')) - \sum_{l \in Q \setminus Q^\cap} \bar{m}_l - \sum_{l \in Q^\cap} (\bar{m}_l - \sum_{k \in K_l \cap K^\cap} d_k) + d(K) -$
$d(K^\cap) =$

$\bar{m}_j - \sum_{k\in K_j\cap K^\cap} d_k - \sum_{l\in J\setminus Q} \min(m_l, d(K')) - \sum_{l\in Q\setminus Q^\cap} \bar{m}_l - \sum_{l\in Q^\cap}(\bar{m}_l - \sum_{k\in K_l\cap K^\cap} d_k) + \sum_{l\in J} \bar{m}_l -$
$\lambda - d(K^\cap) =$
$\bar{m}_j - \lambda + \sum_{l\in J\setminus Q}(\bar{m}_l - \min(m_l, d(K')) + \sum_{j\in Q\setminus\{j\}} \sum_{k\in K_l\cap K^\cap} d_k - d(K^\cap) \geq \bar{m}_j - \lambda.$ ■

Remark: Consider an effective capacity inequality based on sets J, K, and K_j for all $j \in J$. If only two client sets K_{q_1} and K_{q_2}, $q_1, q_2 \in Q$ have a nonempty intersection, and if Conditions (2)-(5) of Theorem 4 hold, then the effective capacity inequality based on the depot set J, and the client sets K', and K'_j for all $j \in J$, as defined in the proof of necessity of Condition (1), defines a facet. This is true since Condition (1) that was previously violated now holds, and since the modification of the client set does not cause other conditions to become violated. By referring to Example 1, we see that inequality (14) defines a facet.

Outline of the proof that Condition (4) is necessary.

Let $Q^- \subseteq Q$ be the set of depots in Q with $\bar{m}_j \leq \lambda$, and assume that Condition (1) of Theorem 4 holds. Here we will show that by removing all depots in Q^- and their respective client sets K_j, and by adding the inequality $\sum_{j\in M}\sum_{k\in K^-} v_{jk} \leq d(K^-)$, where K^- is the set of removed clients, we obtain an inequality that is stronger than the original effective capacity inequality based on sets J, K and K_j, $j \in J$. Let

$$\begin{aligned} J' &= J \setminus Q^-, \\ K^- &= \cup_{j\in Q^-} K_j, \\ K' &= K \setminus K^-, \\ K'_j &= K_j \setminus K^-,\ j \in J'. \end{aligned}$$

First, note that J' can be written as $J' = (J\setminus Q)\cup(Q\setminus Q^-)$. The values of the effective capacities $\bar{m}_j$, $j \in J'$, given the the new client sets K' and K'_j for all $j \in J'$ are:

$$\begin{aligned} \bar{m}'_j &= \min(m_j, d(K')),\ j \in J \setminus Q, \\ \bar{m}'_j &= \bar{m}_j,\ j \in Q \setminus Q^-. \end{aligned}$$

The second equation holds since we assume that Condition 1 is satisfied, i.e., if we delete a client set K_l, it only affects the effective capacity of depot l. The new excess capacity $\lambda' = \sum_{j\in J'} \bar{m}'_j - d(K')$. If $\min(m_j, d(K')) = m_j$ for all $j \in J\setminus Q$, then $\lambda' = \lambda > 0$. Let $(J\setminus Q)' \subseteq (J\setminus Q)$ be the set of depots for which $d(K') < m_j$. Then

$$\lambda' = \sum_{j\in(J\setminus Q)\setminus(J\setminus Q)'} m_j + (|(J \setminus Q)'| - 1)d(K') + \sum_{j\in Q\setminus Q^-} \bar{m}_j \geq 0.$$

We want to show that the inequality

$$\sum_{j\in J'}\sum_{k\in K'_j} v_{jk} + \sum_{j\in M}\sum_{k\in K^-} v_{jk} + \sum_{j\in J'}(\bar{m}'_j - \lambda')^+(1 - y_j) \leq d(K') + d(K^-) \tag{17}$$

is stronger than the effective capacity inequality based on sets J, K and K_j.

Since $\lambda' \geq 0$, inequality (17) is valid, following the same argument as in the proof of necessity of Condition(1). All coefficients of the flow variables v_{jk} are equal in the two inequalities except for the coefficients for $v_{jk}, j \in Q \setminus Q^-$, $k \in K^-$ and $j \in M \setminus J$, $k \in K^-$, which have value one in inequality (17) and zero in the effective capacity inequality. The right-hand sides of the equalities are equal as $d(K) = d(K') + d(K^-)$. It remains to show that $(\bar{m}'_j - \lambda')^+ \geq (\bar{m}_j - \lambda)^+$ for all $j \in J'$ (recall that $(\bar{m}_j - \lambda)^+ = 0$ for $j \in J \setminus J'$), which is done in a similar way as in the proof of necessity of Condition (1). ■

If $\rho_j(J \setminus \{j\}) = (\bar{m}_j - \lambda)^+$, it means that $f(J \setminus \{j\}) = \min(\sum_{l \in J \setminus \{j\}} \bar{m}_l, d(K))$. The case where $\rho_j(J \setminus \{j\}) > (\bar{m}_j - \lambda)^+$ is difficult to analyze since, in general, we have no closed-form expression for $f(J \setminus \{j\})$. Consider a submodular inequality for which $f(J) = d(K)$ and whose support graph is connected. Aardal et al. (1993) observed that if $\rho_j(J \setminus \{j\}) > (\bar{m}_j - \lambda)^+$ for some $j \in J$, then there exists a non-trivial partition of the clients $(\bar{K}, K \setminus \bar{K})$ and the depots $(\bar{J}, J \setminus \bar{J})$ with $j \in \bar{J}$, where the clients in $\bar{K}$ are uniquely served by the depots in $\bar{J}$. This observation led to the development of two classes of facet-defining submodular inequalities having $\rho_j(J \setminus \{j\}) > (\bar{m}_j - \lambda)^+$ for at least one $j \in J$. We describe one of these classes, the single depot inequalities, below.

Lemma 5 (Aardal et al. (1993).) *Let C^{EC} be an effective capacity component with client set K^{EC}, depot set J^{EC}, and arc set $\{(j,k) : j \in J^{EC}, k \in K_j \subseteq K^{EC}\}$, and such that J^{EC}, K^{EC} and $\{Kj\}_{j \in J^{EC}}$ satisfy the conditions of Theorem 4. The set $Q^{EC} \subset J^{EC}$ is the set of depots in J^{EC} having $\bar{m}_j < m_j$. Let $P = \{1, ..., |P|\}$ be a set of additional depots with client set $K_p = K^{EC} \cup \bar{K}_p$, $\bar{K}_p \neq \emptyset$, $p \in P$, where the clients in $\bar{K}_p$ are served uniquely by depot p, and such that $m_p > d(\bar{K}_p)$ for all $p \in P$. Let $J = J^{EC} \cup P$, $K = K^{EC} \cup_{p \in P} \bar{K}_p$ and $\lambda = \sum_{j \in J} \bar{m}_j - d(K)$.*

Then $\rho_j(J \setminus \{j\}) = (\bar{m}_j - \lambda)^+$ for all $j \in J^{EC}$ and $\rho_p(J \setminus \{p\}) = d(\bar{K}_p) > (\bar{m}_p - \lambda)^+$ for all $p \in P$.

In Figure 2 we show the structure of the support graph corresponding to a single depot inequality.

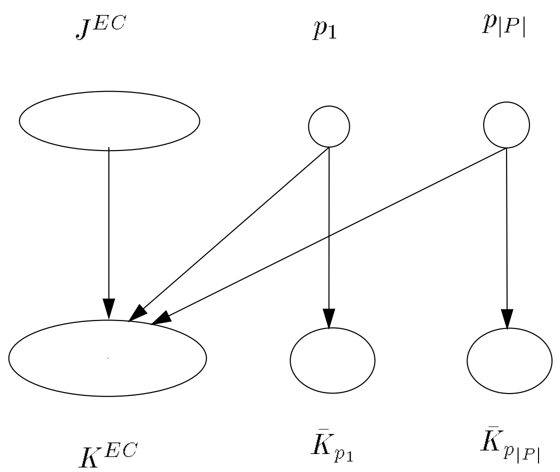

Figure 2: Single depot structure.

Theorem 6 (Aardal et al. (1993).) *Assume that $f(J) = d(K) < \sum_{j \in J} \bar{m}_j$, and that $\sum_{j \in M} m_j > d(N) + m_r$ for all $r \in J$. Then the submodular inequality (10) based on the single depot structure, as defined in Lemma 5, defines a facet of conv(X^{CFL}) if and only if*

(1) $\rho_q(J \setminus \{q\}) > 0$ *for all* $q \in Q^{EC}$,

(2) *if* $|Q^{EC}| \leq 1$, *then* $\exists\ j \in J^{EC} \setminus Q^{EC}$ *with* $\rho_j(J \setminus \{j\}) > 0$.

3 The Combinatorial Inequalities

The family of combinatorial inequalities were developed by Cornuéjols and Thizy (1982), and Cho et al. (1983a) for the uncapacitated facility location problem. Given $J \subseteq M$, $K \subseteq N$ and $K_j \subset K$ for all $j \in J$, let β be the minimum number of depots in J needed to reach, or cover, all clients in K. β is called the *covering number*. Let S be a zero-one $|J| \times |K|$ matrix with $s_{jk} = 1$ if $k \in K_j$ and $s_{jk} = 0$ otherwise. The matrix S is the *adjacency matrix* corresponding to the bipartite graph $G^S = (V^S, U^S, E^S)$ with $V^S = \{j \in J\}$, $U^S = \{k \in K\}$, and $E^S = \{(j,k) : j \in J,\ k \in K_j\}$. Assume that (i) $|J|$, $|K| \geq 3$, that (ii) the graph G^S is connected, and that (iii) each column of S has at least one 0 and one 1. Changing an element s_{jk} from zero to one corresponds to adding an edge (j,k) in G^S. If no such change can take place without decreasing β, S is said to be *maximal*. Let X^{UFL} denote the set of feasible solutions to the uncapacitate facaility location problem.

Theorem 7 (Cho et al. (1983a,b).) *Let $J \subseteq M$, $K \subseteq N$, and let $K_j \subset K$ for all $j \in J$. Let β be the minimum number of depots in J needed to cover all the clients in K, given the client sets K_j, $j \in J$. The combinatorial inequality*

$$\sum_{j \in J} \sum_{k \in K_j} \frac{1}{d_k} v_{jk} - \sum_{j \in J} y_j \leq |K| - c, \tag{18}$$

is valid for X^{UFL} for all $c \leq \beta$. Inequality (18) defines facets of conv(X^{UFL}) *if and only if $c = \beta$, and the adjacency matrix S corresponding to the support graph of the inequality is maximal and satisfies conditions (i)-(iii) given above.*

To provide some intuition behind the combinatorial inequalities we give an example below.

Example 2 Consider the structure given in Figure 3. Let $K = \{1,2,3\}$, $J = \{1,2,3\}$, and let $K_1 = \{2,3\}$, $K_2 = \{1,3\}$, and $K_3 = \{1,2\}$.
Since we need at least two of the depots in J open to cover the clients in K, given the client sets K_j, $j \in J$, the covering number $\beta = 2$. The inequality

$$v_{12} + v_{13} + v_{21} + v_{23} + v_{31} + v_{32} - y_1 - y_2 - y_3 \leq 3 - 2 = 1 \tag{19}$$

is valid, and defines a facet since the adjacency matrix corresponding to the support graph is maximal, and satisfies Conditions (i)–(iii) given above. Inequality (19) belongs to the class of so-called *odd hole inequalities,* which form a subclass of the combinatorial inequalities. ■

Since the uncapacitated facility location problem is a relaxation of CFL, the inequalities (18) are also valid for conv(X^{CFL}). Aardal et al. (1993) showed that inequalities (18) define facets of conv(X^{CFL}) if, in addition to S being maximal and $c = \beta$, the part of the instance that

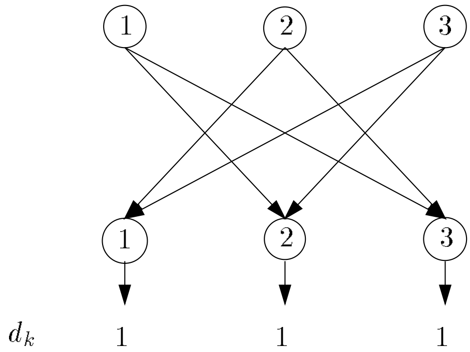

Figure 3: Example of a support graph corresponding to a combinatorial inequality.

is defined by the sets J, K and K_j has enough slack capacity. For details see Aardal et al., Theorem 20.

The class of combinatorial inequalities is very general, and subsumes all classes of valid inequalities developed so far for the uncapacitated facilty location problem. The previously known special cases, see Padberg (1973), Cornuéjols et al. (1977), Guignard (1980) and Cornuéjols and Thizy (1982), all have the property that the adjacency matrix corresponding to the bipartite support graph of the inequality is cyclic. A cyclic adjacency matrix has the same number of ones in every row, and it is possible to permute the rows and the columns such that in every row all ones appear in sequence and are shifted one position to the right for every new row. This implies that all depots serve equally many clients and that all clients are served by equally many depots.

4 Separation Heuristics

The separation problem based on a certain family $\mathcal{F}$ of valid inequalities can be formulated as follows.

Given a fractional solution (v^*, y^*), show that (v^*, y^*) satisfies all valid inequalities in $\mathcal{F}$, or find an inequality belonging to $\mathcal{F}$ that is violated by (v^*, y^*).

The separation problem based on the submodular inequalities (10) is hard as we need to choose a depot set, a client set, and an arc set simultaneously. Before we describe a heuristic for finding violated, more general, submodular inequalities we consider the flow cover case.

Given a client set $K \subseteq N$, let $z_j = 1$ if $j \in J$, and $z_j = 0$ otherwise, and let $v'_j = \sum_{k \in K} v_{jk}$ for $j \in M$. The separation problem based on the set of flow cover inequalities (11) for fixed client set K, can be formulated as follows.

$$\max \sum_{j \in M} [v'^*_j + (m_j - \lambda)^+ (1 - y^*_j)] z_j - d(K) \tag{20}$$

$$\begin{aligned} \text{s.t.} \quad \sum_{j \in M} m_j z_j &= d(K) + \lambda, \\ \lambda &> 0, \end{aligned}$$

$$z_j \in \{0,1\}.$$

For a given value of λ problem (20) is an equality constrained knapsack problem and hence NP-hard. A heuristic for solving (20) is given by Van Roy and Wolsey (1987). A similar heuristic is implemented in MINTO (Savelsbergh et al. (1994)). If we consider the case where all capacities are equal, i.e. $m_j = m$ for all $j \in J$, then the flow cover inequality can be written as

$$\sum_{j \in S} v_j - \sum_{j \in S} (m - \lambda) y_j \leq d(K) - (m - \lambda) l, \tag{21}$$

where $l = \lceil d(K)/m \rceil$ is the size of a minimal cover, $\lambda = ml - d(K)$, and S is any cover, i.e., $|S| \geq l$. Such inequalities were used by Leung and Magnanti (1989) for CFL with equal capacities.

Theorem 8 *If $m_j = m$ for all $j \in M$, and if the client set K is fixed, then the separation problem based on the flow cover inequalities can be solved in polynomial time.*

Proof. Assume that $d(K)$ is not an integer multiple of m. Let $r = d(K) - m(l-1)$, i.e., r is the remaining demand that the last depot in the minimal cover must satisfy if all clients in K are to be served, and if $(l-1)$ depots in the minimal cover send flow to K corresponding to their full capacity. Note that $r = (m - \lambda)$, and that $1 \leq r < m$. We now need to find the set S. Let $z_j = 1$ if $j \in S$ and $z_j = 0$ otherwise. If $m_j = m$ for all $j \in J$, and if the client set K is fixed , then the separation problem based on flow cover inequalities can be formulated as:

$$\max\ \xi = \sum_{j \in M} [v_j'^* - r y_j^*] z_j + rl - d(K) \tag{22}$$

$$\text{s.t.} \quad \sum_{j \in M} z_j \geq l,$$

$$z_j \in \{0,1\}.$$

Next, we show that if $\xi > 0$, then $\sum_{j \in M} z_j \geq l$, i.e. all solutions $z \in \{0,1\}^m$ giving $\xi > 0$ are feasible in (22). For given $z \in \{0,1\}^m$ we get

$\xi = \sum_{j \in M} [v_j'^* - r y_j^*] z_j + rl - d(K) \leq m \sum_{j \in M} y_j^* z_j - r \sum_{j \in M} y_j^* z_j + rl - d(K) \leq$
$(m-r) \sum_{j \in M} z_j + rl - d(K) = (m-r) \sum_{j \in M} z_j + rl - r - m(l-1) = (m-r)(\sum_{j \in M} z_j - (l-1)),$

where the first inequality follows from $v_j^* \leq m y_j^*$, and where the second inequality follows from $y_j^* \leq 1$. It now follows that if $\xi > 0$, then $\sum_{j \in M} z_j - (l-1) > 0$, i.e. $\sum_{j \in M} z_j \geq l$, since $(m-r) > 0$, and since l is integral. Solving the separation problem (22) can now be reduced to letting $z_j = 1$ if $(v_j'^* - r y^*) > 0$ and $z_j = 0$ otherwise, and then checking if the resulting value of ξ is positive. ∎

For fixed client set $K \subseteq N$, the coefficients $(\bar{m}_j - \lambda)^+$ and $\rho_j(J \setminus \{j\})$ of the y_j-variables in the effective capacity and the general submodular inequalities respectively, depend on the arc set. Moreover, in order to determine the value of each coefficient $\rho_j(J \setminus \{j\})$, we need to apply a maximum flow algorithm. All these difficulties make the separation problem based on the effective capacity and the general submodular inequalities hard to analyze, and it is not possible to solve any of these problems as a single integer program. We have therefore developed a heuristic for identifying violated effective capacity inequalities, and to further extend the effective capacity inequalities to single depot inequalities. The heuristic consists of two parts; the first part, presented in the next subsection, aims at identifying good candidate sets J and K, and the second part, described in Section 3.2, determines an appropriate arc set and possible P-depots (for the definition of P-depots, see Lemma 5).

4.1 A General Heuristic for Identifying Sets J and K

For all three subclasses of the submodular inequalities presented in the previous section we need to identify an effective capacity component, i.e. a client set K, a cover J and an arc set where at least one depot in J serves all clients in K. To get a single depot structure we also need to find a set P of depots and client sets $\bar{K}_p$ for all $p \in P$, see Lemma 5. The following heuristic produces sets J and K, with J defining a cover with respect to K. Depending on the structure of the graph induced by J, K and the active arcs in the fractional solution, and on the characteristics of the instance, we then try to identify violated flow cover, EC, and single depot inequalities. The heuristic is based on three observations regarding the effective capacity inequalities

$$\sum_{j \in J} \sum_{k \in K_j} v_{jk} + \sum_{j \in J} (\bar{m}_j - \lambda)^+ (1 - y_j) \leq d(K). \tag{12}$$

Let (v^*, y^*) denote the current fractional solution.

Observation 1 *Inequality (12) can be violated only if at least one of the y_j-variables has a fractional value.*

To guarantee that J contains at least one fractional depot, we choose as initial depot set $J^1 = \{j^1\}$ where $0 < y^*_{j^1} < 1$.

Observation 2 *The value of the flow $\sum_{j \in J} \sum_{k \in K_j} v^*_{jk}$ should be as close as possible to $d(K)$.*

At iteration t, given the current client set K^{t-1} and the depot set $J^t = \{j^1, ..., j^t\}$, let K^{t-1} be augmented by the set of clients served by j^t that are not already in K^{t-1}, giving set K^t. Moreover, we regard as candidates for J^{t+1}, all depots, apart from the depots already in J^t, that serve clients in K^t. The choice of depot to include in J^{t+1} is based on the following observation.

Observation 3 *Inequality (12) is more easily violated if depots having a small value of $s_j = \bar{m}_j y^*_j - \sum_{k \in K_j} v^*_{jk}$ are included in J.*

To see that this is true, replace $\sum_{k \in K_j} v^*_{jk}$ by $\bar{m}_j y^*_j - s_j$ and $(\bar{m}_j - \lambda)^+$ by $(\bar{m}_j - \lambda)$ in the effective capacity inequality (12), and use $d(K) = \sum_{j \in J} \bar{m}_j - \lambda$. Replacing $(\bar{m}_j - \lambda)^+$ by $(\bar{m}_j - \lambda)$ makes (12) weaker. This weaker form of (12) can now be written as

$$\lambda - \lambda \sum_{j \in J} (1 - y^*_j) \leq \sum_{j \in J} s_j.$$

In practice it will often be the case, especially if the problem instance has little slack capacity, i.e., if $\sum_{j \in M} m_j / \sum_{k \in N} d_k$ is relatively close to one, that $s_j = 0$ for several depots considered as candidates for set J. To break ties, we choose as depot j^{t+1}, i.e. as the depot by which J^t should be augmented, the depot with largest flow contribution $\sum_{k \in K^t} v^*_{jk}$ among the candidate depots having minimum value of s_j. This is to avoid including a depot that only marginally contributes to the flow whereas the effective capacity is fully counted for.

The algorithm consists of two major loops. At every iteration of the outer loop, we choose a depot with fractional value as the first depot j^1 in the set J. The outer loop is repeated until all depots with fractional value have been considered. The inner loop is initialized with the sets $J^1 = \{j^1\}$ and $K^0 = \emptyset$. The sets K^t and J^t are then iteratively augmented according to the second and third observations. If at a certain iteration t, J^t defines a cover with respect to M and K^t, we check whether a flow cover, EC, or single depot inequality based on (J^t, K^t) is violated. If the set of candidate depots, that can be used to augment J^t, is empty, the inner loop is terminated. We also stop augmenting J^t and K^t if we fail to generate a violated inequality after having generated a cover n times using the same initial depot set J^1. In our experiments we use $n = 2$.

4.2 Identifying Violated Flow Cover, Effective Capacity and Single Depot Inequalities

Given sets J and K we temporarily define K_j as $K_j = \{k \in K : v^*_{jk} > 0\}$, $\bar{m}_j = \min(m_j, d(K_j))$ for all $j \in J$, and $Q = \{j \in J : \bar{m}_j < m_j\}$. We distinguish between the following three cases.

Case 1: $Q = \emptyset$ and $|J| = 2$

We redefine K_j as $K_j = K$ for all $j \in J$, giving a flow cover inequality (11).

Case 2: $Q = \emptyset$ and $|J| > 2$

We try to identify P-depots as described below.

Case 3: $0 < |Q| \leq |J|$

We first check if any of the depots in Q have intersecting client sets, i.e. if $K_{q_1} \cap K_{q_2} \neq \emptyset$ for any $q_1, q_2 \in Q$. If that is the case, we remove all clients belonging to any such intersection. This rule is based on the proof that Condition (1) of Theorem 4 is necessary for (12) to be facet defining. Given the resulting sets J', Q', K' and K'_j for all $j \in J'$, we update the effective capacities $\bar{m}_j$ and check whether there are any depots in Q' with $\bar{m}_j \leq \lambda$. These depots and their client sets K'_j are removed form J' and K', which is a rule based on the proof that Condition (4) of Theorem 4 is necessary for the effective capacity inequalities to be facet defining. We again update the effective capacities and obtain sets J'', Q'', K'' and K''_j for all $j \in J''$. Let K^r denote the set of removed clients and let for simplicity $J = J''$, $Q = Q''$, $K = K''$ and $K_j = K''_j$.

If $|J| > 2$, we try to identify depots that could be considered as P-depots, see Lemma 5. As candidates for P we consider depots p that at the current fractional solution uniquely serve a subset $\bar{K}_p \subset K$ of the clients, and for which $d(\bar{K}_p) < m_p < d(K)$. For a single depot structure we require the depots that constitute the effective capacity component to define a cover with respect to the clients in the effective capacity component. Therefore, after choosing appropriate depots as candidates for P, denoted $\text{cand}(P)$, we check whether (i) the resulting set $J \setminus \text{cand}(P)$ defines a cover with respect to $K \setminus (\cup_{p \in \text{cand}(P)} \bar{K}_p)$ and if (ii) $(J \setminus \text{cand}(P)) \setminus Q \neq \emptyset$. If this is the case we have generated a single depot structure with $P = \text{cand}(P)$, otherwise we need to remove depots from $\text{cand}(P)$ and include them in $J \setminus \text{cand}(P)$ until conditions (i) and (ii) are satisfied. Depots are removed from $\text{cand}(P)$ in order of increasing values of $d(\bar{K}_p)$. This is done in order to keep depots with high values of $\rho_j(J \setminus \{j\}) = d(\bar{K}_p)$ in the set P if possible.

When the algorithm terminates, we have the final depot and client sets J and K, which can be partitioned into the sets $(Q, P, J \setminus (Q \cup P))$ and $(\cup_{p \in P} \bar{K}_p, K \setminus (\cup_{p \in P} \bar{K}_p))$ respectively. Moreover we have the set K^r of clients that in Case 3 were removed from the initial client set. If the inequality based on the above mentioned sets is violated, the following variables are stored.

$$
\begin{array}{ll}
v_{jk}, & j \in Q, \; k \in K_j, \\
 & j \in P, \; k \in K \setminus \cup_{p \in P \setminus \{j\}} \bar{K}_p, \\
 & j \in J \setminus (Q \cup P), \; k \in K \setminus \cup_{p \in P} \bar{K}_p, \\
 & j \in M, \; k \in K^r, \\
 & \\
y_j, & j \in J.
\end{array}
$$

All variables v_{jk} have coefficient 1, and the coefficient of y_j is $-\rho_j(J \setminus \{j\}) = -(\bar{m}_j - \lambda)^+$ for $j \in J \setminus P$, and $-\rho_j(J \setminus \{j\}) = -d(\bar{K}_p)$ for $j \in P$. The right-hand side of the inequality is $d(K) + d(K^r) - \sum_{j \in J} \rho_j(J \setminus \{j\})$.

4.3 Identifying Violated Combinatorial Inequalities

Recall the combinatorial inequalities

$$\sum_{j \in J} \sum_{k \in K_j} \frac{1}{d_k} v_{jk} - \sum_{j \in J} y_j \leq |K| - c, \tag{23}$$

where $c \leq \beta$, and β is the covering number, i.e., the minimum number of depots needed to cover all clients in K, given the client sets $K_j \subset K$, $j \in J$.

Although a fair amount of attention has been given to the various subclasses of combinatorial inequalities, very little work has been done on the separation problem based on the general class, and no computational experience with any of the subclasses is reported in the literature. Here we present a heuristic for generating violated combinatorial inequalities that uses the sets J and K, produced by the general heuristic described in Section 4.1, as input.

Due to the way that the sets J and K are generated, the bipartite graph $G = (V, U, E)$ where $V = \{j \in J\}$, $U = \{k \in K\}$ and $E = \{(j,k) : v^*_{jk} > 0, \; j \in J, k \in K\}$ is connected. Moreover, $\sum_{j \in J} m_j > d(K)$. Determining β, which is equivalent to determining the minimum number of rows of the adjacency matrix corresponding to G, such that there is at least one 1 in every column, is an NP-hard problem. We therefore start by approximating β by the cardinality of a minimum subset $I \subseteq J$ such that $\sum_{j \in I} d(K_j) > d(K)$, where K_j initially is defined as $K_j = \{k \in K : v^*_{jk} > 0\}$. It is clear that $|I| \leq \beta$. Assuming that the right-hand side of the inequality is $|K| - |I|$, and that all depots serve equally many clients, as in the case of cyclic adjacency matrices, it is possible to calculate the maximal outdegree t of each depot by using $|I| = \lceil |K|/t \rceil$. For each depot we sort the values v^*_{jk}, $k \in K$, in nondecreasing order and we redefine K_j to be the set of clients corresponding to the t highest values of v^*_{jk}. If the corresponding inequality is violated, we store variables v_{jk}, $j \in J$, $k \in K_j$, with coefficient $1/d_k$ and variables y_j, $j \in J$ with coefficient -1. The right-hand side of the inequality is $|K| - |I|$.

5 Computational Experience

We solve 60 capacitated facility location problems of four different sizes and three different capacity levels. Each test problem is represented by the notation *v-xxxyyz*, where v is the capacity level, xxx the number of clients, yy the number of depots, and z the number of the instance in the series of test problems having equal values of v, xxx and yy. The different capacity levels we consider are $v = 1, 2, 3$ corresponding to ratios $\sum_{j \in M} m_j / d(N) = 1.5, 2$ and 3 respectively. The test problems of sizes 50×16, 50×33, and 50×50 were kindly made available to us by J.-M. Thizy, and were generated by Cornuéjols et al. (1991), to provide more difficult and general test problems than the classical Kuhn-Hamburger instances. To test on more challenging instances we generated three sets of instances of size 100×75, using the same formulae as in Cornuéjols et al. In particular, the values of the capacities are generated from $U[10, 160]$, and then scaled so as to match the capacity level v. To reflect the economies of scale, the fixed costs are generated using the formula $f_j = U[0, 90] + U[100, 110]\sqrt{m_j}$. For the instances having $v = 1$, and $v = 2$ we multiply f_j by two. For more details concerning the test problems we refer to Cornuéjols et al. The number of variables, constraints and nonzeros corresponding to the different problem sizes are reported in Table 1.

problem type	# variables	# constraints	# nonzeros
50×16	832	883	3,264
50×33	1,716	1,767	6,732
50×50	2,600	2,651	10,200
100×75	7,650	7,751	30,300

Table 1: Problem characteristics.

For the computational experiments we use MINTO 1.6a (Savelsbergh et al. (1994)) with the CPLEX 2.1 callable library, implemented on a SUN Sparc ELC. We refer to the knapsack cover inequalities and the flow cover inequalities with $K = N$, which are automatically generated by MINTO from the substructures (7) and (8), as *MINTO inequalities*. The *user inequalities* are the EC, single depot and combinatorial inequalities. We consider the flow cover inequalities with $K \subset N$ as a special case of the effective capacity inequalities. Three solution strategies are used, all starting from the LP-relaxation of CFL.

I. Branch-and-bound.

II. Branch-and-cut using knapsack cover inequalities generated by MINTO.

III. Branch-and-cut using MINTO inequalities and user inequalities.

Strategy I was chosen in order to obtain a reference point to which we can compare the results obtained after adding the various classes of valid inequalities. By experimenting with the different families of inequalities we could conclude that the knapsack cover inequalities were the single most effective class. This was rather surprising since the surrogate knapsack polytope seems like a quite drastic relaxation of CFL. The knapsack cover inequalities only involve the y_j-variables of the depots, and do not consider the flow variables at all. To illustrate these observations we decided to report on them separately as Strategy II. In Strategy III we investigate the effect of adding inequalities from all classes discussed in Section 2 and 3. The user inequalities were generated in the root node only, since the main purpose was to test how much effect the reduction of the duality gap in the root node has on the size of the branch-and-bound tree and thereby on the total time needed to solve the problems, and since the user inequalities had slightly less effect deeper in the branch-and-bound tree.

The results from using Strategy I–III are reported in Tables 2-4 below. The duality gap is calculated as $(z_{IP} - z_{LP})/z_{IP}$, where z_{IP} and z_{LP} denote the optimal solution to CFL and the LP-relaxation of CFL respectively. The percentage duality gap closed by the inequalities is determined by $(z_{CUT} - z_{LP})/(z_{IP} - z_{LP})$, where z_{CUT} is the optimal value of the LP-relaxation with all inequalities generated at the root node added to the formulation. In Tables 3 and 4, *% time reduction* refers to the percentage reduction in total time of Strategy II and III respectively, compared to Strategy I, whereas *average % time reduction* is the average percentage reduction over the instances in the same set. The *user time* is the time needed to generate user inequalities. Both user time and total time are given in seconds after being rounded down. All results are summarized in Table 5.

From the results in Table 3, we note that the knapsack cover inequalities are quite effective in reducing the total time required to solve the problems. The average time reduction over all instances is 32.0%, and forty-six of the sixty instances are solved at least as fast using Strategy II as by Strategy I. It is worth noticing here that the loosely capacitated problems do not seem easier to solve, something that is often claimed to be the case. For the instances of size 100×75 with $v = 2$ and 3, the results are mixed. For problem 2-100754 we only close 5.9% of the duality gap by adding the knapsack cover inequalities, but the total time decreases by 96.8%. For problem 2-100751, 35.0% of the gap is closed whereas the time increases by more than a factor two. We can see, however, that, seemingly by coincidence, the reduction of 35% of the duality gap for 2-100751 only yields a small reduction in the number of branch-and-bound nodes. Since each branch-and-bound node is expensive for the large instances, we get this net negative effect. For problem 2-100754 the number of branch-and-bound nodes, however, decreases substantially (98.8%). In Table 6, we report for each problem size on the average time spent per branch-and-bound node in the different strategies.

If we consider Strategy III, all instances but one are solved at least as fast as by Strategy I. Seven problems were solved to optimality at the root node and for twenty-one of the problems the percentage time reduction was more than 75%. For the instances of size 50×50 and 100×75 with capacity level $v = 3$, the percentage duality gap closed by the inequalities added at the root node is smaller, and the time required to solve them remains longer than for the more tightly capacitated instances of the same size. This is not unexpected since all inequalities used in this study, except the combinatorial, make use of the capacities. For a relatively uncapacitated instance, a typical fractional solution after adding only knapsack and flow cover inequalities, has very few depots with positive y_j-values and only one with a fractional value. This fractional value is usually very small. The fractional depot serves a small portion of the flow to a set of clients almost completely served by one other depot that is just slightly short of capacity. The structure of such a solution leaves little freedom to generate inequalities of the types we have discussed. If the inequalities generated at that point are not able to cut off the remaining duality gap, the flow tends to spread drastically in the sense that every depot with a positive y_j-value will serve almost every client, and almost all y_j-variables get fractional values, which makes it very hard to identify an isolated substructure on which an inequality that is substantially violated can be based. The consequence is then that quite a large number of inequalities with a small violation are generated and the solution value hardly increases.

Very few violated combinatorial inequalities were generated, and they were all generated for the problems with $v = 3$ as could be expected. The heuristic we applied for these inequalities, however, uses the sets J and K from the general heuristic that was developed for the submodular inequalities. Given a set K, the set J that we generate is likely to be a good choice of depot set, but given the different character of the combinatorial inequalities compared to the submodular inequalities, we may use the wrong client set. Moreover, given the active arcs, we do not compute the exact covering number. As an additional experiment we chose some instances of size 25×8 with $v = 3$ from the test set of Cornuéjols et al. (1991) and generated *all* violated odd hole inequalities based on three depots and three clients, c.f. Figure 3, to see if this subclass of inequalities would help for these relatively uncapacitated instances. No improvement was obtained, however, neither in terms of reduction of the duality gap nor in time. The reason why we chose the small odd hole inequalities was that they are facet defining for the uncapacitated problems, and straightforward to generate. As soon as we base an odd hole inequalities on a larger set of depots and clients the inequality needs to be lifted to become a facet. In order to solve loosely capacitated problems it will therefore be important to develop a good heuristic based on the general class of combinatorial problems, and also to develop new classes of inequalities that use the capacities in a less restrictive way.

problem	duality gap (%)	# B&B nodes	time
1-050161	2.2	61	59
1-050162	0.4	19	17
1-050163	2.3	143	129
1-050164	2.1	141	173
1-050165	0.9	43	43
1-050331	1.5	1,361	1,813
1-050332	1.2	797	860
1-050333	0.2	31	66
1-050334	1.0	1,349	2,606
1-050335	1.6	1,617	2,050
1-050501	0.3	143	278
1-050502	0.1	67	116
1-050503	0.4	361	681
1-050504	0.2	123	283
1-050505	0.0	1	75
1-100751	0.7	4,077	22,977
1-100752	0.6	15,419	74,351
1-100753	0.1	183	761
1-100754	0.3	6,687	40,604
1-100755	0.1	117	621
2-050161	3.7	129	145
2-050162	2.6	95	103
2-050163	0.5	19	30
2-050164	2.5	345	512
2-050165	0.0	1	15
2-050331	1.5	399	686
2-050332	1.2	691	1,560
2-050333	1.5	259	556
2-050334	0.7	239	493
2-050335	1.3	685	1,232
2-050501	0.2	143	296
2-050502	1.0	1,625	4,050
2-050503	0.1	45	164
2-050504	0.1	33	129
2-050505	0.02	3	67
2-100751	0.1	59	413
2-100752	0.3	429	2,540
2-100753	0.3	541	5,312
2-100754	0.3	7,817	69,326
2-100755	0.6	5,049	46,111
3-050161	4.1	127	160
3-050162	5.4	109	189
3-050163	1.3	17	38
3-050164	2.0	31	104
3-050165	2.1	7	23
3-050331	2.5	187	1,727
3-050332	2.9	766	5,188
3-050333	1.6	249	626
3-050334	0.8	57	194
3-050335	2.1	193	844
3-050501	1.4	299	1,479
3-050502	1.1	443	1,949
3-050503	0.9	1,003	3,162
3-050504	0.5	273	666
3-050505	0.7	199	686
3-100751	0.6	477	4,057
3-100752	0.9	1,289	12,834
3-100753	1.0	545	8,808
3-100754	0.6	235	2,250
3-100755	1.1	6,135	117,058

Table 2: Results from using Strategy I.

problem	# cover inequalities	% gap closed	# B&B nodes	time	% time reduction	average % time reduction
1-050161	31	68.1	17	51	13.6	
1-050162	2	48.0	13	20	−17.6	
1-050163	16	82.0	13	48	62.8	22.9
1-050164	33	33.7	35	109	37.0	
1-050165	24	52.3	15	35	18.6	
1-050331	34	81.8	41	173	90.5	
1-050332	118	65.7	129	476	44.7	
1-050333	0	0.0	31	66	0.0	57.1
1-050334	130	10.8	175	975	62.6	
1-050335	46	78.4	49	255	87.6	
1-050501	3	100.0	1	54	80.6	
1-050502	9	37.2	7	93	19.8	
1-050503	10	32.4	7	107	84.3	39.7
1-050504	43	38.9	47	356	−25.8	
1-050505	LP optimal	–	–	–	–	
1-100751	295	40.9	611	10,560	54.0	
1-100752	648	55.4	1,423	20,055	73.0	
1-100753	48	12.5	59	844	−10.9	44.7
1-100754	228	9.1	537	11,076	72.7	
1-100755	9	34.6	23	406	34.6	
2-050161	16	77.6	19	58	60.0	
2-050162	20	30.3	13	64	37.9	
2-050163	4	51.5	9	36	−20.0	41.5
2-050164	12	62.0	23	68	86.7	
2-050165	LP optimal	–	–	–	–	
2-050331	13	86.0	31	125	81.8	
2-050332	58	54.3	51	450	71.2	
2-050333	122	54.1	89	769	−38.3	50.3
2-050334	42	76.6	23	213	56.8	
2-050335	25	78.3	49	248	79.9	
2-050501	8	82.5	7	76	74.3	
2-050502	158	59.4	225	1,551	61.7	
2-050503	6	0.0	31	170	−3.7	22.7
2-050504	5	0.0	31	153	−18.6	
2-050505	0	0.0	3	67	0.0	
2-100751	25	35.0	45	882	−113.6	
2-100752	192	15.8	239	6,835	−169.1	
2-100753	77	25.0	73	1,792	66.3	−4.7
2-100754	73	5.9	95	2,235	96.8	
2-100755	67	66.4	105	1,803	96.1	
3-050161	15	52.3	13	76	52.5	
3-050162	20	53.6	15	98	48.1	
3-050163	9	21.0	9	55	−44.7	9.8
3-050164	22	82.4	15	80	23.1	
3-050165	5	39.7	5	30	−30.4	
3-050331	119	64.4	107	1,076	37.7	
3-050332	78	66.2	83	495	90.5	
3-050333	108	0.2	77	591	5.6	48.2
3-050334	9	27.8	25	120	38.1	
3-050335	45	42.2	45	260	69.2	
3-050501	57	66.0	37	352	76.2	
3-050502	79	17.4	125	987	49.3	
3-050503	105	6.3	125	997	68.5	46.4
3-050504	56	0.0	79	693	−4.1	
3-050505	31	12.0	73	396	42.3	
3-100751	133	0.4	379	8,653	−113.3	
3-100752	90	15.8	213	6,923	56.3	
3-100753	47	19.4	87	2,553	71.0	5.6
3-100754	80	12.1	137	3,925	−74.4	
3-100755	198	18.7	461	13,529	88.4	

Table 3: Results from using Strategy II.

	# MINTO inequalities		# User inequalities								
problem	cover	FC	EC	SD	combi-natorial	time user	%gap closed	# B&B nodes	total time	% time reduct.	average % time reduct.
1-050161	15	23	9	13	0	0	86.5	9	49	16.9	
1-050162	2	4	3	1	0	0	100.0	1	10	41.2	
1-050163	11	19	8	9	0	0	85.0	11	35	72.9	47.7
1-050164	13	20	13	17	0	0	47.1	19	76	56.1	
1-050165	4	5	5	8	0	1	92.2	3	21	51.2	
1-050331	10	31	5	18	0	1	89.3	19	125	93.1	
1-050332	52	65	4	11	0	0	70.1	35	208	75.8	
1-050333	0	2	1	0	0	0	100.0	1	23	65.2	81.9
1-050334	75	97	3	14	0	1	53.8	69	488	81.3	
1-050335	22	39	1	6	0	0	81.8	25	124	94.0	
1-050501	3	5	1	35	0	1	100.0	1	56	79.9	
1-050502	2	4	5	12	0	0	96.8	3	52	55.2	
1-050503	10	11	0	0	0	0	32.4	7	79	88.4	72.4
1-050504	6	8	1	20	0	1	54.1	7	96	66.1	
1-050505	LP	opt.	–	–	–	–	–	–	–	–	
1-100751	107	183	12	19	0	9	44.2	251	6,851	70.2	
1-100752	381	591	13	22	0	11	56.7	1,253	33,186	55.4	
1-100753	15	23	15	26	0	10	43.1	17	712	6.4	52.7
1-100754	125	124	15	7	0	8	11.0	195	5,514	86.7	
1-100755	11	16	5	3	0	5	62.9	5	345	44.6	
2-050161	15	30	11	8	0	1	80.1	13	69	52.4	
2-050162	11	20	4	1	0	0	30.3	13	72	30.1	
2-050163	1	2	2	5	0	0	86.0	3	15	50.0	54.6
2-050164	12	16	7	20	0	0	66.7	15	72	85.9	
2-050165	LP	opt.	–	–	–	–	–	–	–	–	
2-050331	13	24	8	8	0	0	92.1	11	115	83.2	
2-050332	33	36	4	29	0	1	67.2	27	290	81.4	
2-050333	32	43	8	23	0	1	62.1	25	177	68.1	62.6
2-050334	17	25	7	14	0	0	81.9	11	97	80.3	
2-050335	17	23	8	10	0	1	88.0	14	124	89.9	
2-050501	5	5	4	14	0	0	85.1	5	71	76.0	
2-050502	56	71	2	3	0	0	60.6	77	734	81.9	
2-050503	8	9	6	9	0	1	94.3	5	97	40.9	52.0
2-050504	2	3	3	21	0	0	100.0	1	60	53.5	
2-050505	0	1	2	8	0	0	95.6	3	62	7.5	
2-100751	4	8	24	15	0	7	58.4	9	234	43.3	
2-100752	46	62	15	31	0	11	32.2	49	1,590	37.4	
2-100753	34	53	15	21	0	12	48.1	43	1,225	76.9	70.3
2-100754	43	53	18	13	0	9	28.2	77	2,240	96.8	
2-100755	33	26	17	21	0	15	72.8	43	1,407	96.9	
3-050161	7	14	6	2	0	1	64.2	9	58	63.8	
3-050162	15	15	7	3	1	0	60.9	11	69	63.5	
3-050163	4	6	12	1	0	0	99.7	3	26	31.6	53.6
3-050164	5	9	9	5	2	1	91.3	3	31	70.2	
3-050165	1	2	1	0	0	0	100.0	1	14	39.1	
3-050331	48	66	14	3	0	0	67.6	47	474	72.6	
3-050332	56	82	20	8	0	1	73.4	49	521	90.0	
3-050333	37	47	10	2	0	0	47.4	39	394	37.1	62.8
3-050334	9	12	7	4	1	1	45.8	11	108	44.3	
3-050335	23	34	7	4	2	1	53.3	23	254	69.9	
3-050501	10	11	11	25	1	0	85.6	9	212	85.7	
3-050502	68	97	6	8	0	0	21.7	71	1,317	32.4	
3-050503	55	47	7	1	0	0	29.1	67	674	78.7	45.8
3-050504	36	46	9	9	0	0	25.5	55	800	−20.1	
3-050505	30	19	10	17	0	1	54.0	31	326	52.5	
3-100751	90	108	32	12	0	12	46.1	207	5,869	−44.7	
3-100752	78	112	30	13	2	8	36.7	127	5,249	66.8	
3-100753	60	66	27	2	3	11	39.5	53	2,888	67.2	27.7
3-100754	43	60	28	11	0	9	23.9	79	2,988	−32.8	
3-100755	211	339	39	23	5	15	25.2	387	21,318	81.8	

Table 4: Results from using Strategy III.

		Strategy I		Strategy II			Strategy III		
problem	duality gap	# B& B nodes	time	% gap closed	# B& B nodes	time	% gap closed	# B& B nodes	time
1-050161	2.2	61	59	68.1	17	51	86.5	9	49
1-050162	0.4	19	17	48.0	13	20	100.0	1	10
1-050163	2.3	143	129	82.0	13	48	85.0	11	35
1-050164	2.1	141	173	33.7	35	109	47.1	19	76
1-050165	0.9	43	43	52.3	15	35	92.2	3	21
1-050331	1.5	1,361	1,813	81.8	41	173	89.3	19	125
1-050332	1.2	797	860	65.7	129	476	70.1	35	208
1-050333	0.2	31	66	0.0	31	66	100.0	1	23
1-050334	1.0	1,349	2,606	10.8	175	975	53.8	69	488
1-050335	1.6	1,617	2,050	78.4	49	255	81.8	25	124
1-050501	0.3	143	278	100.0	1	54	100.0	1	56
1-050502	0.1	67	116	37.2	7	93	96.8	3	52
1-050503	0.4	361	681	32.4	7	107	32.4	7	79
1-050504	0.2	123	283	38.9	47	356	54.1	7	96
1-050505	0.0	1	75	–	–	–	–	–	–
1-100751	0.7	4,077	22,977	40.9	611	10,560	44.2	251	6,851
1-100752	0.6	15,419	74,351	55.4	1,243	20,055	56.7	1,253	33,186
1-100753	0.1	183	761	12.5	59	844	43.1	17	712
1-100754	0.3	6,687	40,604	9.1	537	11,076	11.0	195	5,514
1-100755	0.1	117	621	34.6	23	406	62.9	5	345
2-050161	3.7	129	145	77.6	19	58	80.1	13	69
2-050162	2.6	95	103	30.3	13	64	30.3	13	72
2-050163	0.5	19	30	51.5	9	36	86.0	3	15
2-050164	2.5	345	512	62.0	23	68	66.7	15	72
2-050165	0.0	1	15	–	–	–	–	–	–
2-050331	1.5	399	686	86.0	31	125	92.1	11	115
2-050332	1.2	691	1,560	54.3	51	450	67.2	27	290
2-050333	1.5	259	556	54.1	89	769	62.1	25	177
2-050334	0.7	239	493	76.6	23	213	81.9	11	97
2-050335	1.3	685	1,232	78.3	49	248	88.0	14	124
2-050501	0.2	143	296	82.5	7	76	85.1	5	71
2-050502	1.0	1,625	4,050	59.4	225	1,551	60.6	77	734
2-050503	0.1	45	164	0.0	31	170	94.3	5	97
2-050504	0.1	33	129	0.0	31	153	100.0	1	60
2-050505	0.02	3	67	0.0	3	67	95.6	3	62
2-100751	0.1	59	413	35.0	45	882	58.4	9	234
2-100752	0.3	429	2,540	15.8	239	6,835	32.2	49	1,590
2-100753	0.3	541	5,312	25.0	73	1,792	48.1	43	1,225
2-100754	0.3	7,817	69,326	5.9	95	2,235	28.2	77	2,240
2-100755	0.6	5,049	46,111	66.4	105	1,803	72.8	43	1,407
3-050161	4.1	127	160	52.3	13	76	64.2	9	58
3-050162	5.4	109	189	53.6	15	98	60.9	11	69
3-050163	1.3	17	38	21.0	9	55	99.7	3	26
3-050164	2.0	31	104	82.4	15	80	91.3	3	31
3-050165	2.1	7	23	39.7	5	30	100.0	1	14
3-050331	2.5	187	1,727	64.4	107	1,076	67.6	47	474
3-050332	2.9	766	5,188	66.2	83	495	73.4	49	521
3-050333	1.6	249	626	0.2	77	591	47.4	39	394
3-050334	0.8	57	194	27.8	25	120	45.8	11	108
3-050335	2.1	193	844	42.2	45	260	53.3	23	254
3-050501	1.4	299	1,479	66.0	37	352	85.6	9	212
3-050502	1.1	443	1,949	17.4	125	987	21.7	71	1,317
3-050503	0.9	1,003	3,162	6.3	125	997	29.1	67	674
3-050504	0.5	273	666	0.0	79	693	25.5	55	800
3-050505	0.7	199	686	12.0	73	396	54.0	31	326
3-100751	0.6	477	4,057	0.4	379	8,653	46.1	207	5,869
3-100752	0.9	1,289	12,834	15.8	213	6,923	36.7	127	5,249
3-100753	1.0	545	8,808	19.4	87	2,553	39.5	53	2,888
3-100754	0.6	235	2,250	12.1	137	3,925	23.9	79	2,988
3-100755	1.1	6,135	117,058	18.7	461	13,529	25.2	387	21,318

Table 5: Summary of the results.

	Strategy I	Stategy II		Strategy III	
problem type	average time/node	average time/node	factor increase	average time/node	factor increase
50×16	1.5	4.1	2.7	5.3	3.5
50×33	3.0	6.0	2.0	8.5	2.8
50×50	3.0	7.1	2.4	12.5	4.2
100×75	8.7	22.7	2.6	32.8	3.8

Table 6: Average time spent per branch-and-bound node, and the factor of increase relative to Strategy I.

6 Concluding Remarks

We have studied the cutting plane approach to solving CFL. The computational experience shows the importance of adding violated inequalities to the linear relaxation of CFL. By adding aggregate variables and constraints to the standard formulation of CFL, general purpose mixed integer optimizers such as MINTO and MPSARX (Van Roy and Wolsey (1987)) will recognize the surrogate knapsack and single-node flow polytopes (7) and (8), and generate knapsack cover and flow cover inequalities as part of the system procedures. Therefore, even if we used redundant variables and constraints, we almost always gained in reduction of the duality gap, and thereby in the total time needed to solve the problem. This supports the choice of formulation (1)–(6). Experience showed that the class of knapsack cover inequalities is the single most important one for closing the duality gap. However, we conclude that in most cases more problem specific inequalities are needed to tighten the duality gap sufficiently. The heuristic suggested in Section 4 for identifying violated submodular inequalities appears effective, in particular for the more tightly capacitated instances. The results were less satisfactory for the larger, relatively uncapacitated instances. Hence, it seems useful to improve the heuristic based on the class of combinatorial inequalities, as well as to develop new classes of strong inequalities. The new classes of inequalities should, ideally, rely less on capacities than the effective capacity or single depot inequalities, and have a support graph that to a greater extent corresponds to the structure of a typical fractional solution than the support graph corresponding to the combinatorial inequalities.

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