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The Complexity of Interval Routing on Random Graphs *

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Abstract. Several methods exist for routing messages in a network without using complete routing tables (compact routing). In k-interval routing schemes (k-IRS), links carry up to k intervals each. A message is routed over certain link if its destination belongs to one of the intervals of the link. We give some results for the necessary value of k in order to achieve shortest path routing. Even though for very structured networks low values of k suffice, we show that for 'general graphs' interval routing cannot significantly reduce the space-requirements for shortest path routing. In particular we show that for suitably large n, there are suitable values of p such that for randomly chosen graphs $G \in \mathcal{G}_{n,p}$ the following holds, with high probability: if G admits an optimal k-IRS, then $k = \Omega(n^{1-\frac{6}{\ln(np)}-\frac{\ln(np)}{\ln n}})$. The result is obtained by means of a novel matrix representation for the shortest paths in a network.

1 Introduction

Routing messages is a fundamental operation in distributed systems. But how much information needs to be stored in the processors in order to do it well? For a better definition of the problem, we will model a network of processors by a connected, undirected graph of n nodes and e edges: the nodes represent the processors, and the edges represent the communication links. Given a cost function on the edges, it is natural to ask for shortest path routes between any source and destination in the network. For the purposes of this paper we will assume that edges have positive cost and that the routing method must identify *some* shortest path for every source-destination pair.

Shortest routes can be represented trivially by storing a complete routing table at each node v that specifies, for each destination u, a link incident to v which is on a shortest path from v to u. In large networks of processors where processor memory is at a premium, this is space-consuming and thus unattractive. A complete routing table has n entries of $\log d$-bits each, where d is the degree of the node where it is stored. For the entire network this means a total of $\sum_{v=1}^{n} n \log d_v = \mathcal{O}\ (n^2 \log \frac{e}{n})$ bits (this follows from the inequality for the geometric and arithmetic means using $d_1 + \ldots + d_n = 2e$). It is an intriguing question whether this bound can be reduced to e.g. $\mathcal{O}\ (n^2)$. Fraigniaud and Gavoille [21] (see also [32]) recently proved that any general shortest-path routing scheme requires $\Omega(n^2)$ bits total at the nodes.

Practical networking environments have introduced the need for devising routing methods with provably smaller tables (*compact routing*), even accepting the possibility of routing messages along

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paths which are not necessarily shortest. Prior research activities have identified useful classes of network topologies for which the shortest path information can indeed be coded more succinctly at the nodes, usually with a different regime for the routing decisions as well. Tables typically use in the order of $d \log n$ bits per node, hence $\mathcal{O}(e \log n)$ bits total. See [41, 42] for an overview. We will consider the most popular of these techniques called 'interval routing' and will demonstrate that for general networks it is not likely to work as well.

Interval routing is based on a suitable naming scheme for the nodes and edges in a network. A node label is any element of the set $\{1, \ldots, n\}$, and an edge label is any 'interval' $[a, b]$ with $a, b \in \{1, \ldots, n\}$. Assuming that the labels 1 through n are arranged clockwise around a circle, an interval $[a, b]$ represents the node labels from a through b in clockwise order. (Thus for $n = 5$, the interval $[4, 2]$ represents the node labels 4, 5, 1, and 2.) A *k-interval labeling scheme* or k-ILS ([40]) consists of two components: (i) a node labeling such that every node gets a unique label, and (ii) at every node, some assignment of up to k intervals to each link leaving that node (an edge labeling). Note that the two endpoint of an edge may be assigned different sets of up to k edge labels. When none of the edge labels contains the interval $[n, 1]$, i.e. no interval 'wraps around', then the k-ILS is called *linear* [6, 7].

Given a k-ILS, a routing strategy is devised as follows. Let a message m be destined for some node v, suppose it has reached some node u on its way, and suppose $u \neq v$. Then an edge $e = (u, u\prime)$ incident to u is determined such that v (or rather, its node label) belongs to one of the edge labels (an interval) assigned to e at u, and m is transmitted over e from u to $u\prime$, to reach the next node on its way to v. If this routing strategy guarantees that messages always arrive at their destination, no matter where they originate and what their destination is, then a k-ILS is termed a *k-interval routing scheme* or k-IRS. If the implied routes are always the shortest, then the k-IRS is called *optimal*. (*Warning.* Earlier definitions of k-interval routing impose stricter requirements on the edge labels at a node and force the routing strategy to be deterministic.)

Interval routing is generally applicable. In fact, it can be shown that every network admits a 'non-trivial' 1-IRS, in which each edge has exactly one label at either end and for every node, the joint edge labels assigned to the edges at that node are a *partition* of $1, \ldots, n$ (thus the 'intervals' at every node are disjoint, all edges are utilized, and the routing strategy is fully deterministic, see [39, 40]). Many interesting processor interconnection networks have optimum k-interval routing schemes for small k (see e.g. [36, 40, 23, 25, 26, 19, 13]) and interval routing has been used as an ingredient in various other routing problems (see [2, 3, 32]). Interval routing is not only of theoretical interest but has also found industrial applications, e.g. in the design of the C104 router chip of the INMOS T9000 Transputer [31] and other practical applications [28, 43, 44, 45, 46]. Several variants are being explored as well, to obtain even more flexible compact routing schemes (like 'prefix routing' [8], 'Boolean routing', see [12, 15] or 'multi-dimensional' interval routing).

Most of results cited above show that interval routing can be efficiently used for specific topologies and, in general, for networks having particular symmetries or regularities. Observe that every network admits an optimal k-IRS for $k = \frac{1}{2}n$ (simply represent 'complete' routing tables and note that any assignment of more than $\frac{1}{2}n$ destinations to an edge can be represented by at most $\frac{1}{2}n$ intervals on the edge). But it appears to be much harder to decide whether a network admits an optimal k-IRS for some smaller value of k (see [16]). Some results exist that show that 1-IRS [34, 35, 37] and 2-IRS schemes [38] cannot even come close to optimality for some classes of networks. Indeed, when no specific assumption about the topology of a network is made, interval routing does not seem to reduce the space requirement for the routing information in the nodes significantly.

In this paper we formalize this observation by developing some lowerbound results for compact routing using intervals on general graphs:

(i) First we develop a technique for proving lowerbounds to the complexity of k-interval routing schemes. As an application we show that for each n there exists a graph G with n nodes and $O(n \log n)$ edges such that for each optimal k-IRS scheme of G one has $k = \Omega(n/\log n)$. Clearly this bound holds also for linear interval routing, and thus it improves the recent $\Omega(n^{1/3})$ lowerbound due to Kranakis, Krizanc, and Ravi [30]. The technique for proving the bound has been used recently by Gavoille and Guévremont [27] to construct graphs G with n nodes that actually require $k = \Theta(n)$.

(*ii*) Next we consider the problem of bounding the values of k for which random graphs on n nodes have an optimal k-IRS. We show that for the random graph model $\mathcal{G}_{n,p}$ there are suitable values of p such that with sufficiently high probability each optimal k-IRS scheme for a graph G has $k \geq \frac{1}{10} n^{1-\frac{6}{\ln(np)}-\frac{\ln(np)}{\ln n}}$. We also show that for each n there exist a p such that with sufficiently high probability each k-IRS scheme for a graph $G \in \mathcal{G}_{n,p}$ has $k \geq \frac{1}{20} n^{1-2\sqrt{6}/\sqrt{\ln n}}$. (Throughout, log denotes the logarithm to the base 2 and ln the natural logarithm.)

The second result can be viewed as a way of *constructing* many concrete networks that require a 'large k' for routing according to the k-IRS routing method in the classical case of uniform edge costs. Large k's are indeed the more common situation if we allow *dynamic* edge costs and require that a graph can be optimally routed by a k-IRS for every cost assignment to the edges. Recent work of Bodlaender *et al.* [9] shows that graphs G which have a node labeling such that for some fixed k, an optimal k-IRS can be designed for G for every choice of positive edge costs, have a *treewidth* bounded in terms of k. It is known from work of Kloks [29] that random graphs that have sufficiently many edges necessarily have a treewidth of $\Theta(n)$ and thus require $k = \Theta(n)$ for routing under this, stronger, dynamic edge costs model.

The paper is organized as follows. In section 2 we develop the technique for proving lower- and upperbounds on the number of edge labels (intervals) needed in an optimal k-IRS. It extends ideas used in Flammini, Gambosi and Salomone [12, 14, 16] for proving complexity results concerning the hardness of devising interval routing schemes with a minimum number of intervals and some approximation algorithms for this problem. We apply the technique to prove result (*i*). In sections 3 and 4 we develop the main results (*ii*) concerning the minimum number of intervals required by IRS schemes for random graphs. Some conclusions follow in section 5. The paper is a revised and extended version of [17].

2 A Lowerbound Technique for k-Interval Routing Schemes

Let $G = (V, E)$ be a graph with $|V| = n$. We want to be able to determine lowerbounds on the required value of k for routing in G according to some k-IRS. We will only consider routes that are optimal by some criterion. (Thus, the techniques in this section apply to minimum-hop paths as well as to other forms of 'shortest' paths.) The simple technique we develop in this section will be essential in the constructions later. In designing optimal k-IRS schemes, a simple observation learns that one can restrict to deterministic schemes. Also, one can project optimal schemes to optimal schemes for subsets $U \subseteq V$.

Proposition 1. (a) *If G admits an optimal k-IRS, then G admits an optimal k-IRS in which at every node v the assigned edge labels are all disjoint.*
(b) *For every $U \subseteq V$, any optimal k-IRS of G can be reduced to a k-IRS that is optimal for routing to destinations in U only.*

Assume that the set *Opt* of optimal routes is given. For nodes $v \in V$, let $I(v)$ be the set of edges incident to v. For nodes v and edges $e \in I(v)$, let $Opt(v, e)$ be the subset of nodes $u \in V$ optimally reachable from v through its outgoing link e. Any pair (v, e) with $e \in I(v)$ will be called a 'source-pair' (the beginning of a route from v over e). If the routes are defined by a k-IRS, then $Opt(v, e)$ is the union of the intervals assigned to e at v. (In fact, by proposition 1 we may assume also that $Opt(v, e) \cap Opt(v, e\prime) = \emptyset$ for $e \neq e\prime$ in this case.) Conversely, given a node labeling, the sets $Opt(v, e)$ will be instrumental in determining the interval labels needed in any optimal IRS for G. Minimizing the number of labels per edge requires that we observe the effect of all node labelings on the 'clustering' into intervals. To analyze this further, we consider a suitable representation of the optimal routes.

Definition 2. Given a set of destinations $U = \{u_1, ..., u_l\} \subseteq V$ and a set of source-pairs $S = \{(v_j, e_j) \mid e_j \in I(v_j), 1 \leq j \leq m\}$, the *Matrix Representation* of *Opt* w.r.t. U and S is the $l \times m$

Boolean matrix $M(U,S)$ with $M(U,S)[i,j] = 1$ if $u_i \in Opt(v_j, e_j)$ and $M(U,S)[i,j] = 0$ otherwise, for every $1 \le i \le l$ and $1 \le j \le m$.

Observe that the j-th column of $M(U,S)$ has a '1' precisely in the rows corresponding to destination nodes $u \in U$ which can be reached optimally from v_j over its outgoing link e_j.

An optimal k-IRS for G need not represent *all* optimal routes for every source-destination pair. Thus we cannot just consider all sets $Opt(v,e)$ and decompose them into unions of intervals, if we want to find a lowerbound on the number of intervals needed. We have to restrict to a 'kernel' of the routes which any k-IRS for G must represent. Thus we essentially look for sets U and S such that optimal routes to U have unique directions at the nodes in S. We also exclude routes 'back to source nodes' from consideration because k-IRS schemes do not send messages away if source and destination of a message are equal. This leads to the following definition.

Definition 3. Given a set of destinations U and a set of source-pairs S, *Opt* is called *unique* w.r.t. U and S if and only if it satisfies the following properties:

- for each $u \in U$ and $(v_j, e_j) \in S$ with $u \in Opt(v_j, e_j)$, there exists no $e \in I(v_j)$ with $e \neq e_j$ such that $u \in Opt(v_j, e)$.
- for each $(v_j, e_j) \in S$, $v_j \notin U$.

Informally, if *Opt* is unique w.r.t. U and S, then any optimal IRS scheme for G must reflect the optimal routes as given by *Opt* if we restrict our observation to U and S. This is exactly what the Matrix Representation of *Opt* w.r.t. U and S lets us do.

Definition 4. Let M be a $l \times m$ Boolean matrix. The Hamming score $c_M(i,j)$ of rows i and j in M is equal to $\frac{1}{2} d_H(i,j)$, where $d_H(i,j)$ is the Hamming distance between the i-th and j-th row as vectors. The Hamming index of M is equal to $Ind(M) = \min\{c_M(i,j) \mid 1 \le i,j \le l\}$.

Lemma 5. *Let Opt be unique w.r.t. U and S and let $M = M(U,S)$ be the Matrix Representation of Opt w.r.t. U and S. Then for any optimal k-IRS scheme for G: $k \ge \frac{l}{m} Ind(M)$, where $\mid U \mid = l$ and $\mid S \mid = m$.*

Proof. Let $U = \{u_1, \ldots, u_l\}$, let $S = \{(v_j, e_j) \mid e_j \in I(v_j), 1 \le j \le m\}$ and let $S\prime = \{v \mid (v,e) \in S \textit{ for some } e\}$. Using proposition 1 any optimal k-IRS for G must reduce to a k-IRS for G that is optimal if we restrict the set of destinations to U. If we now restrict the set of source-pairs to S as well, uniqueness of *Opt* w.r.t. U and S implies that at every node $v \in S\prime$ this k-IRS must assign disjoint interval labels to all edges e with $(v,e) \in S$ leaving v. (We don't care about the edges e with $(v,e) \notin S$.)

Considering the j-th column of M for any $1 \le j \le m$, it is clear that the blocks of 1's separated by 0's down the column 'and around' must precisely correspond to the intervals assigned to the edge e_j at v_j by the scheme, assuming that $u_1, \ldots, u_l$ are labeled in this order. The number of blocks, and hence the number of intervals needed, is equal to half the number of occurrences of the patterns 01 and 10 down the column and around. (Note that the number of occurrences of 01 is equal to the number of occurrences of 10.) This can be counted using Hamming scores. Summing for all columns, gives $\sum_{i=1}^{l} c_M(i, (i+1) mod\ l) \ge l \cdot Ind(M)$ blocks total. Hence there is a column in M which has at least $\frac{l}{m} \cdot Ind(M)$ blocks, and the corresponding edge e_j must have this many labels at v_j under the scheme.

The same argument applies if the nodes $u_1, \ldots, u_l$ are labeled in any other way and the rows of M are permuted accordingly, hence *any* k-IRS for G must assign at least $\frac{l}{m} Ind(M)$ intervals to some edge. The lemma follows.

The Matrix Representation can also be employed to construct k-IRS schemes, by taking $U = V$ and $S = \{(v,e) \mid v \in V, e \in I(v)\}$ (the set of 'all possible source-pairs') and representing *all* routes $\in Opt$ by the scheme. Minimizing the maximum number of 1-blocks per column over all possible permutations of the rows of $M = M(U,S)$ (hence over all possible node-labelings) and translating from blocks to intervals gives an optimal k-IRS with the smallest k this way (but it may represent too many routes). If $c_M(i,j) \leq d$ for all $1 \leq i,j \leq |U| = n$, then by the argument of lemma 5 any node-labeling will lead to an optimal IRS with at most $d.n$ labels total.

Kranakis, Krizanc, and Ravi [30] recently showed that for every n there exists graphs on n nodes for which any optimal *linear* k-IRS requires $k \geq \Omega(n^{1/3})$. Lemma 5 enables us to improve the bound considerably, even for optimal k-IRS that are not necessarily linear.

Proposition 6. *For every n there exist a graph G with n nodes and $O(n \log n)$ edges such that for each optimal k-IRS scheme of G one has $k = \Omega(n/\log n)$.*

Proof. It is sufficient to show that for each integer r there exists a graph G with $n = (3r + 2^r)$ nodes and $(2r + r.2^r)$ edges such that any optimal k-IRS for G has $k \geq \frac{2^{r-1}}{r}$. For integers j let $bin(j)$ denote the binary representation of j. Define $G = (V, E)$ where $V = V_1 \cup V_2 \cup V_3 \cup V_4$ and $E = E_1 \cup E_2 \cup E_3 \cup E_4$ are as follows:
$V_1 = \{v_i \mid i = 1, \ldots, r\}$, $V_2 = \{w_{0,i} \mid i = 1, \ldots, r\}$, $V_3 = \{w_{1,i} \mid i = 1, \ldots, r\}$, $V_4 = \{u_j \mid j = 0, \ldots, 2^r - 1\}$, $E_1 = \{e_{0,i} = (v_i, w_{0,i}) \mid i = 1, \ldots, r\}$, $E_2 = \{e_{1,i} = (v_i, w_{1,i}) \mid i = 1, \ldots, r\}$, $E_3 = \{(w_{0,i}, u_j) \mid i = 1, \ldots, r, j = 0, \ldots, 2^r - 1 \textit{ and the } i-th \textit{ bit of } bin(j) \textit{ is } 0\}$, and $E_4 = \{(w_{1,i}, u_j) \mid i = 1, \ldots, r, j = 0, \ldots, 2^r - 1 \textit{ and the } i-th \textit{ bit of } bin(j) \textit{ is } 1\}$. G has the desired number of nodes and edges.

Consider the optimal routes in G from V_1 to V_4. Given any node $v_i \in V_1$, a node $u_j \in V_4$ is reachable optimally over precisely one of the two outgoing edges $e_{0,i}$ and $e_{1,i}$. Taking $U = V_4$ and $S = \{(v_i, e_{1,i}) \mid 1 \leq i \leq r\}$, it easily follows that the set of optimal routes of G is unique w.r.t. U and S. Now consider $M = M(U, S)$ and observe that its rows are precisely the 2^r different r-bit strings. Thus $Ind(M) = \frac{1}{2}$. By lemma 5 we conclude that any optimal k-IRS for G requires $k \geq \frac{2^r}{r} \cdot \frac{1}{2} = \frac{2^{r-1}}{r}$.

The lowerbound technique of this section has been used recently by Gavoille and Guévremont [27] to construct graphs G with n nodes that actually require $k = \Theta(n)$. They also proved that for bounded-degree networks there are graphs of n nodes that require $k = \Omega(\sqrt{n})$. A similar technique was used in [21]. We need the lowerbound technique in Section 4.

3 Interval Routing in Random Graphs: Preliminaries

In this section and the next we consider the main topic of this paper, the complexity of interval routing in random graphs. We adopt the $\mathcal{G}_{n,p}$ model of random graphs, which consists of all graphs with node set $V = \{1, \ldots, n\}$ and edges chosen independently with probability p (see [10]). We will assume throughout that $\frac{\ln^{1+\epsilon} n}{n} < p < n^{-0.5-\epsilon}$ for some $\varepsilon > 0$. The (lower) bound on p guarantees that a graph in $\mathcal{G}_{n,p}$ is indeed connected, with high probability ([10], Chapter IV). We claim that optimal k-IRS schemes for graphs $G \in \mathcal{G}_{n,p}$ are likely to require a 'large k'. In this section we give some preliminaries and prove some useful facts concerning the $\mathcal{G}_{n,p}$ model. In the next section we prove the claim.

Let G be a random graph from $\mathcal{G}_{n,p}$. Let $d(u, v)$ denote the length of the shortest path between u and v.

Definition 7. Given a node $v \in V$, we say that $u \in V$ is *uniquely reachable* from v if and only if all shortest paths from v to u exit v over the same edge (v, w) for some w. For any $h > 0$, let $D(v, h)$ be the set of nodes u with $d(v, u) = h$ and let $U_{v,h} \subseteq D(v, h)$ be the set of nodes u that are uniquely reachable from v and have $d(v, u) = h$.

Observe that a node $u \in V$ is not uniquely reachable from some node v if there exist at least two optimal paths between v and u that exit at v through two different outgoing edges. Let us call such paths *not v-unique* (or *not i-unique* if nodes are indexed and $v = v_i$).

Now let $c = \lfloor \frac{\ln n}{\ln(np)} \rfloor = \frac{\ln n}{\ln(np)} - \beta$ (for some $0 \leq \beta < 1$). Consider some set of c source nodes $S\prime = \{v_1, ..., v_c\}$ in G, and let $\mathcal{U} = \bigcap_{i=1}^{c} U_{v_i,c}$ be a set of destinations. $\mathcal{U}$ is the intersection of the sets of nodes at distance c uniquely reachable from each v_i. We will exploit the technique of Section 2 to prove a lowerbound on k for any optimal k-IRS for G.

Informally, the intuition behind the proof is that the set $\mathcal{U}$ is sufficiently large with high probability (as shown in theorem 13). Now the number of intervals needed at the nodes $v_1, ..., v_c$ to route messages in G along shortest paths, is at least the number of intervals required to optimally route messages to nodes $\in \mathcal{U}$. By the independence of the choice of the edges in $G \in \mathcal{G}_{n,p}$, the probability that two given nodes $\in \mathcal{U}$ are reachable optimally over the same edge in each v_i vanishes for $n \to \infty$. This implies that if we try to reduce the number of intervals on some of the outgoing edges at the nodes $v_1, ..., v_c$, then we necessarily have to increase the number of intervals on the others. Hence we cannot have an optimal IRS-scheme with a low number of intervals on each edge. To make this argument precise, we need to analyze the structure of G. Informally, random graphs G are rather 'tree-ish' in the neighborhood of a node v_i up to distance $c-1$.

Bollobás ([10], p.230-233) proved very precise bounds on the cardinality of the set of nodes that are at a given distance (or within a given distance) from a node v.

Lemma 8. *[10] Let $K > 14$ be a constant. If n is sufficiently large, then with probability $\geq 1-n^{-K}$, for every node v and every natural number h with $1 \leq h \leq c-1$:*

$$\left|\, | D(v,h) | - (np)^h \,\right| \leq (np)^h/4$$

and

$$\sum_{j=1}^{h} | D(v,j) | \leq 2(np)^h.$$

The lemma enables us to prove bounds on the probability that a random node u is at some specified distance from any of the source nodes.

Lemma 9. *Given a random node u then, if n is sufficiently large, for each vertex v_i, $i = 1, 2..., c$ and for any natural number h with $1 \leq h \leq c-1$:*

$$\frac{2(np)^h}{3n} \leq Prob(d(v_i,u) = h) \leq \frac{4(np)^h}{3n}$$

and

$$Prob(d(v_i,u) \leq h) \leq \frac{7(np)^h}{3n}.$$

Proof. Let $v = v_i$. Observe that $Prob(d(v,u) = h) = E(|D(v,h)|)/n$. By lemma 8 we have, for K large enough:

$$E(D(u,h)) \geq (1-n^{-K})\frac{3}{4}(np)^h > \frac{2}{3}(np)^h$$

and

$$E(D(u,h)) \leq (1-n^{-K})\frac{5}{4}(np)^h + n^{-K}.n < \frac{4}{3}(np)^h.$$

The lemma follows.

In order to prove that U is 'large enough', we need two further lemmas. We will use that

$$(np)^{c-1}p = e^{(\frac{\ln n}{\ln(np)} - \beta - 1)\ln(np)}p = \frac{1}{(np)^{\beta}},$$

which implies that for n sufficiently large:

$$(np)^{c-1} \leq \frac{1}{p} \leq \frac{n}{\ln^{1+\epsilon} n} \leq \frac{n}{4},$$

$$(np)^{c-1} \geq \frac{1}{np^2} \geq n^{2\epsilon},$$

and, since $np \geq \ln^{1+\epsilon} n$:

$$\frac{c(np)^{c-1}}{n} = \frac{c}{(np)^{1+\beta}} \leq \frac{\ln n}{np\ln(np)} \leq \frac{1}{(1+\varepsilon).\ln^{\varepsilon} n.\ln\ln n} = o(1).$$

Observe from lemma 8 that $\sum_{j=1}^{c-1} | D(v,j) | \leq 2(np)^{c-1} \leq n/2$, hence with probability $\geq (1-n^{-K})$ there are at least $n/2$ nodes at distance $\geq c-1$ from v for n sufficiently large.

Lemma 10. *Given $v_i \in S\prime$ and $u \in V$, the probability that there exist at least two not i-unique paths of length $c-1$ between v_i and u is $\leq 2(np)^{2c-2}n^{-2}$ $(= \frac{2}{(np)^{2+2\beta}})$.*

Proof. The probability that there exist at least two *completely disjoint* not i-unique paths of length $c-1$ between v_i and u is less than or equal to the expected value of the number of pairs of completely disjoint, not i-unique paths of length $c-1$ between v_i and u, which is:

$$\binom{n-2}{c-2}(c-2)!p^{c-1}\binom{n-c}{c-2}(c-2)!p^{c-1} \leq \left[\binom{n}{c-2}(c-2)!p^{c-1}\right]^2 \leq (np)^{2c-2}n^{-2}.$$

Let two not i-unique paths between v_i and u intersect first at the j-th edge (that is, the j-th edge is the first common edge) for some $j \leq c-1$. This happens precisely in the case of event A defined by: "there exists a node v such that there are at least two completely disjoint not i-unique paths p_1 and p_2 of length $j-1$ between v_i and v, and there is a path of length $c-j$ between v and u that is completely disjoint from p_1 and p_2". *Prob*(A) is bounded by:

$$(n-2)\left[\binom{n-2}{j-2}(j-2)!p^{j-1}\right]\left[\binom{n-j}{j-2}(j-2)!p^{j-1})\right]\left[\binom{n-2j+2}{c-j-1}(c-j-1)!p^{c-j}\right] <$$

$$(n-2)\left[\binom{n-2}{j-2}(j-2)!p^{j-1}\right]^2\left[\binom{n-2j+2}{c-j-1}(c-j-1)!p^{c-j}\right] \leq (np)^{c+j-2}n^{-2}.$$

We can now prove the lemma. If two paths are not i-unique, they can intersect only somewhere *after* the second edge or not at all. Thus, for n sufficiently large, the probability that there are at least two not i-unique paths between v_i and u is bounded by:

$$(np)^{2c-2}n^{-2} + \sum_{j=3}^{c-1}(np)^{c+j-2}n^{-2} \leq 2(np)^{2c-2}n^{-2}.$$

As $(np)^{c-1}p = \frac{1}{(np)^{\beta}}$ it follows that $2(np)^{2c-2}n^{-2} = \frac{2}{(np)^{2+2\beta}}$.

Lemma 11. *Given $v_i \in S\prime$ and $u \in V$ then, for n sufficiently large:*

$$Prob(d(v_i,u) = c-1 \, and \, for \, each \, j \neq i \;\; d(v_j,u) > c-1) \geq \frac{(np)^{c-1}}{2n} \; (= \frac{1}{2(np)^{1+\beta}}).$$

Proof. Assume there exists a path $p(v_i,u)$ of length $c-1$ between v_i and u, and let the nodes on the path be numbered $u_0 = v_i, u_1, \ldots .u_{c-1} = u$. Consider a node v_j with $j \neq i$. A path from v_j to u of length $\leq c-1$ can exist only if for some l, $1 \leq l \leq c-1$, there exists a completely disjoint path of length $\leq l$ from v_j to u_l.

Let $A(i,j,u)$ be the event: "there exists a path of length $c-1$ between v_i and u, and a path of length $\leq c-1$ between v_j and u", and let $X(i,j,u)$ be the random variable denoting "the number of pairs of paths such that the first one is a path of length $c-1$ between v_i and u, and the second one is a path of length $\leq l$ between v_j and u_l for some l". Then:

$$Prob(d(v_i,u) = c-1 \; and \, for \; each \, j \neq i, \;\; d(v_j,u) > c-1) =$$

$$Prob(d(v_i,u) = c-1 \; and \, for \; each \, j \neq i, \;\; \neg A(i,j,u)) \geq$$

$$Prob(d(v_i,u)=c-1) - Prob(\textit{there exists a } j \neq i \textit{ s.t. } A(i,j,u)),$$

where we have used that $Prob(X \cap \neg Y) \geq Prob(X) - Prob(Y)$. $Prob(A(i,j,u))$ can be bounded as follows:

$$Prob(A(i,j,u)) = Prob(X(i,j,u) \geq 1) \leq E(X(i,j,u)) \leq$$

$$\binom{n-2}{c-2}(c-2)!\,p^{c-1}\sum_{l=1}^{c-1}\left(\sum_{k=1}^{l}\binom{n-c-1}{k-1}(k-1)!p^k\right) \leq$$

$$n^{c-2}p^{c-1}\sum_{l=1}^{c-1}2n^{l-2}p^{l-1} \leq n^{c-2}p^{c-1}4n^{c-2}p^{c-1} \leq 4(np)^{2c-2}n^{-2}.$$

Summing over all possible nodes $v_j \in S\prime$, we obtain

$$Prob(\textit{there exists a } j \neq i \textit{ s.t. } A(i,j,u)) \leq 4c(np)^{2c-2}n^{-2}.$$

Using lemma 9 and the fact that $\frac{c(np)^{c-1}}{n} = o(1)$ we obtain, for suitably large n:

$$Prob(d(v_i,u) = c-1 \textit{ and for each } j \neq i,\ \neg A(i,j,u)) \geq$$

$$Prob(d(v_i,u)=c-1) - Prob(\textit{there exists a } j \neq i \textit{ s.t. } A(i,j,u)) \geq$$

$$\frac{2(np)^{c-1}}{3n} - 4c(np)^{2c-2}n^{-2} = \frac{(np)^{c-1}}{n}\left(\frac{2}{3} - \frac{4c(np)^{c-1}}{n}\right) \geq \frac{(np)^{c-1}}{2n} = \frac{1}{2(np)^{1+\beta}}.$$

We can now put things together and prove that, with sufficiently large probability, the set $\mathcal{U}$ is sufficiently large. In the sequel we will use the following inequalities: $(1-x) \leq e^{-x}$, $(1-x/2) \geq e^{-x}$ for $0 < x < 1$, $(1-\delta)^h > 1 - \delta h$ for $h > 1$ and $0 < \delta < 1$. We will also use the following estimates due to Chernoff [11] (the 'Chernoff bounds', see also [5]) on the tails of the binomial distribution:

Lemma 12. *Let $0 \leq p \leq 1$, let X be a Bernoulli variable with success probability p, let $0 \leq \gamma \leq 1$, and let m be any positive integer. The probability of at most $(1-\gamma)mp$ successes in m independent trials of X is less than $e^{-\gamma^2 mp/2}$. The probability of at least $(1+\gamma)mp$ successes is less than $e^{-\gamma^2 mp/3}$.*

Recall that $\mathcal{U} = \bigcap_{i=1}^{c} U_{v_i,c}$ is the intersection of the sets of nodes at distance c uniquely reachable from each node $v_i \in S\prime$.

Theorem 13. *For n sufficiently large we have, with probability $1 - o(1)$:*

$$|\,\mathcal{U}\,| \geq \frac{1}{4}\left(\frac{e^{-\frac{5}{2(np)^{\beta}}}}{8}\right)^c n^{1+\beta^2-\beta}p^{\beta^2}.$$

Proof. For each i write $U_{v_i,c-1} = U_i' \cup U_i''$, where U_i' is the set of nodes $u \in U_{v_i,c-1}$ with $d(v_j,u) > c-1$ for each $j \neq i$, and U_i'' is the set of nodes $u \in U_{v_i,c-1}$ with $d(v_j,u) \leq c-1$ for some $j \neq i$. Now observe that, if a node u has distance at least c from v_i and is connected to exactly one node in U_i' and *not* connected to any node in U_i'' for each i, then it has distance c from each v_i and is uniquely reachable from each v_i and, consequently, $u \in \mathcal{U}$. Thus, a lower bound on the number of nodes u with the stated property is also a lower bound on $|\,\mathcal{U}\,|$.

We will first estimate the size of U_i' and U_i'', and then estimate the number of u's. By definition, U_i' is the set of nodes u satisfying the event $A(i,u)$ defined by: "$d(v_i,u) = c-1$ and u is uniquely reachable from v_i and for each $j \neq i$ $d(v_j,u) > c-1$ ". Thus, given $v_i \in V$ and using lemmas 10 and 11 and the fact that $\frac{(np)^{c-1}}{n} < \frac{c(np)^{c-1}}{n} = o(1)$, we have for suitably large n:

$$Prob(A(i,u)) \geq$$

$$Prob(d(v_i,u) = c-1 \textit{ and for each } j \neq i,\ d(v_j,u) > c-1) -$$

$Prob(there\ exist\ at\ least\ two\ not\ i-unique\ paths\ of\ length\ c-1\ from\ v_i\ to\ u\ )\geq$

$$\frac{(np)^{c-1}}{2n}-\frac{2(np)^{2c-2}}{n^2}\geq\frac{(np)^{c-1}}{4n},$$

where we have used the inequality $Prob(X\cap Y\cap Z)\geq Prob(X\cap Z)-Prob(\neg Y)$.

This implies that $E(|U_i'|)\geq\frac{(np)^{c-1}}{4}$, where $E(|U_i'|)$ denotes the expected value of $|U_i'|$. By applying Chernoff's bound it follows that for suitably large n, the probability that $|U_i'|\leq\frac{(np)^{c-1}}{8}$ is at most $e^{-\frac{(np)^{c-1}}{32}}$. Thus the probability that for each i, $|U_i'|\leq\frac{(np)^{c-1}}{8}$ is at most $ce^{-\frac{(np)^{c-1}}{32}}$ and, hence, the probability that for each i $|U_i'|\geq\frac{(np)^{c-1}}{8}$ is at least $1-ce^{-\frac{(np)^{c-1}}{32}}$.

An upper bound on $|U_i'|$ and $|U_i''|$ is immediately obtained using lemma 8. In fact, $|U_i'|$ and $|U_i''|$ are both $\leq|D(v_i,c-1)|$ for each i and the probability that for each i, $|U_i'|$, $|U_i''|\leq\frac{5(np)^{c-1}}{4}$ is at least $1-cn^{-K}$, with K as in the lemma. In fact, if we let $m_i'=|U_i'|$ and $m_i''=|U_i''|$, then the given bound is actually a bound on $m_i'+m_i''$.

Let us now use the above bounds on $|U_i'|$ and $m_i'+m_i''$, and count the number of 'special' nodes $u\in\mathcal{U}$. For any node u such that for each i $d(v_i,u)\geq c$, the probability that u is connected to exactly one node in U_i' and no nodes in U_i'' for each i is equal to:

$$m_1'p(1-p)^{m_1'-1}(1-p)^{m_1''}\cdot\ldots\cdot m_c'p(1-p)^{m_c'-1}(1-p)^{m_c''}\geq$$

$$(\frac{(np)^{c-1}}{8}p(1-p)^{\frac{5(np)^{c-1}}{4}})^c=(\frac{(np)^{c-1}p}{8}(1-p)^{\frac{5}{4}(np)^{c-1}})^c\geq$$

$$(\frac{(np)^{c-1}p}{8}e^{-\frac{5}{2}(np)^{c-1}p})^c=(\frac{e^{-\frac{5}{2(np)^\beta}}}{8(np)^\beta})^c=(\frac{e^{-\frac{5}{2(np)^\beta}}}{8})^c(\frac{1}{(np)^c})^\beta=$$

$$(\frac{e^{-\frac{5}{2(np)^\beta}}}{8})^c(\frac{1}{(np)^{c-1}p})^\beta(\frac{p}{np})^\beta=(\frac{e^{-\frac{5}{2(np)^\beta}}}{8})^c(np)^{\beta^2}\frac{1}{n^\beta}=(\frac{e^{-\frac{5}{2(np)^\beta}}}{8})^c n^{\beta^2-\beta}p^{\beta^2},$$

where we have used that $(np)^{c-1}p=\frac{1}{(np)^\beta}$.

We have noted before that for n suitably large, with probability $\geq(1-n^{-K})$ there are at least $n/2$ nodes u at distance $\geq c-1$ from v, with K as before. Hence an expected number of $(\frac{e^{-\frac{5}{2(np)^\beta}}}{8})^c n^{1+\beta^2-\beta}p^{\beta^2}$ nodes u will have the desired special property. By applying Chernoff's bound with $\gamma=1/2$, $m=n/2$ and $p'=(\frac{e^{-\frac{11}{4(np)^\beta}}}{8})^c n^{\beta^2-\beta}p^{\beta^2}$, it follows that the probability that $|\mathcal{U}|\geq\frac{(\frac{e^{-\frac{5}{2(np)^\beta}}}{8})^c n^{1+\beta^2-\beta}p^{\beta^2}}{4}$ is at least $1-e^{-\frac{(\frac{e^{-\frac{5}{2(np)^\beta}}}{8})^c n^{1+\beta^2-\beta}p^{\beta^2}}{16}}$.

Finally, the probability that for each i, $m_i'\geq\frac{(np)^{c-1}}{8}$ and $m_i'+m_i''\leq\frac{5(np)^{c-1}}{4}$, that there exist at least $\frac{n}{2}$ nodes u such that for each i, $d(v_i,u)\geq c$ and, consequently, that $|\mathcal{U}|\geq\frac{(\frac{e^{-\frac{5}{2(np)^\beta}}}{8})^c n^{1+\beta^2-\beta}p^{\beta^2}}{4}$, is at least

$$(1-ce^{-\frac{(np)^{c-1}}{32}})(1-cn^{-K})(1-cn^{-K})(1-e^{-\frac{(\frac{e^{-\frac{5}{2(np)^\beta}}}{8})^c n^{1+\beta^2-\beta}p^{\beta^2}}{16}})=1-o(1).$$

4 Interval Routing in Random Graphs: A Lower Bound

In this section we prove the main result of this paper: for n sufficiently large and for suitable values of p, an optimal k-IRS scheme for a random graph $G\in\mathcal{G}_{n,p}$ requires $k=\Omega(n^{1-\frac{6}{\ln(np)}-\frac{\ln(np)}{\ln n}})=\Omega(\frac{1}{p}n^{-\frac{6}{\ln np}})$ with high probability. This leads to classes of random graphs for which optimal k-IRS schemes require $k=\Omega(n^{1-\delta})$, for any constant $\delta>0$.

The definitions and notations of Section 3 remain in effect. In the proof of lemma 15 we will need a version of the classical 'Coupon Collectors Problem' (see [33]). In this version of the problem there are N cells, and balls are distributed one by one into the cells. Let a trial consist of placing one ball into one cell. The trials are independent and in each trial the probability that a ball lands in a specified cell is N^{-1}. We are interested in T_s, the number of trials needed until for the first time precisely s cells have received at least one ball. Recall that, given a random variable X of expectation $E(X) = \mu$ and variance $Var(X) = \sigma^2$, $Prob(| X - \mu |< \varepsilon) \geq 1 - \frac{\sigma^2}{\varepsilon^2}$ (Chebyshev's inequality).

Proposition 14. *If $s \leq \frac{N}{4}$, then $Prob(T_s > 2s) \leq \frac{4}{N}$.*

Proof. By elementary probability theory one has:

$$E(T_s) = N\sum_{j=0}^{s-1} \frac{1}{N-j} \text{ and } VAR(T_s) = N\sum_{j=1}^{s-1} \frac{j}{(N-j)^2}.$$

For $s \leq \frac{N}{4}$ the following estimates can be made:

$$E(T_s) \leq N\sum_{j=0}^{s-1} \frac{4}{3N} = \frac{4}{3}s < \frac{3}{2}s$$

and

$$VAR(T_s) \leq \frac{16}{9}\frac{\sum_{j=1}^{s-1} j}{N} = \frac{8}{9}\frac{s(s-1)}{N} < \frac{s^2}{N}.$$

By applying Chebyshev's inequality one obtains:

$$Prob(T_s > 2s) \leq \frac{s^2}{N}\frac{4}{s^2} = \frac{4}{N}.$$

Let $G \in \mathcal{G}_{n,p}$ be a random graph, with $\frac{\ln^{1+\epsilon} n}{n} < p < n^{-0.5-\epsilon}$. Let $S\prime$ and $\mathcal{U} = \bigcap_{i=1}^{c} U_{v_i,c}$ be as defined in Section 3. We use the technique from Section 2, and some additional probabilistic considerations, to prove a lowerbound on the value of k that is likely to be needed in any optimal k-IRS scheme for G. Define s by

$$s = \frac{1}{8}(\frac{e^{-\frac{5}{2(np)^\beta}}}{8})^c n^{1+\beta^2-\beta} p^{\beta^2},$$

and observe that $s \leq \frac{1}{8}e^{-2c} n^{1+\beta^2-\beta} p^{\beta^2}$.

Lemma 15. *For n sufficiently large, any optimal k-IRS scheme for G must have*

$$k \geq \frac{1}{10}\frac{\ln(np)}{np\ln n}(\frac{e^{-\frac{5}{2(np)^\beta}}}{8})^c n^{1+\beta^2-\beta} p^{\beta^2},$$

with probability $1 - o(1)$.

Proof. Each node $u \in \mathcal{U}$ is optimally reachable from each v_i through a unique incident link, thus if we let $U = \mathcal{U}$ and $S = \{(v_i, e) \mid e \in I(v_i)\}$, then the set of optimal paths in G is unique w.r.t. U and S. Let $M = M(U, S)$. We will estimate $| U |$, $| S |$ and $Ind(M)$ so as to apply lemma 5. We know that $| U |= l \geq 2s$ with probability $1 - o(1)$.

To estimate $| S |$ observe from lemma 8 that with probability $1 - cn^{-K}$ we have for each i, $| D(v_i, 1) |\leq \frac{5}{4}np$. Thus $| S |= m \leq \frac{5}{4}cnp \leq \frac{5np\ln n}{4\ln(np)}$ with probability $1 - cn^{-K}$, with K as in the lemma. Bounding $Ind(M)$ poses more difficulties, because one might in fact have $Ind(M) = 0$. We will get around this by selecting a sufficiently large submatrix with all rows different.

By lemma 8 we know that for every i $|D(v_i,1)| \geq \frac{3}{4}np$ with probability at least $1-cn^{-K}$, with K as in the lemma. Assume that this bound on each $|D(v_i,1)|$ holds. We can view the columns of M as being divided into c disjoint groups, each group corresponding to the source-pairs $(v_i, e) \in S$ for some fixed v_i. By the common bound on the $|D(v_i,1)|$, each group will have $\geq \frac{3}{4}np$ columns. Considering any row of M as it is divided by the c groups of columns into sectors of at least $\frac{3}{4}np$ bits each, it follows from the uniqueness w.r.t. U and S that each sector of a row has exactly one of its bits set to 1, while the other bits in the sector are equal to 0. How many mutually different rows is M likely to have.

Consider the set R of all possible rows that satisfy the stated condition. It follows that $|R| \geq (\frac{3}{4}np)^c \geq (\frac{1}{e})^c \frac{n}{(np)^\beta}$, where we use that $(np)^{c-1}p = \frac{1}{(np)^\beta}$. By symmetry, each row $\in R$ occurs with equal probability as a row in M. Estimating the number of different rows in M can now be viewed as an instance of the Coupon Collectors Problem in the following way. The cells are the elements of R and a trial that places a ball into a certain cell corresponds to the actual occurrence (placement) of the corresponding row in M. In the notation used earlier we have $N \geq (\frac{1}{e})^c \frac{n}{(np)^\beta}$. Consider a series of trials and let's see how it fills up M, where we note that with probability $1-o(1)$, M has at least $2s$ rows and thus needs at least $2s$ trials with probability $1-o(1)$. Note that for n sufficiently large, $\frac{N}{s} \geq 8e^c n^{\beta^2} p^{\beta^2-\beta} \geq 4$ (using the above inequalities for N and s) and hence $s \leq \frac{N}{4}$. By proposition 14, $Prob(T_s > 2s) \leq \frac{4}{N}$ and it follows that with high probability M will have at least s mutually different rows.

Let $U\prime$ be the subset of U consisting of the s mutually different rows which are likely to occur in M. Then the set of optimal paths is unique w.r.t. $U\prime$ and S as well but now, with $M\prime = M(U\prime, S)$, we have $Ind(M\prime) \geq 1$ (because different rows will differ in at least two positions). By lemma 5 we conclude that necessarily

$$k \geq \frac{s}{m} \geq \left(\frac{(\frac{e^{-\frac{5}{2(np)^\beta}}}{8})^c n^{1+\beta^2-\beta} p^{\beta^2}}{8}\right) / \left(\frac{5np\ln n}{4\ln(np)}\right) = \frac{1}{10}\frac{\ln(np)}{np\ln n}\left(\frac{e^{-\frac{5}{2(np)^\beta}}}{8}\right)^c n^{1+\beta^2-\beta} p^{\beta^2}.$$

To end the proof, note that the probability that the above bounds on $|D(v_i,1)|$ for each i and on the number of different rows in $M(U,P)$ hold is at least

$$(1-cn^{-K})(1-cn^{-K})(1-o(1)) = (1-o(1)).$$

We now use the lemma to obtain the following theorems.

Theorem 16. *For any n, let $p = n^{-\frac{t-1}{t}}$ for an integer $t > 0$ such that $\frac{\ln^{1+\epsilon} n}{n} < p < n^{-0.5-\epsilon}$. Then, for n sufficiently large, an optimal k-IRS scheme for a random graph $G \in \mathcal{G}_{n,p}$ has $k \geq \frac{1}{10} n^{1-\frac{6}{\ln(np)}-\frac{\ln(np)}{\ln n}}$ $(= \frac{1}{10} n^{1-6t-\frac{1}{t\ln n}})$ with probability $1-0(1)$.*

Proof. In this case $c = \frac{\ln n}{\ln(np)} = t$ and $\beta = 0$. The theorem follows directly from lemma 15 by observing that for n sufficiently large and with probability $1-o(1)$, an optimal k-IRS scheme for G has

$$k \geq \frac{1}{10}\frac{\ln(np)}{np\ln n}\left(\frac{e^{-\frac{5}{2(np)^\beta}}}{8}\right)^c n^{1+\beta^2-\beta} p^{\beta^2} \geq \frac{1}{10}\frac{\ln(np)}{np\ln n} e^{-5c} n =$$

$$\frac{1}{10}\frac{\ln(np)}{np\ln n} n^{1-\frac{5c}{\ln n}} = \frac{1}{10}\frac{\ln(np)}{np\ln n} n^{1-\frac{5}{\ln(np)}} =$$

$$\frac{1}{10} n^{1-\frac{5}{\ln(np)}-\frac{\ln(np)}{\ln n}-\frac{\ln(\ln n/\ln(np))}{\ln n}} \geq \frac{1}{10} n^{1-\frac{6}{\ln(np)}-\frac{\ln(np)}{\ln n}}.$$

By a further choice of p one obtains:

Theorem 17. *For all n sufficiently large, there exist $p > 0$ such that for any random graph $G \in \mathcal{G}_{n,p}$, an optimal k-IRS scheme for G has $k \geq \frac{1}{20} n^{1-2\sqrt{6}/\sqrt{\ln n}}$ with probability $1-o(1)$.*

Proof. Take $t = \lceil \sqrt{\ln n}/\sqrt{6} \rceil$. In fact let $t' = \sqrt{\ln n}/\sqrt{6}$ and $\beta = t - t'$. By theorem 16 one has, with probability $1 - o(1)$:

$$k \geq \frac{1}{10} n^{1 - \frac{6}{\ln(np)} - \frac{\ln(np)}{\ln n}} = \frac{1}{10} n^{1 - \frac{6t}{\ln n} - \frac{1}{t}} = \frac{1}{10} n^{1 - \frac{6t'}{\ln n} - \frac{1}{t'}} n^{-\frac{6\beta}{\ln n} + \frac{\beta}{t'(t'+\beta)}}.$$

But for n sufficiently large, substituting the value of t' into the formula leads to

$$n^{-\frac{6\beta}{\ln n} + \frac{\beta}{t'(t'+\beta)}} \geq n^{-\frac{6}{\ln n} - \frac{1}{t'(t'+1)}} = n^{-\frac{1}{s'^2(s'+1)}} = e^{-\sqrt{6}/(t'+1)} \geq 1/2,$$

and the theorem follows.

5 Conclusion

In this paper we considered the complexity of optimal routing in processor networks by means of k-interval routing schemes. The technique has gained some popularity, because it is applicable to a reasonable range of commonly used networks, with reasonably small values of k. These networks are usually quite regular.

We have shown that in 'general' networks small values of k are the exception rather than the rule for optimal k-interval routing schemes. In fact, theorem 17 implies that for any small $\delta > 0$, there exist classes of random graphs $\mathcal{G}_{n,p}$ for any n sufficiently large such that with high probability an optimal k-IRS for a graph $G \in \mathcal{G}_{n,p}$ requires $k = \Omega(n^{1-\delta})$. The result is proved by means of a technique that seems to be useful for other lowerbound problems in interval routing and related problems as well.

Optimal k-interval routing on random graphs might actually require $k = \Theta(n)$ in many cases, but this remains to be proved. A result of this kind does hold in slightly stronger version of the problem, namely for dynamic optimal k-interval routing [9]. Many interesting problems seem to remain in the area.

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