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Abstract

When grasping a part with a parallel-jaw gripper, the part will generally rotate due to kinematic constraints. Predicting the part's final orientation is useful for grasping and for planning sequences of gripper motions to orient parts. For a given part geometry, the *grasp function* maps initial orientations to final orientations. Previously, we studied polygonal and algebraic parts in the absence of friction. This paper considers how Coulomb friction affects the grasp function.

We consider two models of Coulomb friction. For a deterministic model, we show that the grasp function of any polygonal part can be represented with a piecewise linear function that we call a step-ramp function. We then show that any step-ramp function is the grasp function of a curved part operating under zero friction. Both contain ranges of orientations where the part does not rotate when grasped. This yields our primary result, that any part with deterministic friction has equivalent grasp mechanics to a "dual" part under zero friction. We then apply previous results to derive grasp plans that orient parts in the presence of friction. We also give an algorithm for planning under a non-deterministic model of Coulomb friction, and give bounds on the friction coefficient needed to insure the existence of such plans.

1 Introduction

The parallel-jaw gripper is perhaps the most common type of gripper in industry. The quality of a grasp configuration depends, among other factors, on the orientation of the part with respect to the gripper. In previous papers we studied the problem of grasping and orienting parts in the plane under the simplifying assumption that there is zero friction between the part and the gripper [8, 18]. Since it is impossible to completely eliminate friction in practice, it is useful to consider how Coulomb friction, characterized by a coefficient of friction μ, affects these results.

From elementary physics, Coulomb's model of friction gives a bound on tangential friction force that is a linear function of the applied normal force. For a point in contact with a planar surface, consider an infinite cone, orthogonal to this surface, with half-angle $\alpha = \tan^{-1} \mu$, where μ is the coefficient of friction. Contact forces directed outside of the "friction cone" will result in slip at the contact point. Contact forces inside the cone are capable of sticking. It is important to note that Coulomb's model only provides a bound on friction forces; thus contacts directed into the cone may stick or slip: both are consistent with the model. When grasping a part between the jaws of a parallel-jaw gripper, point contact with the jaws may cause the part to become *wedged*, after which it will no longer rotate to permit further closing of the jaws.

In this paper we consider two models of Coulomb friction. Under the *deterministic* model, we assume that contact forces directed into the friction cone result in sticking. That is, wedging occurs whenever possible, yielding a unique final orientation of the part in the gripper. Under this model, the grasp function is single-valued and contains finite intervals where part does not rotate. This is also the case for curved parts operating under zero friction, which suggests a duality between friction and curvature.

We show that for any part grasped with Coulomb friction, there is a curved part with zero friction that has the same grasp function. In particular, this curved part under zero friction will have only linear or circular bounding arcs; we

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call such part shapes *generalized-polygonal (GP)*. See Fig. 1. In the limit where $\mu = \infty$, any planar part behaves, when grasped, like a disk with $\mu = 0$, *i.e.* it does not rotate. This observation allow us to use our previously developed planning algorithms to generate plans that orient planar parts under Coulomb friction.

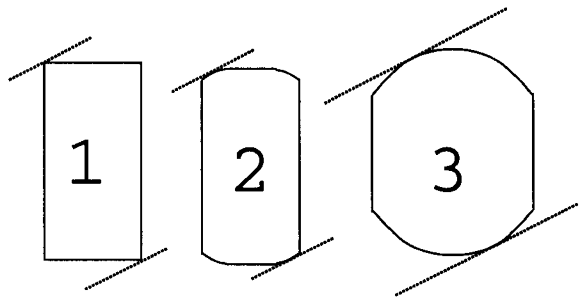


Figure 1: Three parts with the same grasp mechanics. Part 1 with $\mu = 0.1$ is dual to Part 2 operating under zero friction. Part 1 with $\mu = 0.5$ is dual to Part 3 operating under zero friction.

In a non-deterministic model of Coulomb friction, the final orientation of a grasped part cannot be uniquely predicted: contacts may stick, slip, or slip and then stick. Under this model, we describe a new planning algorithm for finding a plan, if one exists, for orienting a given part. We give bounds on the coefficient of friction required to insure the existence of such a plan.

1.1 Grasp Functions

Consider a given part ¶ on a planar worksurface and a parallel-jaw gripper surrounding the part and oriented at some angle with respect to the part. A *grasp action* at angle β is the combination of closing the jaws as far as possible (until further deformation would violate part rigidity), and then opening the jaws.

We assume that all motion is restricted to the plane and represent parts and the gripper by their planar projections. To predict parallel-jaw grasp mechanics, it is sufficient to consider the convex hull of the part, a simple closed planar convex curve, P. If P is polygonal, we say that ¶ is a polygonal part. If P is composed of arcs that can be described with polynomial equations of the form $g(x, y) = 0$, we say that ¶ is an algebraic part. The convex hull of a planar part can be computed in time linear in the number of its bounding arcs [20].

Let S^1 denote the space of planar orientations. A connected subset of S^1 with non-zero measure is called an *interval* of orientations. A *maximal interval* is an interval of maximal measure. Let $\Gamma : S^1 \to S^1$ be a function mapping angles to angles. A *fixed-point* of Γ is any angle b that is unchanged by the application of Γ: $\Gamma(b) = b$. A *step* in Γ is a maximal interval I satisfying

$$\exists b \in I \; \forall x \in I \; \Gamma(x) = b.$$

In other words, a step is a maximal interval containing a *single* included fixed-point. Γ is a *step function* if it partitions its domain S^1 into a finite number of steps. A *ramp* in Γ is a maximal interval I such that

$$\forall x \in I \; \Gamma(x) = x.$$

That is, every element of a ramp is a fixed-point. Finally, Γ is a *step-ramp function* if it partitions S^1 into a finite number of steps and ramps.

Under zero friction, the rotation of a planar part during grasping is represented by the *grasp function*, $\Gamma : S^1 \to S^1$, that maps initial orientations of the part into final orientations. See Fig. 2. Fixed-points in a grasp function are orientations of the part that do not change when it is grasped, *i.e.* stable orientations. Goldberg [8] proved the following.

Proposition 1 *The grasp functions of polygonal parts are step functions.*

In [16, 18], we extended this analysis to algebraic parts and showed that:

Proposition 2 *The grasp function of an algebraic part is a step-ramp function.*

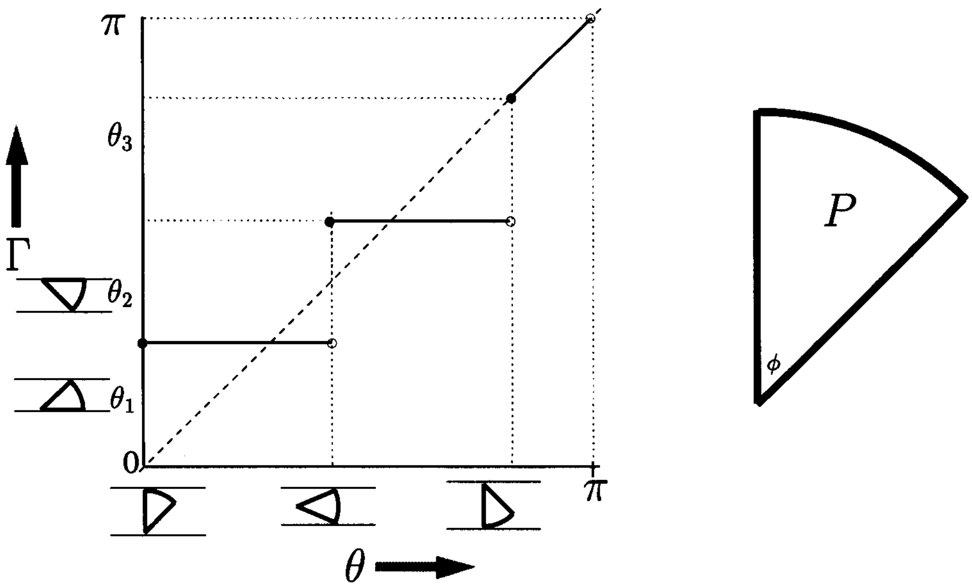


Figure 2: Grasp function for a pie-shaped part with $\phi = 40°$. Part is shown on the right oriented at 0° with respect to the gripper. The horizontal and vertical axes represent the initial and final orientations of the part, respectively. Due to symmetry in the gripper, the grasp function from $[\pi, 2\pi)$ is a copy of the function from $[0, \pi)$. Intersections of the function with the 45° line (shown dotted) give fixed-points. By a simple modification to the grasping process, we can show that the steps and ramps may be assumed left-closed and right-open [18] as indicated in the figure.

1.2 Frictional Contacts

Our previous analysis was simplified assuming zero friction between the part and gripper, *i.e.* the part never wedges. This was justified by adding a linear bearing between the jaws to greatly reduce friction [7]. However, since friction can never be eliminated completely, it is useful to consider its effect.

Consider a grasp contact configuration between the two parallel jaws of the gripper and the part as shown in Fig. 3. If the (open) line segment joining the contact points lies inside both (open) friction cones, we call this a *frictional configuration.*

For any $\mu > 0$, the set of frictional orientations, denoted as $\mathcal{F}$ in this paper, are a set of disjoint intervals from S^1, the space of planar orientations; it may include the whole of S^1, $\mathcal{F} \subseteq S^1$. In Section 2, we show that the set of frictional orientations is a *finite* set of intervals for algebraic parts. In that section, we also discuss the computation of these intervals of frictional orientations.

We use $\mathcal{F}$ to denote the set of frictional configurations. In Section 2, we show that for algebraic parts, $\mathcal{F}$ consists of a finite set of intervals.

1.3 Models of Coulomb Friction

We distinguish three models of Coulomb friction. See Fig. 4.

1. **zero friction model:** There is no friction between the part and the gripper. The part's final orientation is uniquely determined except at points where two contacts are exactly aligned. These orientations occur with probability zero and can be avoided with a a three-phase grasp as described in [18].
2. **deterministic friction model:** Wedging occurs whenever the part is in a frictional configuration. Again the part's final orientation can be uniquely predicted.
3. **non-deterministic friction model:** For a frictional orientation, the part either: (a) gets wedged at that orientation, (b) rotates to the orientation as if friction were absent, or (c) gets wedged at some intermediate orientation.

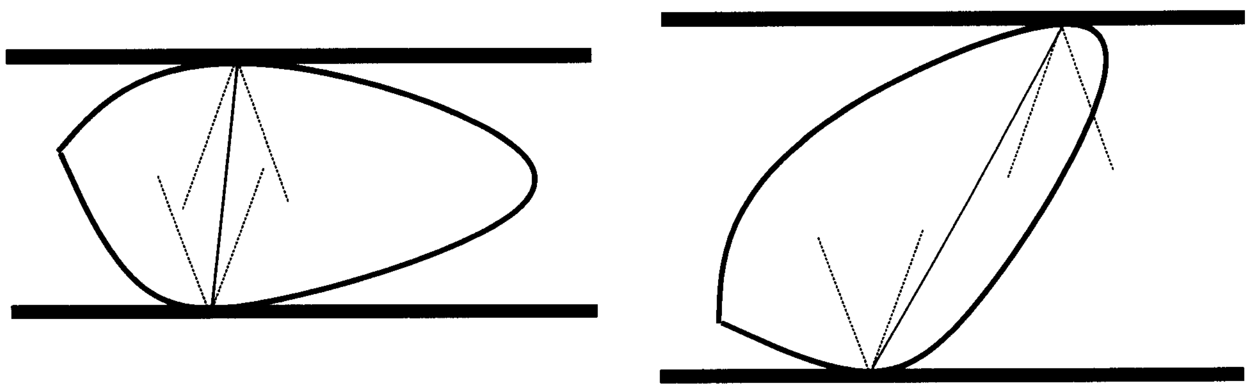


Figure 3: Contact configurations between a parallel-jaw gripper and a planar part. The planar part is shown by the convex hull of its planar projection. At each point of contact, draw a "friction cone" of width 2α pointing into the part and symmetrical about the normal to the part at that point. If two contact points exist such that the open line segment joining them lies in both contact friction cones, the contact configuration is called frictional. The configuration on the left is frictional, while the other is not.

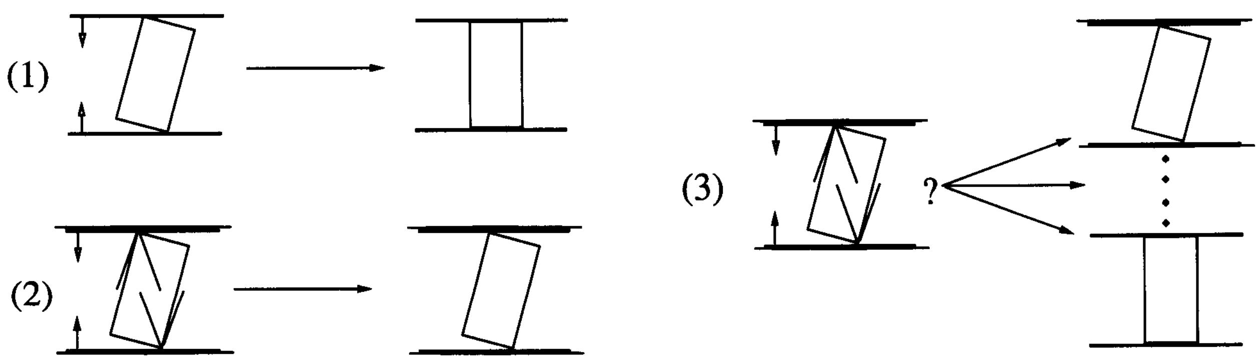


Figure 4: Three different friction models for parallel-jaw grasping. The initial orientation of the part is the same but in (1) there is zero friction, and the part rotates into a stable configuration; in (2) there is deterministic friction as indicated by the friction cones, and the part wedges; and in (3) there is non-deterministic friction, the part's final orientation is indeterminate: it could remain wedged at the initial orientation, it could rotate into a stable orientation, or could get wedged anywhere in between these two orientations.

1.4 Contributions of this paper

Our primary results are:

Theorem 1 *Under deterministic friction, the grasp map of an algebraic part is a step-ramp function.*

We also present an algorithm in Section 2 to compute the map given a parametric representation of the curves bounding the part's convex hull.

Theorem 2 Friction/Curvature Duality
For any planar algebraic part operating under deterministic Coulomb friction, there is a curved part operating under zero friction with an identical grasp function.

In Section 3.1, we give a constructive proof. A corollary of Theorem 2 is that given the coefficient of friction, we can orient any planar part with a series of parallel-jaw grasps.

Theorem 3 *Given a planar part, let Γ denote its grasp function under zero friction and Γ_n denote its grasp map under non-deterministic Coulomb friction with coefficient μ. Then:*

- *Suppose that all fixed-points of Γ are non-frictional in Γ_n. If $\mu < \mu_1$, there is a plan that is guaranteed to orient the part. We give a constructive proof.*
- *If $\mu \geq \mu_2$, no plan is guaranteed to orient the part.*

Notice that the non-deterministic friction model with a coefficient μ can be interpreted as Coulomb friction operating with a unknown, possibly changing, but bounded coefficient. Thus the plan found by the proof of this theorem (see Section 3.3) is robust to uncertainty in the friction coefficient so long as it never exceeds μ_1.

1.5 Related Work

The entire field of Tribology is concerned with the study of friction. Whitney and his colleagues [21] published pioneering work that analyzed the role of Coulomb friction in part insertion. In the presence of friction, part motion is notoriously difficult to characterize, especially in pushing when the distribution of support pressure is unknown. Friction between the pusher and the part must be distinguished from friction between the part and the support surface. An excellent summary of related work can be found in [6, 13].

For cases where the exact support pressure is known, Goyal *et al.* [9] presented two functions that characterize the net frictional force and moment between a rigid body and a planar surface on which it slides. The moment function description applies only to ordinary isotropic Coulomb friction, while the limit surface description applies a wider class of friction laws.

For cases where the support distribution is not known precisely, Mason [11] investigated the mechanics of planar pushing and derived a rule for determining which direction the part will rotate. Peshkin and Sanderson [15] bounded the rate of rotation by characterizing the the locus of possible centers of rotation (COR) for all possible pressure distributions. Alexander and Maddocks [2] assumed only the geometric extent of the support area, not the precise pressure distribution, and give bounds on the possible motions. While Alexander and Maddocks consider cases where the friction between the pusher and the part is either dominated by or dominates that between the part and the surface, Peshkin and Sanderson consider the complementary case of the two frictional forces being comparable.

Several robot planning algorithms have been developed that explicitly consider frictional indeterminacy. Brost [3] gave an algorithm for planning single-step grasping motions with a parallel jaw gripper in the presence Coulomb friction and bounded uncertainty in the orientation of a given part. Peshkin and Sanderson [14] gave an algorithm for finding a sequence of fence angles that will orient a given part as it passes on a conveyor belt.

To avoid frictional indeterminacies, Lynch and Mason [10] identify pushing directions that maintain planar contact between a part edge and the pushing surface. They show how these pushing directions can be composed to reorient parts that may be too heavy to be grasped. Akella and Mason [1] proved that in the absence of obstacles, it is always possible to position and orient a polygonal part by a finite sequence of pushing operations.

In this paper, we focus on friction between the part and the gripper, using Mason's rule to predict the direction of part rotation. In addition to frictional indeterminacies, we consider the uncertainty in the initial orientation of the part

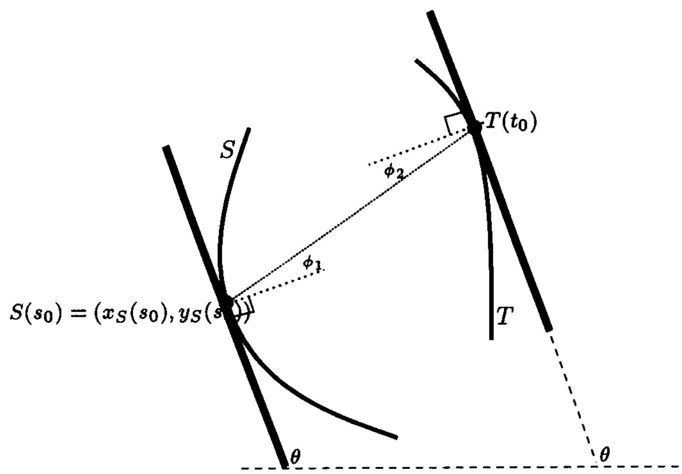


Figure 5: Test for frictional contact configurations: the case of arc-arc configurations is shown. S, T are two parametric curves and the orientation θ defined by the contact points $S(s_0), T(t_0)$, strictly interior to the respective arcs. The tangents at the two contact points have to be parallel for it to be a valid contact configuration. Furthermore, the configuration is frictional if and only if the angles ϕ_1, ϕ_2 are both less than α, the friction angle.

in the gripper. For a given part, we give bounds on the coefficient of friction that permits the existence of plans to orient parts when their initial orientation is unknown.

This paper is a revised and improved version of [19]. In this revised form, it has been submitted for publication in *Journal of Robotic Systems.*

2 Computing Frictional Orientations

We will first set up the precise mathematical conditions that need to be satisfied for an orientation to be frictional. These conditions are valid for both the deterministic or non-deterministic models and directly lead to a proof that $\mathcal{F}$ induces a *finite* (Lemma 1) and *computable* (Lemma 2) partition of S^1 for an algebraic part. Then we prove Theorem 1.

Let S denote an arbitrary arc in the convex boundary of part P. Let S lie on a curve with equation $g(x, y) = 0$. This curve is often called the "carrier" of S. The function g is assumed to be smooth over the region corresponding to the arc; the only discontinuities in slope along P exist at end-vertices of arcs. If g is polynomial, then the carrier curve is algebraic and by inclusion, so is S. The expression $g(x, y) = 0$ is called the implicit representation of the carrier. For convenience, we'll express our equations below assuming a parametric representation. Under this representation, the x, y coordinates of a point on the curve are given as functions of a third variable t. Most curve segments used in engineering have parametric representations; in particular, most useful algebraic curves have a rational parameterization, *i.e.* $x(t), y(t)$ are ratios of polynomials.

For an arc S, let $x_S(s), y_S(s)$ be the parametric representation of the carrier. The points on the arc itself are assumed (without loss of generality) to correspond to $0 \le s \le 1$, clockwise, and $s = 0, 1$ being the end-vertices of the curve. We assume, again without loss of generality, that every arc is "small" in that it does not contain two points with parallel tangents. If an arc does not have this property, it can be split into two arcs with the addition of a (pseudo) vertex so that it would.

The contact configurations corresponding to frictional orientations are of three types: arc-arc, arc-vertex, and vertex-vertex, which will be considered separately.

Arc-arc frictional configurations Let S, T be two parametric arcs as shown in Fig. 5. To be a contact configuration, the following conditions are necessary and sufficient:

$$0 < s_0 < 1,\ 0 < t_0 < 1, \tag{1}$$

and

$$y'_S(s_0)x'_T(t_0) = y'_T(t_0)x'_S(s_0). \tag{2}$$

$x'_S(s_0)$ denotes the derivative of x_S with respect to s taken at s_0, and so on. The first conditions require that the points be strictly interior to the arcs and the second condition requires that the tangents be parallel.

For such a contact configuration to be *frictional*, the line joining $S(s_0)$ and $T(t_0)$ must make angles less than α with the normals at $S(s_0)$ and $T(t_0)$ as shown in the figure:

$$|(y_S - y_T)y'_S + (x_S - x_T)x'_S| < \mu|(x_S - x_T)y'_S - (y_S - y_T)x'_S|, \tag{3}$$

and

$$|(y_S - y_T)y'_T + (x_S - x_T)x'_T| < \mu|(x_S - x_T)y'_T - (y_S - y_T)x'_T|. \tag{4}$$

Parameters s_0, t_0 have been dropped from the above two equations for brevity. If the subscript is S, then the argument is understood to be s_0; *i.e.* y_S means $y_S(s_0)$ and so on.

The orientation θ of this contact configuration can be taken to correspond to the slope of the tangent at the two contact points (which are the same due to parallel-jaw grasp - Equation 2):

$$\theta = \tan^{-1}\frac{y'_S}{x'_S}.$$

Arc-vertex configurations Let v be a vertex formed at the intersection of arcs S, T and let U be another arc. The contact configuration defined by contact points $v, U(u_0)$ needs to be tested for frictionality. Let θ_S, θ_T be defined as before. Analogously, define θ_U, θ_V for vertex w. For a orientation θ to correspond to a valid vertex-vertex contact configuration, θ must lie in both intervals $[\theta_S, \theta_T]$ and $[\theta_U, \theta_V]$. Finally, define ϕ to be the angle formed by the line joining u, v with the line with orientation $\pi/2 + \theta$. Then θ is frictional iff $\phi < \alpha$.

These conditions hold for any parametric curve, not necessarily algebraic. With this background, we are ready to present the following lemma related to algebraic curves.

Lemma 1 *For an algebraic part, $\mathcal{F}$ is a finite set of intervals from S^1.*

Proof: We'll restrict the proof, for conciseness, to the frictional orientations defined by arc-arc configurations. The other cases are easier. Recall that an orientation defined by two contact points $S(s_0), T(t_0)$ is frictional iff conditions in 1-4 hold. Intervals in $\mathcal{F}$ are *bounded* by orientations that correspond to solutions s_0, t_0 from $(0, 1)$ to either of the systems:

$$y'_S(s_0)x'_T(t_0) = y'_T(t_0)x'_S(s_0),\quad |(y_S - y_T)y'_S + (x_S - x_T)x'_S| = \mu|(x_S - x_T)y'_S - (y_S - y_T)x'_S|. \tag{5}$$

or

$$y'_S(s_0)x'_T(t_0) = y'_T(t_0)x'_S(s_0),\quad |(y_S - y_T)y'_T + (x_S - x_T)x'_T| = \mu|(x_S - x_T)y'_T - (y_S - y_T)x'_T|. \tag{6}$$

It remains to show that these systems have a finite number of solutions if x_S, y_S *etc.* are ratios of polynomials. Let κ be the maximum degree of any polynomial occurring in the parametric representation of any of the n curves forming the boundary of P.

Consider System 5. The first equation is of degree at most $2\kappa - 2$ in s_0, t_0 and the second is of degree at most $2\kappa - 1$. So we have two polynomial equations in two variables and each polynomial is of degree at most $2\kappa - 1$. The number of coefficients in each polynomial is therefore at most $(2\kappa)(2\kappa + 1)/2$. Thus, if the polynomials are distinct, they cannot have more than $(2\kappa)(2\kappa + 1)/2 - 1$ solutions.

System 6 is similar. Thus, two arcs S, T can define at most $2(2\kappa^2 + \kappa - 1)$ solutions. This implies that the number of boundary configurations in intervals of $\mathcal{F}$ defined by arc-arc pairs is $n^2(2(2\kappa^2 + \kappa - 1))$. This clearly bounds the number of intervals in $\mathcal{F}$ from arc-arc contact configurations. Similar polynomial functions of κ and n bound the number of intervals in $\mathcal{F}$ from other contact configurations. □

In [18] we show that end-points and fixed-points of steps and ramps in a zero friction grasp function correspond to orientations in which the angles ϕ_1, ϕ_2 in the explanations above (see also Fig. 5) are simultaneously zero. Frictional configurations correspond to ϕ_1 and ϕ_2 being both less than α. Therefore frictional intervals are always in the neighborhood of a step fixed-point or end-point. In [18] we also showed that the number of step fixed-points or end-points is finite. This is alternative proof of Lemma 1.

Assume that a system of equations as in 5 requires $O(1)$ time to solve on a Computer Algebra system such as *Maple*. Strictly, this will take time which is some function of κ, the maximum degree of any polynomial in the representation of any curve segment; but κ is a constant due to the algebraic assumption. Also assume that other simple geometric/geometric computations between a constant number of entities (points, lines, arcs) takes $O(1)$ time. Then we have the following lemma.

Lemma 2 *The set of frictional orientations $\mathcal{F}$ can be computed in $O(n)$ time for an algebraic part with n arcs forming its convex hull, given the arcs in order.*

Proof: By looking at all arc-arc, arc-vertex, and vertex-vertex pairs, and solving systems such as 5, it is clear that $\mathcal{F}$ can be computed in $O(n^2)$ time. However, in [16] we show that only $O(n)$ event pairs need be considered to look at every possible orientation. That same paper also proved how to determine these $O(n)$ pairs in $O(n)$ time. Now we need to solve systems such as 5 only for $O(n)$ events and so the entire algorithm can be made to run in linear time. □

These lemmas yield a

Proof of Theorem 1 In Proposition 2, we saw that the (zero friction) grasp function Γ of an algebraic part is a step-ramp function. In [16], we give an $O(n)$ procedure to compute Γ. The extents of the intervals in the set of frictional orientations $\mathcal{F}$ depend on the friction coefficient as we have seen above. Let Γ_d denote deterministic friction grasp map. We need to show that Γ_d is a step-ramp function. The differences between Γ and Γ_d are only in $\mathcal{F} \subseteq S^\infty$: for every $\theta \in \mathcal{F}$, $\Gamma_d(x) = x$ irrespective of what $\Gamma(x)$ is. For a non-frictional orientation θ, $\Gamma_d(x) = \Gamma(x)$. Since $\mathcal{F}$ can be computed in $O(n)$ time, it is clear that Γ_d is computable in the same time and contains a finite set of steps and ramps. □

3 Grasping parts with friction

Under non-zero friction, the grasp function must be modified suitably to reflect the changed mechanics. The map from initial to final orientations under friction is called the *grasp map*.

We assume:

1. The part is a rigid planar part of known shape and finite description.
2. The part is presented to the gripper in isolation; we do not address the related problem of isolating parts from a bin (commonly known as *singulating*).
3. The motion of the gripper G is orthogonal to the two parallel jaws L, H.
4. The part's initial position is unconstrained as long as it wholly lies somewhere between the two jaws. The part remains between the jaws throughout grasping.
5. Both jaws make contact simultaneously (pure squeezing). This can be relaxed by considering push-grasping [3] as is done in [18].
6. All motion occurs in the plane and is slow enough that inertial forces are negligible. The scope of this *quasi-static* model is discussed in [12] and [15].
7. Non-zero Coulomb friction given by a co-efficient μ is allowed under one of the models: deterministic and non-deterministic.

These are the same assumptions used in [8, 18]) with the exception of Assumption 7.

For an interval Θ, let $|\Theta|$ denote its measure. We define the *image of an interval* Θ, $\Gamma(\Theta)$, to be the convex hull of the set $\{\Gamma(\theta)|\theta \in \Theta\}$. Similar definitions hold for the deterministic and non-deterministic grasp maps Γ_d, Γ_n.

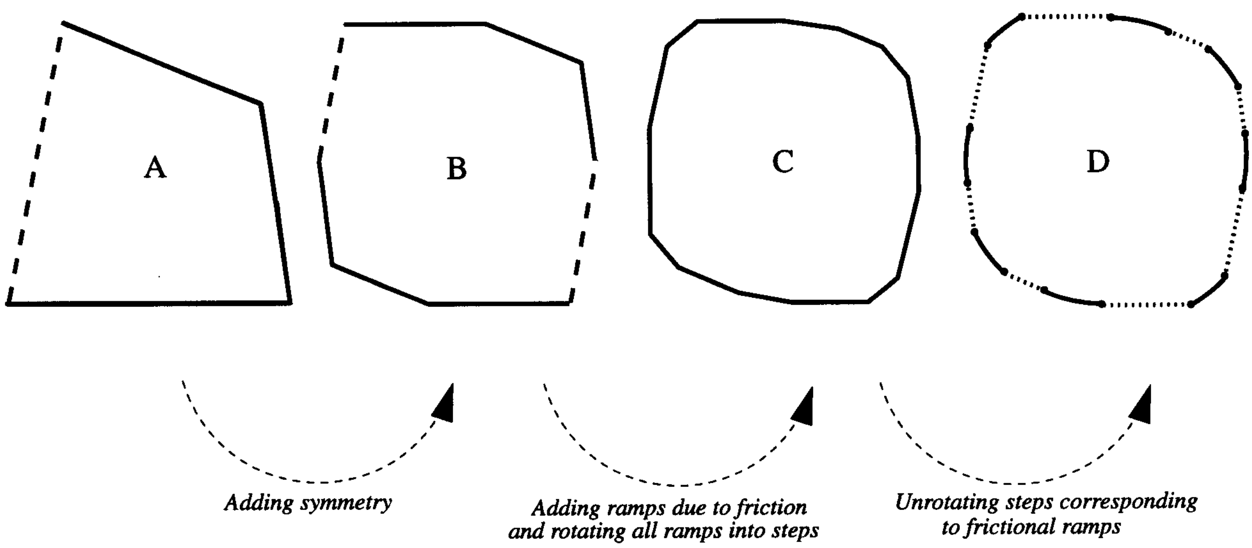


Figure 6: Illustrating the proof of Theorem 2 for a polygonal part A. Given quadrilateral A, and a coefficient of friction, we first construct a symmetric octoganal part, B, with the same grasp function (and perimeter: the dashed edge in A is shown split into opposite dashed edges in B). This grasp function is a step-function. Adding in the ramps that correspond to frictional orientations and then rotating all ramps in the grasp map into steps yields 16-gon C. Finally, D is the curved part formed by changing edges in C corresponding to frictional ramps into circular arc segments. For ease of visualization, circular edges are shown bold and linear edges dotted in D. The 0-friction grasp function of D is the same as the frictional grasp map of A (or B).

3.1 Deterministic Friction and Part Curvature

In this section we prove Theorem 2, which demonstrates a duality between friction and curvature. A schematic of the construction is illustrated in Figure 6. Another example of this construction is given in Figure 8.

Proof of Theorem 2: We give a constructive proof. Recall from Theorem 1 that the grasp map of a planar part P under deterministic friction is a step-ramp function. Let this grasp map, denoted by Γ_d have n steps and ramps.

First we modify given step-ramp function Γ_d into a step function Γ'. Then we construct a polygonal part Q' consistent with Γ'. Finally, we show that had the modification of Γ_d to Γ' been done more carefully, then the Q' obtained can be modified to obtain a *GP*-part Q consistent with Γ.

Modification of $\Gamma_d \rightarrow \Gamma'$: Modification of step-ramp grasp function Γ_d into a step function Γ' is first done by rotating every ramp $[u, v)$ into a step $[u, v)$ about some point a in its interior. Thus, a becomes the fixed-point of this step. See the top of Fig. 7.

Construction of polygonal part Q' consistent with step function Γ' : Process steps in Γ' from left to right. Consider an arbitrary step $[u, v)$ of Γ'. Let its fixed-point be a, $u < a < v$; *i.e.* $\forall\theta \in [u, v), \Gamma'(\theta) = a$. For every such step we construct a parallelogram that has Γ' as its grasp function *between* orientations $[u, v)$. We term this parallelogram as the one *corresponding* to the step. See Fig. 7. For correctly scaling the parallelogram corresponding to step $[u, v)$, we need the length of one of the diagonals of the parallelogram corresponding to the previous step $[x, u)$. In the figure, the parallelogram formed by vertices *KLK'L'* is the parallelogram corresponding to step $[x, u)$. Its diagonal *LL'* (say of length *d*) is the diagonal in question and is also a diagonal of the parallelogram corresponding to step $[u, v)$. Let O be the mid-point of this diagonal *LL'* (it will turn out that O is the centroid of all the parallelograms we construct and also of the final polygon Q').

The parallelogram corresponding to step $[u, v)$ is shown as *LML'M'*. *M,M'* are defined as follows. *M* is the point such that $\angle LOM = v - u$ and $\angle MLO = \frac{\pi}{2} - (\alpha - u)$. *M'* is defined as the point diametrically opposite *M* about O, *i.e.* *MO* = *OM'*. This completely defines the parallelogram *LML'M'* corresponding to step $[u, v)$. Now, *MM'* is the

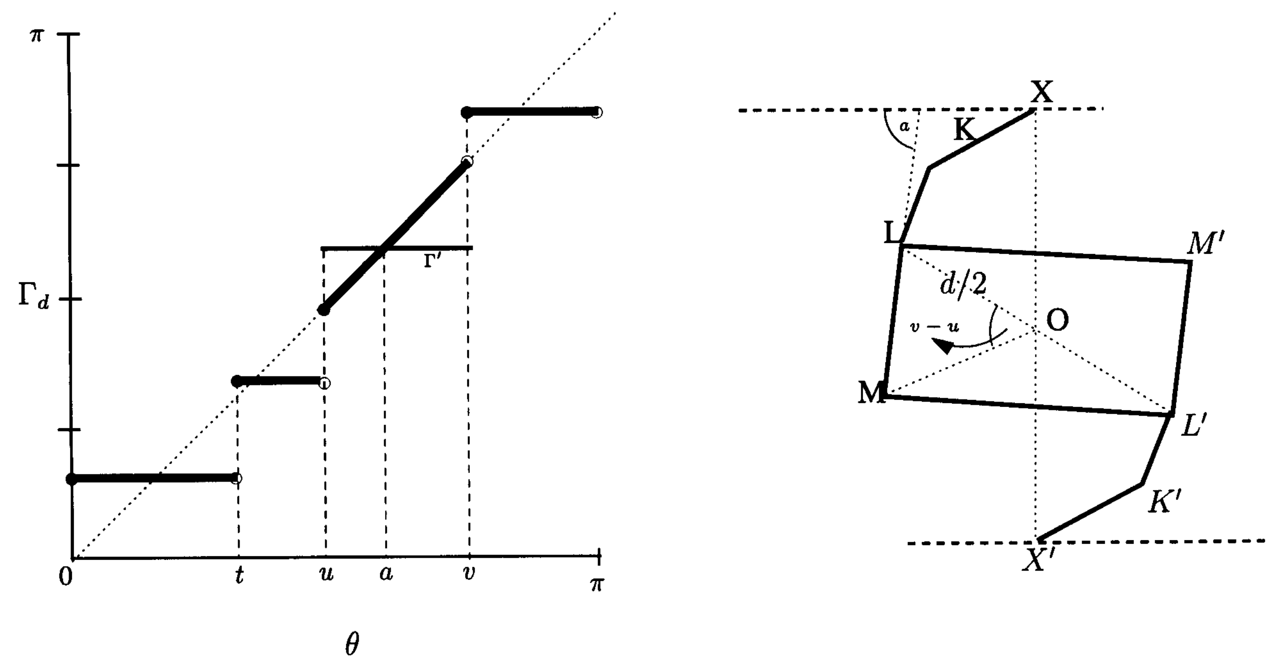


Figure 7: Construction of a *GP*-part polygon compatible with a given step-ramp function. At the top is shown a step-ramp function with three steps and one ramp, all left-closed and right-open. a is the fixed-point of a step formed by rotating the ramp. The bottom shows a portions of a polygon compatible with the grasp function obtained by rotating every ramp into a step about some point in its interior (resulting in a step function). By choosing to rotate every ramp into a step about its *center*, the polygon so obtained can be easily transformed into a *GP*-part consistent with the given step-ramp grasp function. See the proof of Theorem 2.

diagonal to consider when building the parallelogram corresponding to the next step. For the first step, we can choose any vertical segment (shown as *XX'* in the figure) as the diagonal from the parallelogram corresponding to the previous step. Its midpoint will also be *O*.

It is easy to verify that this parallelogram *LML'M'* is consistent with Γ' for the step $[u, v)$ with fixed-point a once we note that the parallelogram is stable at this fixed-point. For verifying stability at a it suffices to observe that O is the centroid of the parallelogram and that $\angle LMO$ and $\angle MLO$ are both acute.

The sequence of outer edges of all these n parallelograms forms the required polygon Q' consistent with Γ'. Observe that

$$\sum_{\text{parallelograms}} \angle LOM = \sum_{\text{steps}} (v - u) = \pi.$$

To verify that the polygon is convex, simply observe that $\angle KLM < \pi$ (to see this, note that each of the angles $\angle KLO$ and $\angle MLO$ are acute which follows from the stability of the parallelograms).

Constructing *GP*-part Q consistent with step-ramp function Γ_d : Recall how Γ_d was changed to Γ'. Every ramp $[u, v)$ in Γ_d was rotated into a step about some arbitrary point a in its interior. In polygon Q', constructed as above, with centroid O, there exists an edge (an outer edge of the parallelogram corresponding to the step $[u, v)$ in Γ') *LM* such that $\angle MLO = \pi/2 - (a - u)$ and $\angle LMO = \pi/2 - (v - a)$. Now, if $a = (v + u)/2$, *i.e.* a, the point about which to rotate the ramp $[u, v)$, was chosen to be at the mid-point of the ramp, then these two angles are equal, which in turn implies that *LO* and *MO* are equal in length. We could then replace the edge *LM* of Q' by a circular arc centered at O with end-points *L,M*. Repeat this for the diametrically opposite edge *L'M'*.

Now we have a *GP*-part Q which we claim is consistent with Γ_d. It is easy to verify that Q has constant diameter between orientations $[u, v)$ (it is the portion of a circle of radius $|LO|$ between these orientations). We only have to verify that Q is convex which essentially follows from the stability argument above, *i.e.* because $\angle KLO < \pi/2$, the edge-arc sequence $K\ \widehat{LM}$ is convex at L. □

3.2 Multi-Step Plans for Orienting Parts

In this subsection, we present a brief overview of the zero-friction results: planning algorithms of [8, 18] that compute a shortest-length sequence of open-loop grasp actions to orient planar parts under zero friction. See Fig. 9. Our planning algorithm for the non-deterministic case uses some of these ideas and is discussed in the proof of Theorem 3. The planning algorithm for the deterministic case is the same as that discussed in [18] in light of Theorem 2. See Fig.s 9,10 for example plans.

The planning algorithms of [8, 18] essentially work as follows. Work backward from a single optimal grasp orientation. Find a grasp (squeeze or push) action that collapses some interval of orientations (of maximum measure if we are interested in length optimality of the plan) into this final orientation. Now search for a larger interval (the largest possible interval, for plan optimality) whose image under grasp is smaller than the current interval. This continues until the algorithm finds an interval of length T, the period of symmetry in the grasp function of the part. For more details, see [8, 18].

Goldberg [8] presented an algorithm that generates plans to orient polygonal parts. For convenience, we call this Algorithm A. For a polygonal part with n sides operating under zero-friction, Algorithm A runs in time $O(n^2 \log n)$ and finds a guaranteed optimal plan of length $O(n)$ to orient the part up to symmetry.[1]

For non-polygonal planar parts, without loss in generality, we assume that its frictionless grasp function, Γ, has at least one step. (Otherwise P is always oriented up to symmetry in its grasp function).[2]

In [18], we presented Algorithm A' for generating plans to orient non-polygonal planar parts operating under zero friction. See Fig. 9 for an example grasp plan. Let n denote the number of steps and ramps making up the grasp function of a planar part. Algorithm A' runs in $O(\min(Nn, N + n^2 \log n))$ time and produces a guaranteed optimal plan of length N to orient the part up to symmetry. Further, unlike the case of polygonal parts, there is no upper-bound on N, the optimal plan length, in terms of n. However, there is a bound on N in terms of a metric quantity.

Since the grasp map of a planar part under deterministic friction is a step-ramp function, Algorithm A' can be applied to orient planar parts operating under deterministic friction (so long as there is at least one step in the grasp map; if it is a single ramp, then it always wedges, behaves as a disk and can never be oriented).

3.3 Non-deterministic friction

The changes to the grasp function caused by non-deterministic friction are described below. Recall that the mapping between initial and final orientations is no longer single valued; the multi-valued grasp map is denoted as Γ_n. As before, Γ denotes the zero-friction grasp function. Clearly, Γ_n, Γ differ only in the frictional orientations $\mathcal{F}$ as computed in Section 2. For a frictional orientation $x \in \mathcal{F}$, $\Gamma_n(x)$ is from the closed interval between[3] $\Gamma(x)$ and x denoted as $[x, \Gamma(x)]$.

We allow the final orientation of the part to be any orientation in that interval. In other words,

$$x \in \mathcal{F} \Rightarrow \Gamma_n(x) \in [x, \Gamma(x)].$$

If initially in orientation x, the part could end up at any orientation in $[x, \Gamma(x)]$. The two ends of this interval, $x, \Gamma(x)$, correspond to the extreme cases of zero compliance (wedging) and maximum compliance (zero friction). See Fig. 11. (In contrast, notice that deterministic wedging was characterized by $x \in \mathcal{F} \Rightarrow \Gamma_n(x) = x$.)

Now we give a conditions for the existence of a plan to orient a part up to symmetry in the presence of non-zero non-deterministic friction. We do not address the optimality in plan length. The following is a more precise restatement of Theorem 3 defining the bounds as $\mu_1 = \tan(\alpha_1)$ and $\mu_2 = \tan(\alpha_2)$.

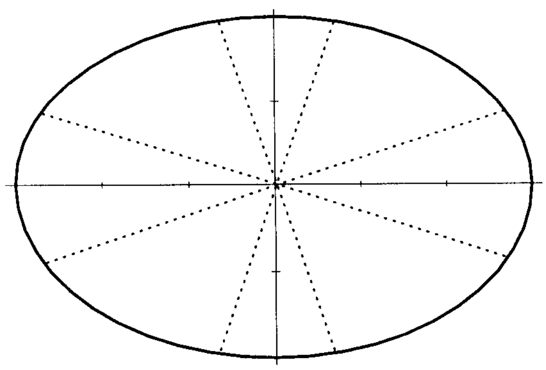
Suppose Γ is a frictionless grasp function with an arbitrary step $[a, c)$ having fixed-point b. The measure of this step is simply $|c - b|$ while its ***fixed-point offset*** is defined to be $\min(|c - b|, |b - a|)$.

[1] The running time of this algorithm was improved to $O(n^2)$ in [5].

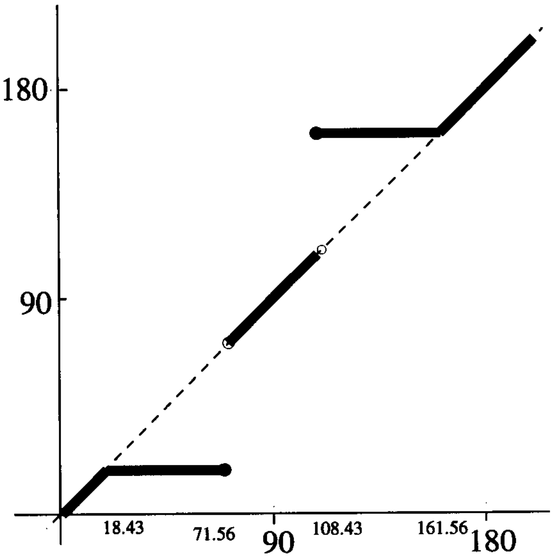
[2] Circles and other constant diameter curves exhibit such behavior[4, 22]

[3] Since S^1 is a circular set of orientations, there are two closed intervals with end-points as $\Gamma(x)$ and x: $[\Gamma(x), x]$ and $[x, \Gamma(x)]$. We intend the interval that contains $(x + \Gamma(x))/2$. For simplicity, we denote this as $[x, \Gamma(x)]$.

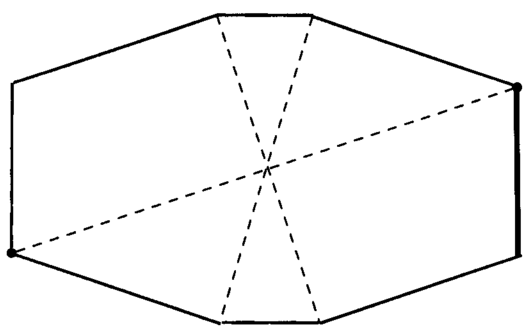
Ellipse $(x/a)^2 + (y/b)^2 = 1$ with $a = 3, b = 2$.



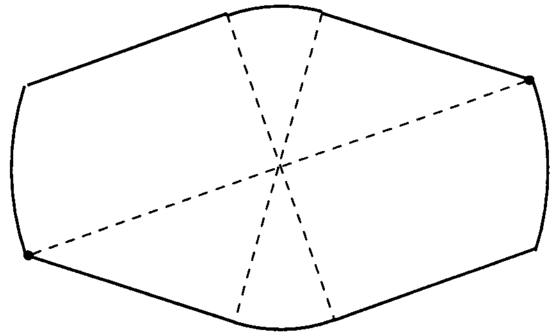
Frictional (rampy) regions shown for $\mu = 0.25$.
If (and only if) $|2\mu \frac{ab}{a^2 - b^2}| < 1$
then there exists a step in Γ_d
and the ellipse is orientable (otherwise
Γ = a single ramp and ellipse behaves like a disk).



Γ_d for $\mu = 0.25$
For $\mu > 0.42$, a single ramp results
(for this ellipse).



Make every ramp in Γ_d into a step by rotating about mid point.
Then, this is the polygon with that grasp function.



This is the GP-part correction for the polygon on left.

Figure 8: An example of an elliptical part operating under friction coefficient $\mu = 0.25$ having the same grasp mechanics as that of the generalized-polygonal part shown at the bottom-right.

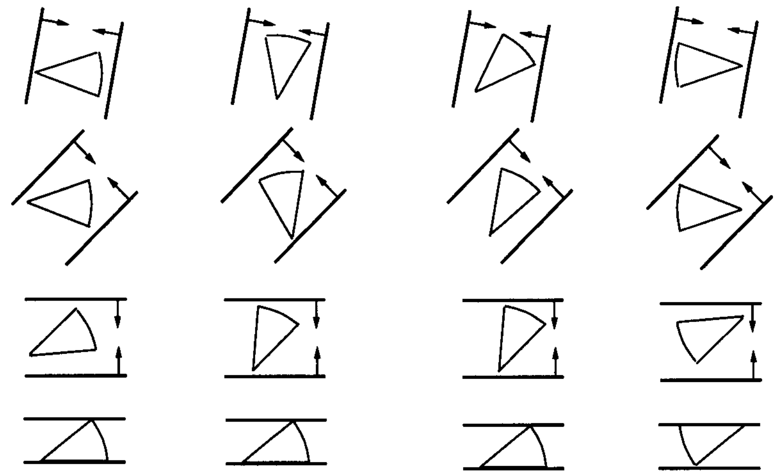


Figure 9: Top view of a plan for optimally grasping pie-shaped parts (half angle = 20 degrees). Frictionless gripper orientation is shown by two parallel lines. Three traces of the three-grasp plan are shown. Each trace runs from top to bottom. The plan is open-loop: commanded actions do not depend on sensor data. Although the part's initial orientation is different in each trace, its final orientation is the same. For the class of squeeze actions, we can only orient this part up to 180 degrees symmetry: Flipping an initial orientation by 180 degrees causes the final orientation also to be shifted by the same amount. However, both these final orientations achieve the same (optimal) measure of grasp stability. This ambiguity in final orientation can be resolved with push-grasp actions. Notice that in the first column, the first squeeze grasp action does not change part orientation.

Restatement of Theorem 3 Let Γ be the step-ramp (with at least one step) grasp function of a part under zero-friction. Let α_1 be the minimum fixed-point offset over all steps in Γ and let α_2 be the maximum step measure.
Then,

1. If no fixed-point of Γ is frictional, there exists a plan to orient the part up to symmetry when $\alpha < \alpha_1$;
2. there does not exist a guaranteed plan if $\alpha \geq \alpha_2$.

Proof: Statement 2 is easy to prove. $\alpha \geq \alpha_2$ implies that the part could get wedged *at any* orientation in S^1 (*i.e.* $\mathcal{F} = S^1$). Suppose that the friction adversary always "decided" that the part should get wedged. Then it is impossible to change the orientation of the part under gripper action (even if Γ, the frictionless grasp function of the part, has a non-zero period of symmetry). Therefore, no guaranteed plan can exist.

Proof of 1.: We have that $\alpha < \alpha_1$. Let Γ_n denote the grasp map under non-deterministic friction. This implies that every step in Γ_n has a fixed-point not disturbed by the friction and all frictional orientations are around the end-points of steps and ramps. We use a variant of the box-placement algorithm described in [18] adapted against a friction adversary which, at every stage of the algorithm, tries to provide maximum hindrance to the progress of the algorithm. The hindrance is provided by choosing the worst possible outcome of a grasp made in the frictional orientations. Fortunately, it turns out that this worst possible outcome of a grasp made at a frictional orientation x is always one of x or $\Gamma(x)$, the end-points of the interval of non-deterministic outcomes. This will become clear in the following.

Let Γ_1, Γ_2 be two *single-valued* grasp functions derived from Γ_n as follows. Both coincide with Γ at non-frictional orientations. However, in the frictional orientations, $\Gamma_1 = \max(\Gamma_n)$ and $\Gamma_2 = \min(\Gamma_n)$. In other words, Γ_1 takes on the maximum allowed value of Γ_n and Γ_2, the minimum possible orientation. See Fig. 12.

The planning algorithm begins by choosing the largest step in Γ_n as interval Θ_1. By "step" in Γ_n, we mean a step in the non-shaded portion of Γ_n. Intervals $\Theta_2, \Theta_3, \ldots$, are built successively using a variation of the box-placement algorithm as follows. In the original algorithm [18], after iteration j, if $|\Theta_j| < T$, the period of symmetry in Γ, we need to determine Θ_{j+1} with $|\Theta_{j+1}| > |\Theta_j|$ and $|\Gamma(\Theta_{j+1})| < |\Theta_j|$. This was done by placing an imaginary square box of width $|\Theta_j|$ with its bottom-left corner at the left-end point of a step or ramp and then considering a Θ_{j+1}

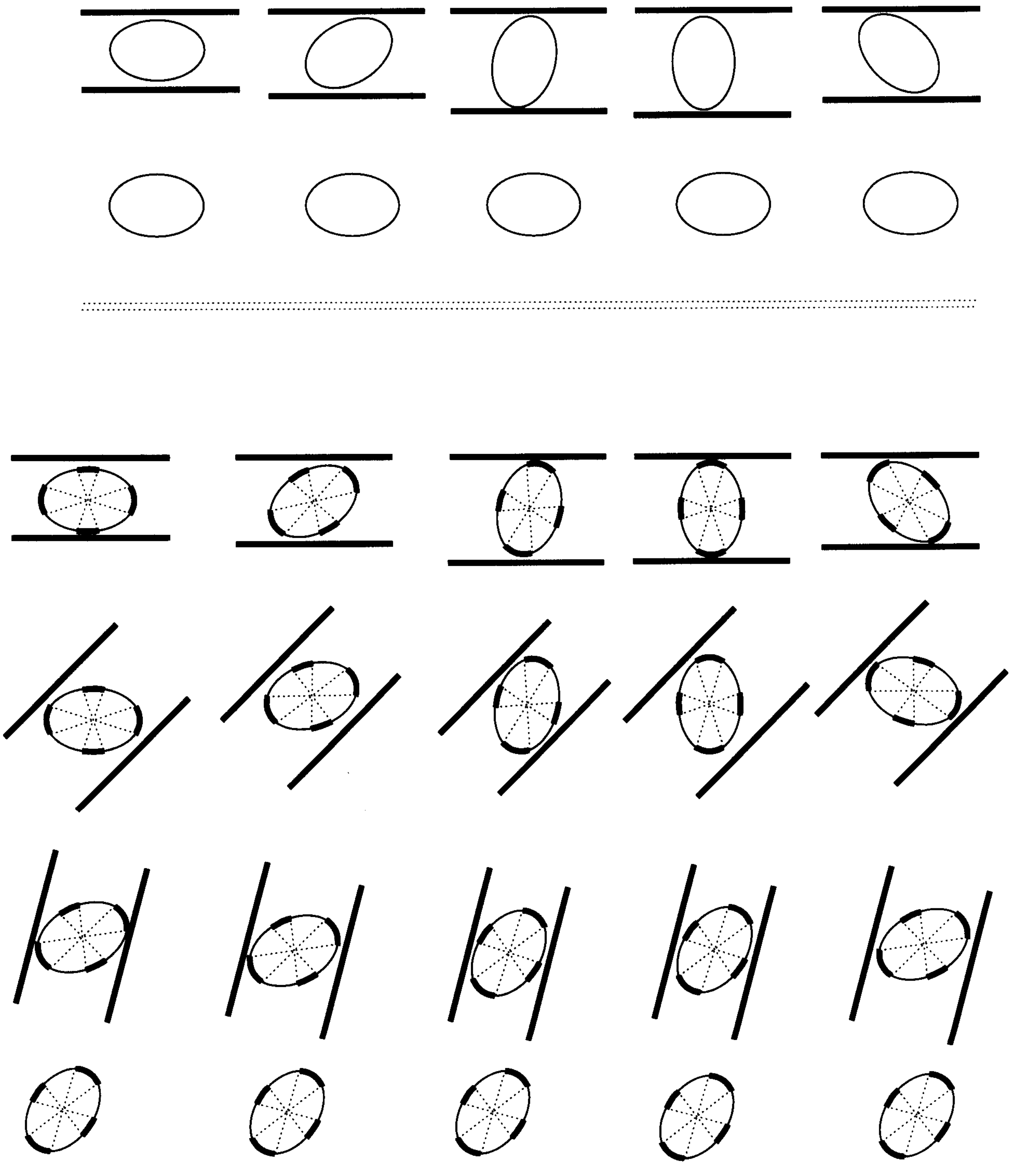


Figure 10: An example plan for the ellipse shown in Fig. 8. Initial orientations of the parts are 0, 30, 60, 75, 135 degrees. At the top is the case of zero friction. In this case the grasp function consists of a single step and the part falls into orientation in one grasp irrespective of its initial orientation. At the bottom is the case of $\mu = 0.25$. In this case the part gets wedged when it is between certain orientations as discussed in Fig. 8. Therefore, more grasp actions are required. Our planning algorithm in [18] computes the optimal plan which requires three grasp actions as shown.

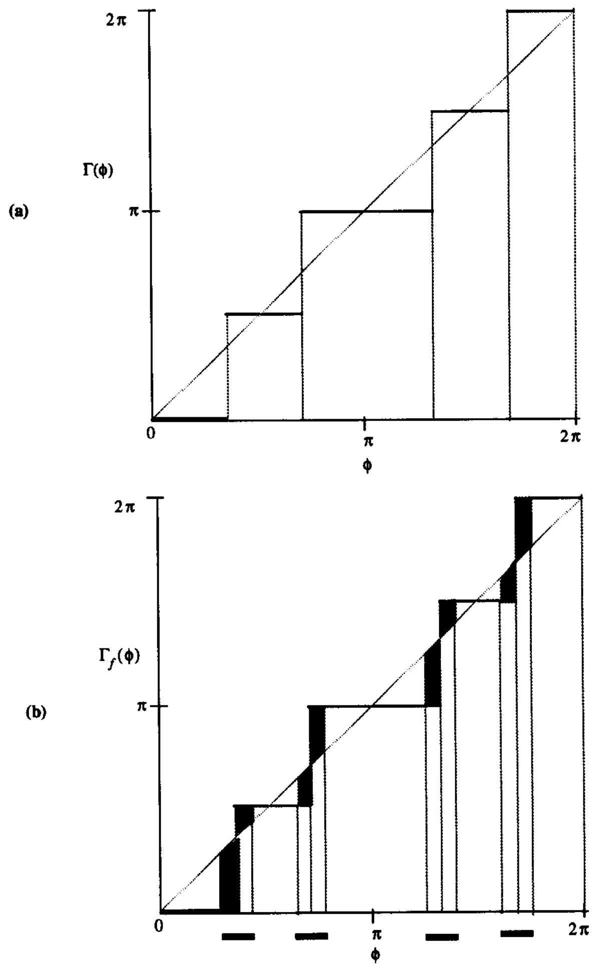


Figure 11: Change in grasp function under non-zero friction. Figure (a) shows the zero-friction grasp function of a rectangle and (b) shows the grasp map Γ_n under non-deterministic friction given by $\mu = 0.24$ ($\alpha = 13°$). The frictional orientations in S^1 are shown in thick lines below the ϕ axis in (b). Notice that every fixed-point (intersection of the 45° line with the steps) is undisturbed by the friction and hence (by Theorem 3) there exists a plan to orient this part.

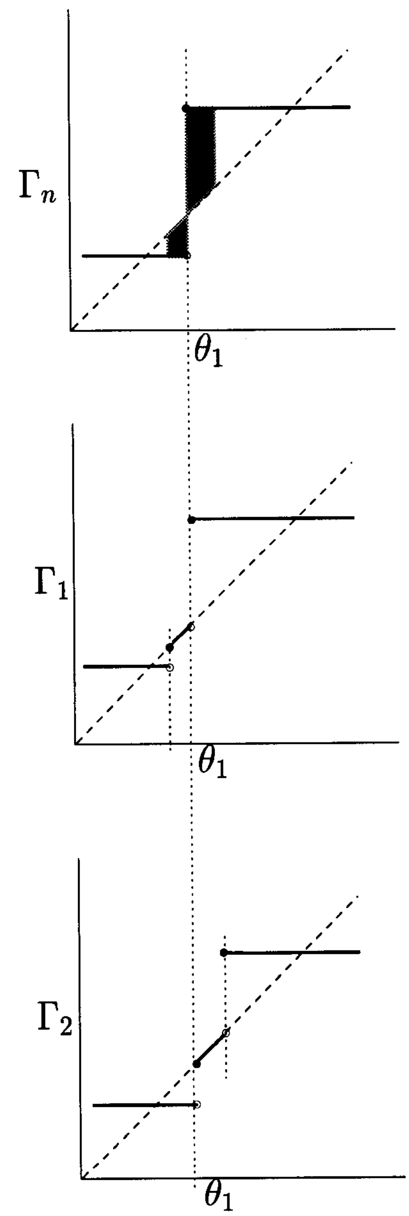


Figure 12: At the top is shown a non-deterministic multi-valued grasp map Γ_n. Shaded regions indicate the frictional orientations, a neighborhood around the orientation θ_1, and the allowed range of possible outcome at each frictional orientation. Taking the maximum of the allowed values results in the single valued grasp map Γ_1 shown in the middle. Replacing maximum by minimum gives us Γ_2 shown at the bottom. We devise a planning algorithm that deals with Γ_1 and Γ_2 instead of the multi-valued function.

beginning at this left-end point. (Or by placing the box with its top-right corner at the right-end point of a step or ramp and considering a Θ_{j+1} ending at this right-end point). Such a Θ_{j+1} was always shown to exist for some placement of the box whenever $|\Theta_j| < T$, so that the algorithm will be correct and terminate. We have to give a similar proof in our case.

The change we make in the box-placement algorithm is that in considering bottom-left (*resp.* top-right) box placements, we consider the grasp function to be Γ_1 (*resp.* Γ_2). Note that this is the most conservative assumption possible, and assumes that friction is an adversary that strives to make orienting as slow as possible. Now all we have to show is that Θ_{j+1} always exists (satisfying the conditions described above) unless $|\Theta_j| \geq T$.

Towards a contradiction, assume that $\Theta_j < T$ and for all bottom-left box-placements in Γ_1, Θ_{j+1} cannot be found. If the part is polygonal, then the only ramps in Γ_1 are those of width $\alpha/2$ and each of these can be rotated into a step about any point in its interior. Let the grasp function so obtained be Γ_1'. Γ_1' is piecewise constant and so is the grasp function of some polygonal part. The completeness proof of [8] shows that the planning algorithm on Γ_1', using bottom-left placements alone, would not halt until an interval of width T is reached. This implies that some portion of some ramp in Γ_1 must be reached by bottom-left placements and cannot halt with $|\Theta_j| < T$.

If the part is non-polygonal and Γ_1 has other ramps (not just the ones caused by friction) as well, then each ramp is cut into "small" portions and each part rotated into a step to get a grasp function Γ_1'. How "small" the ramps must be chopped up into is described in the completeness proof in [18]. Now essentially the same argument as that in the previous paragraph shows that the algorithm cannot halt at Θ_j.

What we have shown is that considering bottom-left placements alone, we can always find Θ_{j+1} unless $|\Theta_j| < T$. Considering top-right placements and grasp function Γ_2 is essential only if we are interested in shortening the length of the plans. Note that the resulting plans are not guaranteed to be optimal unlike the plans output by Algorithms A, A'. □

Special Condition on non-frictional fixed-points in Statement of Theorem 3 The condition that no fixed-point of a step in Γ are frictional is required by the proof of completeness of the box-placement algorithm presented in [18]. Are these parts that exhibit such a behavior? We show that every polygon is of this type for a sufficiently small value of the friction coefficient.

We know from Proposition 1 that a polygon has a grasp function made up of steps alone. In fact, an n-gon has at most n steps since an n-gon can have at most n stable orientations (fixed-points), each stable orientation containing at least one edge of the part flush against one of the jaws [17]. Fig. 13 shows a procedure for computing a critical value for the friction coefficient below which the grasp map under friction will leave all step fixed-points undisturbed.

4 Discussion

In previous work we modeled the grasp mechanics of the parallel-jaw gripper under the assumption that $\mu = 0$. We treated polygonal parts in [8] and algebraic parts in [18]. This paper extends that analysis to the case where $\mu \neq 0$. We discovered that frictional grasps of polygonal parts are equivalent to frictionless grasps of algebraic parts. For the non-deterministic friction model, we presented a bound on the coefficient of friction under which it is possible to find a plan to orient the part. This duality thus ties together results from both previous papers.

Some open problems remain. One is the whether the converse of Theorem 2 holds: *i.e.* given a step-ramp function Γ, is there a polygonal part P and a coefficient of deterministic Coulomb friction μ such that Γ is the grasp function of P under friction μ? There also remains the question of finding the shortest plan in the non-deterministic friction case. Finally, there is a gap of friction coefficients (μ_1, μ_2) between which it is presently uncertain whether there exists a guaranteed plan or not. Can this gap be narrowed?

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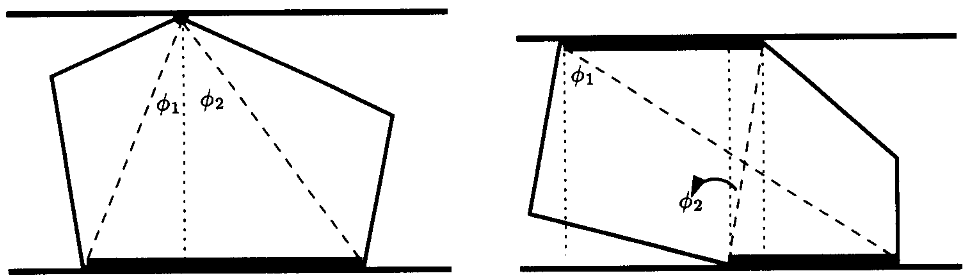

Figure 13: The condition that fixed-points must not be frictional in the statement of Theorem 3 is satisfied for polygons for sufficiently small values of the frictional coefficient. This bound on the frictional coefficient, given the part geometry, is derived here. Fixed-point orientations in polygons arise either from vertex-edge contacts (left) or parallel edge-edge contacts (right). (However, not all such contacts are indeed fixed-points. Treating edge-contacts are open sets, project one contact onto the other. The contact configuration is a fixed-point iff the projection is non-empty [17].) The dashed lines connect the contact configurations in the immediate neighborhood of the current fixed-point orientation. These dashed lines make angles ϕ_1, ϕ_2 with the vertical, as shown. If both ϕ_1 and ϕ_2 are larger than α, the friction angle, then the fixed-point is not frictional. So if ϕ denotes the minimum of all the ϕ_1, ϕ_2 taken over all fixed-points, then the special condition is satisfied iff $\alpha < \phi$. However, notice that the special condition is redundant for polygons. This is because ϕ_1, ϕ_2 are also the distances of the fixed-point to the neighboring end-points; *i.e.* $\min(\phi_1, \phi_2)$ is the fixed-point offset. Therefore $\alpha_1 = \phi$ and the conditions "all fixed-points being non-frictional" is the same as $\alpha < \alpha_1$.

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