Rectilinear Decompositions with Low Stabbing Number

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Abstract

Let $\mathcal{P}$ be a rectilinear polygon. The stabbing number of a decomposition of $\mathcal{P}$ into rectangles is the maximum number of rectangles intersected by any axis-parallel segment that lies completely inside $\mathcal{P}$. We prove that any simple rectilinear polygon with n vertices admits a decomposition with stabbing number $O(\log n)$, and we give an example of a simple rectilinear polygon for which any decomposition has stabbing number $\Omega(\log n)$. We also show that any rectilinear polygon with $k \geqslant 1$ rectilinear holes and n vertices in total admits a decomposition with stabbing number $O(\sqrt{k} \log n)$. When the holes are rectangles, then a decomposition exists with stabbing number $O(\sqrt{k} + \log n)$, which we show is tight. All the above decompositions consist of $O(n)$ rectangles.

1 Introduction

Let $\mathcal{P}$ be a rectilinear polygon, that is, a polygon all of whose edges are parallel to either the x-axis or the y-axis. The *stabbing number* of a decomposition $\mathcal{D}$ of $\mathcal{P}$ into axis-parallel rectangles is the maximum number of rectangles intersected by any axis-parallel segment whose interior is completely contained in the interior of $\mathcal{P}$. The *size* of the decomposition $\mathcal{D}$ is the number of rectangles it consists of. In this paper we study the problem of finding decompositions with low stabbing number and small size. A similar problem has been studied by Chazelle et al. [1]. They have shown that any simple polygon $\mathcal{P}$ with n vertices can be decomposed into $O(n)$ triangles such that any segment inside $\mathcal{P}$ intersects only $O(\log^2 n)$ triangles. Recently this result has been improved to $O(\log n)$ by Hershberger and Suri [4].

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In this paper we give a simple algorithm for the rectilinear case which achieves a stabbing number of $O(\log n)$. Observe that we cannot use the result of Hershberger and Suri to obtain a decomposition with $O(\log n)$ stabbing number, because we are looking for a decomposition into rectangles. We also prove that the $O(\log n)$ bound is tight in the worst case, that is, there are axis-parallel polygons for which any decomposition has $\Omega(\log n)$ stabbing number.

We extend our results to polygons with holes. If the number of holes is $k \geqslant 1$ and the total number of vertices of the polygon (including its holes) is n, then we show that a decomposition of size $O(n)$ exists whose stabbing number is $O(\sqrt{k} \log n)$. Since there are polygons for which any decomposition has stabbing number $\Omega(\sqrt{k} + \log n)$ this result is almost tight. If all the holes are rectangles then we are able to close this small gap by proving that a decomposition with stabbing number $O(\sqrt{k} + \log n)$ always exists.

In the following sections we deal with decompositions of simple rectilinear polygons (Section 2), rectilinear polygons with rectilinear holes (Section 3) and rectilinear polygons with rectangular holes (Section 4). The conclusions and open problems are given in Section 5.

2 Simple Rectilinear Polygons

Let $\mathcal{P}$ be a simple rectilinear polygon with n vertices.[1] In this section we prove that there exists a decomposition for $\mathcal{P}$ with size $O(n)$ and stabbing number $O(\log n)$, and that this bound is tight in the worst case. We start with the lower bound.

Theorem 2.1 *For any $n \geqslant 4$ there exists a simple rectilinear polygon $\mathcal{P}$ with n vertices such that any decomposition of $\mathcal{P}$ has stabbing number $\Omega(\log n)$.*

Proof: Consider a staircase polygon $\mathcal{P}$ with n vertices of the form as depicted in Figure 1. Let $\mathcal{D}$ be a decomposition of $\mathcal{P}$. Define the *vertical stabbing number* of $\mathcal{P}$ (with respect to $\mathcal{D}$) to be the maximum number of rectangles in $\mathcal{D}$ that is intersected by any vertical segment inside the polygon. The *horizontal stabbing number* of $\mathcal{D}$ is defined similarly. Starting at the lower left vertex we number every other vertex of $\mathcal{P}$, thus obtaining a subset $v_1, \ldots, v_m$ of vertices. Notice that $m = \Theta(n)$. We define $\mathcal{P}_{i,j}$ to be the staircase polygon whose lower left vertex is v_i and whose upper right vertex is v_j. The decomposition $\mathcal{D}$ of $\mathcal{P}$ induces in a natural way a decomposition of every $\mathcal{P}_{i,j}$. Thus we can speak of the vertical or horizontal stabbing number of $\mathcal{P}_{i,j}$ (with respect to $\mathcal{D}$). To prove the theorem we will show by induction on k that the following holds for all $0 \leqslant k \leqslant \lfloor \log n \rfloor$: for any $1 \leqslant i \leqslant m - 2^k$, the sum of the vertical stabbing number of $\mathcal{P}_{i,i+2^k}$ and the horizontal stabbing number of $\mathcal{P}_{i,i+2^k}$ is at least $k + 1$.

[1]Note that any rectilinear polygon has an even number of vertices, unless it has adjacent edges that are collinear. For simplicity we will therefore assume in the sequel that n is even.

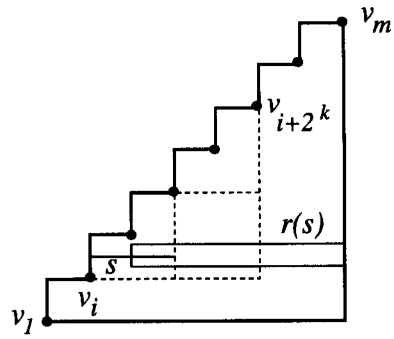

Figure 1: The lower bound construction.

The statement is obviously true in the base case $k = 0$. So let $k \geqslant 1$ and consider some $\mathcal{P}_{i,i+2^k}$. By induction we know that for both $\mathcal{P}_{i,i+2^{k-1}}$ and $\mathcal{P}_{i+2^{k-1},i+2^k}$ the sum of the vertical and the horizontal stabbing number is at least k. Now suppose for a moment that the following holds: *any* horizontal segment inside $\mathcal{P}_{i,i+2^{k-1}}$ will, when extended to the right until it touches the right boundary of $\mathcal{P}_{i,i+2^k}$, intersect a rectangle in $\mathcal{D}$ that it did not intersect inside $\mathcal{P}_{i,i+2^{k-1}}$. Then the horizontal stabbing number of $\mathcal{P}_{i,i+2^k}$ is one larger than the horizontal stabbing number of $\mathcal{P}_{i,i+2^{k-1}}$ and, hence, the horizontal plus the vertical stabbing number of $\mathcal{P}_{i,i+2^k}$ is at least $k+1$. It remains to handle the case where there is a horizontal segment s inside $\mathcal{P}_{i,i+2^{k-1}}$ that does not intersect a new rectangle in the decomposition when extended to the right. Let $r(s)$ be the rectangle in $\mathcal{D}$ that contains the right endpoint of the extended segment. Clearly $r(s)$ must intersect the interior of $\mathcal{P}_{i,i+2^{k-1}}$. But then $r(s)$ cannot intersect the interior of $\mathcal{P}_{i+2^{k-1},i+2^k}$. Hence, *any* vertical segment inside $\mathcal{P}_{i+2^{k-1},i+2^k}$ will, when extended downward until it touches the bottom boundary of $\mathcal{P}_{i,i+2^k}$, intersect a rectangle in $\mathcal{D}$ that it did not intersect inside $\mathcal{P}_{i,i+2^{k-1}}$, namely $r(s)$. So in this case the vertical stabbing number of $\mathcal{P}_{i,i+2^k}$ is one larger than the vertical stabbing number of $\mathcal{P}_{i+2^{k-1},i+2^k}$ and, hence, the horizontal plus the vertical stabbing number of $\mathcal{P}_{i,i+2^k}$ is at least $k+1$. $\square$

Next we show that a decomposition with $O(\log n)$ stabbing number exists for any rectilinear polygon $\mathcal{P}$. The construction of such a decomposition is done in two phases. In the first phase $\mathcal{P}$ is decomposed into histograms in such a way that any axis-parallel line segment inside $\mathcal{P}$ intersects at most three histograms. In the second phase a decomposition is computed for each histogram with $O(\log n)$ stabbing number. Below the two phases are described in more detail.

For the first phase we use the partitioning into histograms described by Levcopoulos [5]. This partitioning can be described recursively as follows. Let e be an edge of $\mathcal{P}$. We call e the *base edge* for the partitioning. Initially the base edge is an arbitrary edge of $\mathcal{P}$; in recursive calls the base edge will be prescribed. Let $\mathcal{H}(e)$ be the maximal histogram inside $\mathcal{P}$ with e as its base. For a horizontal edge e, for example, $\mathcal{H}(e)$ contains all points which are vertically visible from e. Note that $\mathcal{P} \setminus \mathcal{H}(e)$ consists of a number of rectilinear subpolygons $\mathcal{P}_1, \mathcal{P}_2, \ldots$ each of which contains exactly one edge that is also an edge of $\mathcal{H}(e)$ (and which was not an edge of $\mathcal{P}$). This edge is called the *window* of $\mathcal{P}_i$ to $\mathcal{H}(e)$. The partitioning proceeds by partitioning each $\mathcal{P}_i$ recursively, using its window as the base edge. An example is given in Figure 2. Levcopoulos [5] has shown that the total number of vertices of the resulting histograms is $O(n)$. Observe that any axis-parallel segment inside $\mathcal{P}$

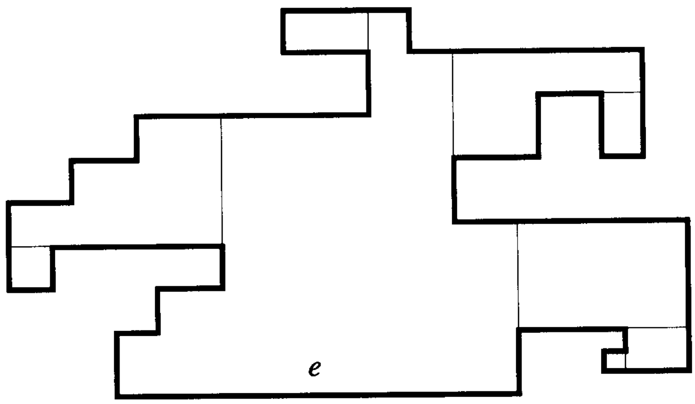

Figure 2: The partitioning into histograms when e is the first base edge.

can intersect no more than two windows. The following lemma readily follows.

Lemma 2.1 *Any simple rectilinear polygon $\mathcal{P}$ with n vertices can be partitioned into a set of histograms with $O(n)$ vertices in total such that any axis-parallel line segment inside $\mathcal{P}$ intersects at most three histograms.*

Next we describe how to construct a decomposition with low stabbing number for a histogram $\mathcal{H}$. With a slight abuse of notation, we denote the number of vertices of $\mathcal{H}$ by n. Assume without loss of generality that the base edge e of $\mathcal{H}$ is horizontal and that $\mathcal{H}$ lies above e. If $\mathcal{H}$ is a rectangle then there is nothing to do; otherwise we proceed as follows. Let $v_1, v_2, \ldots, v_n$ be a clockwise enumeration of the vertices of $\mathcal{H}$, starting at the lower left vertex. First, we split $\mathcal{H}$ into two subhistograms $\mathcal{H}_1, \mathcal{H}_2$ by connecting $v_{n/2}$ to e with a vertical segment ℓ. Second, we split each subhistogram $\mathcal{H}_i$ using a horizontal segment ℓ_i through the lowest edge(s) of $\mathcal{H}_i$ above the base edge e. See Figure 3 for an illustration. This splits $\mathcal{H}_i$ into one or more histograms $\mathcal{H}_{i,j}$ and a rectangle r_i, which has a part of e as its bottom edge. Each $\mathcal{H}_{i,j}$ is decomposed recursively in the same way.

Lemma 2.2 *For any histogram $\mathcal{H}$ with n vertices there exists a decomposition into $O(n)$ rectangles with stabbing number $O(\log n)$.*

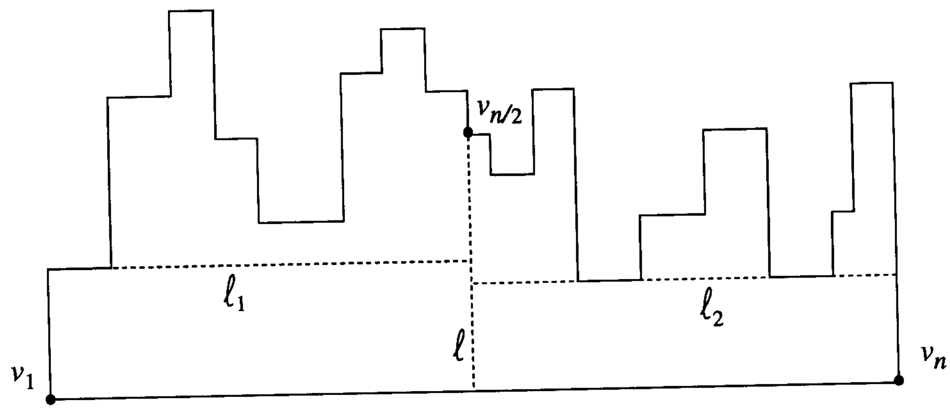

Figure 3: Decomposing a histogram.

Proof: Assume without loss of generality that the base edge of $\mathcal{H}$ is horizontal, and that $\mathcal{H}$ lies above e. Consider the decomposition resulting from the algorithm described above. It is straightforward to check that this decomposition consists of $O(n)$ rectangles. It remains to prove the bound on the stabbing number. To this end we first prove a bound on the quantity $\tilde{\sigma}(n)$, which is defined as the maximum stabbing number, for the decomposition as described above on a histogram with n vertices, of any horizontal segment that touches either the edge v_1v_2 or the edge $v_{n-1}v_n$. Because of the definition of the horizontal split segments ℓ_1 and ℓ_2, we know that such a segment s intersects at most one of the subhistograms $\mathcal{H}_{i,j}$. Also observe that if s intersects one of these subhistograms then s may intersect only one of the rectangles r_1, r_2; if it does not intersects any subhistogram then s may intersect both r_1 and r_2. From the definition of the split segments ℓ, ℓ_1 and ℓ_2 it follows that each $\mathcal{H}_{i,j}$ has at most $n/2$ vertices. So for $n > 4$ we have $\tilde{\sigma}(n) \leqslant 1 + \tilde{\sigma}(n/2)$. Obviously $\tilde{\sigma}(4) = 1$. Hence, $\tilde{\sigma}(n) = O(\log n)$.

From the bound on $\tilde{\sigma}(n)$ we now derive a bound on $\sigma(n)$, the maximum stabbing number of any axis-parallel segment inside $\mathcal{H}$. The important observation is that if a segment intersects two of the subhistograms $\mathcal{H}_{i,j}$ then it must intersect a vertical edge of each subhistogram that is adjacent to its base. We thus have $\sigma(n) \leqslant \max\{2\tilde{\sigma}(n/2), 1 + \sigma(n/2)\}$. Since $\sigma(4) = 1$ it follows that $\sigma(n) = O(\log n)$. □

By combining Lemmas 2.1 and 2.2 we obtain the following result.

Theorem 2.2 *For any simple rectilinear polygon $\mathcal{P}$ with n vertices there exists a decomposition into $O(n)$ rectangles that has stabbing number $O(\log n)$.*

3 Rectilinear Polygons with Rectilinear Holes

We consider the problem of decomposing a rectilinear polygon $\mathcal{P}$ with $k \geqslant 1$ rectilinear holes and n vertices in total (including the vertices of the holes). The algorithm works in two phases. In the first phase, we connect the holes to the boundary to obtain a subdivision into one or more rectilinear regions without holes, and we use the result of the previous section for each region to obtain a decomposition into rectangles with stabbing number $O(\sqrt{k}\log n)$. However, the decomposition may have size $\Omega(n\sqrt{k})$. Therefore, we use a second phase to reduce the number of rectangles by merging adjacent rectangles, without increasing the stabbing number.

Let $\widehat{\mathcal{P}}$ be the simple rectilinear polygon obtained if we remove the holes from $\mathcal{P}$. Let H be the set of $k \geqslant 1$ holes of $\mathcal{P}$, and let $\mathcal{R}$ be a rectangle that encloses the whole polygon $\mathcal{P}$. We start by taking a set S of k points consisting of one vertex of every hole. We take a *kd*-tree approach to connect the points of S, so that we obtain a spanning graph with the property that any horizontal or vertical line intersects $O(\sqrt{k})$ of its edges. The edges of this spanning graph induce a decomposition $\mathcal{D}_{\mathcal{R}}$ of $\mathcal{R}$ into rectangles whose stabbing number is $O(\sqrt{k})$ and which is such that no rectangle has points of S in its interior.

The construction of the spanning graph is done as follows. We partition $\mathcal{R}$ recursively into smaller rectangles using alternately x-splits and y-splits, until no rectangle of the partitioning has a point of S in its interior. In a generic step in the partitioning we have to split a rectangle A containing a subset of the points of S. We describe an x-split to partition A; y-splits are performed analogously with the roles of x- and y-coordinates reversed. Let $S_A \subset S$ be the subset of points lying in the interior of A. If S_A is empty then we are ready with A. Otherwise, we split A into two new subrectangles using a vertical segment through the median x-coordinate of the points in S_A. In the next step of the partitioning we apply a y-split to these two new subrectangles. When the process has finished we are left with a decomposition $\mathcal{D}_{\mathcal{R}}$ of $\mathcal{R}$ into rectangles such that no point of S lies in the interior of a rectangle in $\mathcal{D}_{\mathcal{R}}$.

Lemma 3.1 *The partitioning algorithm described above results in a decomposition of $\mathcal{R}$ which has stabbing number $O(\sqrt{k})$.*

Proof: We prove that any axis-parallel segment s lying inside $\mathcal{R}$ intersects $O(\sqrt{k})$ of the segments used in the partitioning process, which proves the bound on the stabbing number. After we apply an x-split to some rectangle A with k points of S in its interior, the algorithm applies y-splits to the two new subrectangles. Observe that an axis-parallel segment can intersect at most two of the four resulting subsubrectangles. Since the number of points in each of these subsubrectangles is at most $k/4$, the stabbing number $\sigma(k)$ satisfies $\sigma(k) \leqslant 2\sigma(\lfloor k/4 \rfloor)$. It follows that $\sigma(k) = O(\sqrt{k})$. □

Next, we overlay the polygon $\mathcal{P}$ (with its holes) and the decomposition $\mathcal{D}_{\mathcal{R}}$. We remove all parts of $\mathcal{D}_{\mathcal{R}}$ that lie outside $\mathcal{P}$. The remainder of $\mathcal{D}_{\mathcal{R}}$ induces a subdivision of the interior of $\mathcal{P}$ into rectilinear regions. Note that these regions cannot have holes, because then the vertex in S corresponding to the hole would lie in the interior of a rectangle in $\mathcal{D}_{\mathcal{R}}$. For each region in this subdivision, we use the method of the previous section to obtain a decomposition $\mathcal{D}$ into rectangles.

Lemma 3.2 *The decomposition $\mathcal{D}$ has stabbing number $O(\sqrt{k}\log n)$.*

Proof: Let s be any horizontal or vertical line segment inside $\mathcal{P}$. Since the spanning graph has stabbing number $O(\sqrt{k})$, s can intersect at most $O(\sqrt{k})$ regions in the subdivision induced by the overlay. Each region has $O(n)$ vertices, and by Theorem 2.2 it is decomposed into rectangles such that the stabbing number is $O(\log n)$. The lemma follows. □

Although we now have a decomposition $\mathcal{D}$ with low stabbing number we are not finished yet, as it may consist of too many rectangles. To reduce the number of rectangles in $\mathcal{D}$ we apply the following merging step as long as possible. Whenever there are two rectangles $r_1, r_2 \in \mathcal{D}$ that either share a vertical edge or a horizontal edge, we replace r_1 and r_2 by one rectangle $r_1 \cup r_2$. Clearly the merging step can only decrease the stabbing number of $\mathcal{D}$.

Lemma 3.3 *After the merging step $\mathcal{D}$ consists of $O(n)$ rectangles.*

Proof: Since the decomposition is a planar map it suffices to bound its number of vertices. Let $\mathcal{G}$ denote the planar graph corresponding to the decomposition, that is, the nodes of $\mathcal{G}$ are the vertices of $\mathcal{D}$ and the arcs of $\mathcal{G}$ are the edges of $\mathcal{D}$. There are three types of nodes in $\mathcal{G}$: (i) vertices of the polygon $\mathcal{P}$, (ii) endpoints of splitting segments that are used in the partitioning $\mathcal{D}_{\mathcal{R}}$, (iii) intersections of a splitting segment with an edge of $\mathcal{P}$. We have only $O(n)$ nodes of type (i) in $\mathcal{G}$. It is also not difficult to see that the partitioning algorithm uses $O(k)$ splitting segments, so the total number of type (ii) endpoints is also $O(n)$. We will argue that all the remaining nodes in $\mathcal{G}$ are adjacent to a node of type (i) or (ii). Because all nodes in $\mathcal{G}$ have degree at most four this will prove the lemma. So consider a type (iii) node v, and let e be the edge of $\mathcal{P}$ that contains v. Consider the neighbor w of v in $\mathcal{G}$ which lies on the split segment s that intersects e at v. Assume for a contradiction that w is of type (iii); thus w is the intersection of s with some other edge e' in $\mathcal{P}$. See Figure 4. But then there are two rectangles r_1 and r_2 which share the edge vw, contradicting the fact that vw is still present after the merging step. □

From the above two lemmas we obtain the following theorem.

Theorem 3.1 *For any rectilinear polygon $\mathcal{P}$ with $k \geqslant 1$ rectilinear holes and n vertices in total (including the vertices of its holes) there exists a decomposition into $O(n)$ rectangles that has stabbing number $O(\sqrt{k}\log n)$.*

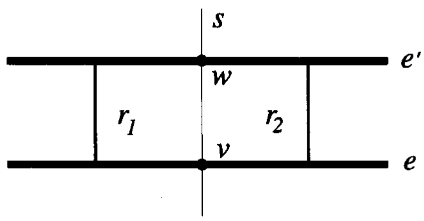

Figure 4: Illustration of the proof of Lemma 3.3.

4 Rectilinear Polygons with Rectangular Holes

In this section we show that for a rectilinear polygon $\mathcal{P}$ with n vertices and with k rectangular holes, we can obtain optimal decompositions of linear size and stabbing number $O(\sqrt{k} + \log n)$. We start with the lower bound, which is based on the following lemma and on Theorem 2.1.

Lemma 4.1 *For any k there exists a rectangle $\mathcal{R}$ and a set H of k rectangular holes such that any decomposition of $\mathcal{R} \setminus (\bigcup H)$ has stabbing number $\Omega(\sqrt{k})$.*

Proof: Let us assume for simplicity that $k = m^2$ for some integer m; the adaptation to the general case is straightforward. The construction that we give is basically the same construction as was used by Edelsbrunner et al. [3] and Chazelle and Welzl [2] to prove lower bounds on the spanning trees with low stabbing number for sets of points in the plane. Let H consist of k equal-sized disjoint rectangles, whose top right vertices are placed in an $m \times m$ grid and let $\mathcal{R}$ be a rectangle enclosing all the rectangles in H. See Figure 5. Observe that for each rectangle there must be at

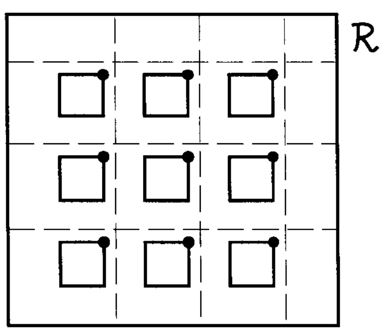

Figure 5: Placement of the rectangles for the lower bound.

least one edge of the decomposition that is not an edge of a rectangle in H and that is incident to the top right vertex of the rectangle. Let δ be the minimum length of such an edge, and consider the set of $2m$ axis-parallel line segments which are at distance $\delta/2$ above or to the right of a top right vertex, and clipped within $\mathcal{R}$. Notice that these line segments do not intersect the rectangles. Each of the m^2 edges incident to a top right vertex of a rectangle is intersected by at least one of the $2m$ line segments. Hence, one of the line segments must have stabbing number $\Omega(m)$. □

Theorem 4.1 *For any $n \geqslant 4$ and any $k \leqslant n/4-1$, there exists a rectilinear polygon $\mathcal{P}$ with k rectangular holes and n vertices in total (including the vertices of its holes) such that any decomposition of $\mathcal{P}$ has stabbing number $\Omega(\log n + \sqrt{k})$.*

To show the upper bound of $O(\sqrt{k} + \log n)$, we first show that if the enclosing rectilinear polygon is a rectangle, then a decomposition of linear size and stabbing number $O(\sqrt{k})$ exists. Then we combine this result with the decomposition result on simple rectilinear polygons. We overlay the two decompositions to obtain a decomposition with the required stabbing number, and then we apply a variation of the merging technique to get a linear size decomposition.

The algorithm that produces a decomposition of a rectangle $\mathcal{R}$ with a set H of rectangular holes is based on the same *kd*-tree approach that we used earlier. However, this time we use all $4k$ vertices of the rectangles in the *kd*-tree construction. We obtain a decomposition with $O(\sqrt{k})$ stabbing number by recursively subdividing every region A that contains at least one vertex of a rectangle of H. If a region is crossed by rectangles, but it contains no vertices, then we do not go into recursion, because this region is already decomposed into rectangles. The resulting decomposition has stabbing number $O(\sqrt{k})$, which can be shown in the same way as Lemma 3.1. We apply the merging technique of the previous section to this decomposition to obtain a decomposition of linear size.

Now consider a polygon $\mathcal{P}$ with k rectangular holes and n vertices in total. Let $\widehat{\mathcal{P}}$ be the polygon $\mathcal{P}$ with its holes removed. Let H be the set of holes of $\mathcal{P}$ and let $\mathcal{R}$ be a rectangle which encloses $\mathcal{P}$ and thus all holes of H. By Theorem 2.2 there exist a decomposition $\mathcal{D}_{\widehat{\mathcal{P}}}$ of the polygon $\widehat{\mathcal{P}}$ with stabbing number $O(\log n)$, and by the above algorithm there exists a decomposition $\mathcal{D}_{\mathcal{R}}$ of the rectangle $\mathcal{R}$ with rectangular holes H with stabbing number $O(\sqrt{k})$. Moreover, both decompositions have linear size. If we superimpose $\mathcal{D}_{\widehat{\mathcal{P}}}$ and $\mathcal{D}_{\mathcal{R}}$ (and remove the parts of the decompositions which are not relevant because they are outside the polygon or inside a hole) we obtain a decomposition $\mathcal{D}$ for $\mathcal{P}$. Moreover the stabbing number of $\mathcal{D}$ is $O(\log n + \sqrt{k})$. However, $\mathcal{D}$ may consist of too many rectangles. So we delete edges of the decomposition to reduce the number of rectangles, in a way that is similar to the merging step that we used in the previous section. More precisely, the removal is done as follows. For each rectangle $r \in \mathcal{D}_{\widehat{\mathcal{P}}}$ we remove the part inside r of every edge of $\mathcal{D}_{\mathcal{R}}$ which does not have an endpoint inside r, provided that this part is not on the boundary of a hole. Similarly, for each rectangle $r \in \mathcal{D}_{\mathcal{R}}$ we remove the part inside r of every edge of $\mathcal{D}_{\widehat{\mathcal{P}}}$ which does not have an endpoint inside r, provided that this part is not on the boundary of $\widehat{\mathcal{P}}$. Notice that the resulting decomposition still consists of rectangles, and that its stabbing number is $O(\log n + \sqrt{k})$.

Lemma 4.2 *After the removal step $\mathcal{D}$ consists of $O(n)$ rectangles.*

Proof: As before we prove the lemma by bounding the number of nodes in the planar graph $\mathcal{G}$ that corresponds to $\mathcal{D}$. There are $O(n)$ nodes that correspond to vertices which were already present in $\mathcal{D}_{\widehat{\mathcal{P}}}$ or $\mathcal{D}_{\mathcal{R}}$. Call these type (i) nodes. The remaining nodes, type (ii) nodes, are intersections between edges of $\mathcal{D}_{\widehat{\mathcal{P}}}$ and $\mathcal{D}_{\mathcal{R}}$. Let v be such an intersection, and let e_1, e_2 be the edges of $\mathcal{D}_{\widehat{\mathcal{P}}}$ and $\mathcal{D}_{\mathcal{R}}$, respectively, whose intersection is v. Clearly, if e_1 is contained in the boundary of $\widehat{\mathcal{P}}$, then e_2 is not contained in the boundary of a hole, and vice versa. Assume that e_1 is not contained in the boundary of $\widehat{\mathcal{P}}$; the argument is similar in the other case. Let r be a rectangle in $\mathcal{D}_{\mathcal{R}}$ which is bounded by e_2 and which is intersected by e_1. We know that e_1 must have an endpoint inside r; otherwise we would have removed e_1. Hence, v is a neighbor of a type (i) node. It follows that there $O(n)$ nodes of type (ii) as well. □

Theorem 4.2 *For any rectilinear polygon $\mathcal{P}$ with k rectangular holes and n vertices in total (including the vertices of its holes) there exists a decomposition into $O(n)$ rectangles that has stabbing number $O(\sqrt{k} + \log n)$.*

5 Conclusions

We have shown several results on partitioning rectilinear polygons without and with holes into a linear number of rectangles, such that any horizontal or vertical segment inside the polygon intersects only few rectangles of the decomposition. For polygons without holes these results complement the results of Hershberger and Suri [4] for decompositions of simple polygons into triangles.

The size and stabbing number of the decompositions for simple rectilinear polygons and for rectilinear polygons with rectangular holes are optimal. For arbitrary rectilinear holes our bounds are not tight; it would be interesting to close the small gap that remains between the upper and lower bound in this case. Another direction for further research is to generalize the result of Hershberger and Suri to polygons with holes, as we did in this paper for the rectilinear case. Finally, we have not considered algorithmic issues in this paper. Some of our methods require quadratic time for constructing a decomposition when implemented naively, and it would be nice to improve this.

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