L_p Optimal d Dimensional Triangulations for Piecewise Linear Interpolation: A New Result on Data Dependent Triangulations

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Abstract

In this paper we address the problem of optimum piecewise linear approximation of general quadratic functions in R^d for $d \geq 2$. In particular the problem is posed as follows: Given a quadratic function $f : R^d \longrightarrow R$ of the form $f(x) = x^T Ax + b^T x + c$, where matrix A symmetric and positive definite and $x \in R^d$. Let $X = \{x_1, x_2, ..., x_n\}$ a set of n points in R^d and let $F = \{f_1, f_2, ..., f_n\}$ the set of values of function f at each of the points of set X (i.e $f(x_i) = f_i$ for $i = 1, 2, ..., n$).

For each triangulation T of the convex hull $CH(X)$ of X and for each set of corresponding values F there exists a unique piecewise linear function π_F^T : $R^d \longrightarrow R$ which approximates function f. We address the problem of finding a triangulation T_0 of $CH(X)$ such that $\pi_F^{T_0}$ best approximates f with respect to the L_p error in any finite dimension d. Since there are exponentially (in n) number of possible triangulations of the point set X it would be impossible to find the optimum using brute force approach. Recently Rippa [7] has provided a solution for the same problem for two dimensions ($d = 2$) only. First we show that the approach of [7] does not generalize in more than two-dimensions. Then we present a completely different technique, than of [7], which enable us to generalize the optimality result of [7] in any finite dimension d. As a consequence of our proofs the optimum triangulation may have long and thin tetrahedra something that people used to avoid. Additionally our proofs are simpler than those in [7]. According to our knowledge this the first result about higher dimensional optimum data dependent triangulations.

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1 Introduction

The problem of piecewise linear interpolation of functions from sampled data is well known and of significant importance. The problem can be posed as follows:

Let $f : R^d \longrightarrow R$ be a function. Let $X = \{x_1, x_2, ..., x_n\}$ a set of n points in R^d and let $F = \{f_1, f_2, ..., f_n\}$ the corresponding values of function f at each of the points of set X (i.e $f(x_i) = f_i$ for $i = 1, 2, ..., n$). We call F as the *sampling vector.*

The standard way to solve the problem is to a) triangulate the convex hull $CH(X)$ of the point set X b) for each point p_i we define a linear function ϕ_i such that $\phi_i(p_i) = 1$ and $\phi_i(q_j) = 0$ for every neighbor q_j of p_i in the underlying triangulation. Then a piecewise linear interpolant of f is a function $\pi_F^T = \sum_{i=1}^n f_i \, \phi_i$.

It is clear that given the point set X and the *sampling vector* F then for each triangulation T of the convex hull $CH(X)$ of X there exists a unique piecewise linear function π_F^T which approximates f. It is also well known that there are exponential number of possible triangulations of a point set in R^d. A reasonable question is which triangulation is optimal in some sense.

In two dimensions, Azevedo and Simpson [13], proved optimal triangulations with respect to L_∞ norm using coordinate transformation. Rippa [7], generalizes the result of [13] for L_p norm with $1 \leq p < \infty$, however his results holds for two dimensions only.

It was believed that the number of "long" and "thin" triangles or tetrahedra should be minimized. This comes from the theoretical analysis of the interpolation error.

Specifically, Bramble and Zlamal [11] and Ciarlet [12] show that the interpolation error with respect to some norms depends on the smallest angle of the underlying triangulation. Babuska and Aziz [10], proved that the small angle condition is too strong and what matters is that no angle should be "very large". Krizek [36], generalized the result of [10] in three dimensions.

Since the Delaunay triangulation (tetrahedrization) usually consists of "well shaped" triangles (tetrahedra) it became the default triangulation for piecewise linear interpolation problem.([23], [35] , [17], [9]).

However recent experimental results [13], [31] show that "bad shaped" triangles are not always bad for linear interpolation. Specifically the experimental results in [13], [31] show instances where the interpolation error we get if we have a triangulation with many "long" and "thin" triangles is much smaller than the error we get if we use the Delaunay triangulation.

As Rippa [7] reports, this inconsistency between the theoretically derived error estimates of [11] and the experimental results of [13], [31] has to do with the way the error estimates were derived. According again to [7], these error estimates were derived with the implicit assumption that all the second derivatives of the function f (which we want to approximate) have magnitude of the same order. If this is not true, then we cannot apply the theoretical result and thus the Delaunay triangulation may not be appropriate.

In an other work Nadler [37], shows that among triangles of given area, the triangle which gives the optimum L_2 approximation of a quadratic function f by a linear polynomial, is "long" in the direction of minimum second directional derivative of f and "narrow" in the direction of maximum second directional derivative of f. However

Nadler [37] does not address the problem of constructing optimal triangulations.

Therefore whenever the function f has some "directionality", if for example

$$\frac{\partial^2 f}{\partial x^2} \gg \frac{\partial^2 f}{\partial y^2}$$

then for better approximation it is advisable to "stretch" the triangles (tetrahedra) along the y direction.

These observations about the accuracy of piecewise linear interpolation depending on the underlying triangulation, led to a newly developed area of *data dependent* triangulations.

In other words, this paper appears as an attempt of finding optimum data dependent triangulations.

"Stretching" the triangles or tetrahedra has not only been considered by approximation theory people. Independently has been considered by researchers working on Finite Element Methods for Computational Fluid Dynamics (CFD). Specifically, in numerical simulations of fluid flow around an airfoil using unstructured meshes, it is desirable to "stretch" the triangles along the direction of fluid flow in the boundary layer region of the airfoil. See for example [19], [28].

In this paper we address the problem of optimum piecewise linear interpolation of general quadratic functions in R^d for $d \geq 2$. In particular the problem is posed as follows: Assume that we have sampled a quadratic function of the form $f(x) = x^\tau A x + b^\tau x + c$ where matrix A is symmetric and positive definite and $x \in R^d$ at each of the n x_i $i = 1,..n$ points of a set $X \subset R^d$. Let $f_i = f(x_i)$ $i = 1,..,n$ and let $F = \{f_1, f_2, ..., f_n\}$ be the sampling vector. Given the sampling vector F, for each triangulation T of the convex hull $CH(X)$ of X there exists a unique piecewise linear function π_F^T which approximates function f. The question is to find the triangulation T_0 of $CH(X)$ such that $\pi_F^{T_0}$ best approximates f with respect to the L_p error. Since there are exponentially (in n) number of possible triangulations of the point set P it would be impossible to find the optimum using brute force approach. In this paper we address the problem of finding the optimum triangulation with respect to L_p error in polynomial time. Specifically we prove that the optimum triangulation may have "long" and "thin" tetrahedra something that people used to avoid. In such a way we generalize the recent result of Rippa [7] in higher dimensions.($d \geq 3$). So far all the results for optimal data dependent triangulations are two-dimensional. According to our knowledge this is the first result about higher dimensional optimal data dependent triangulations.

The problem of approximating functions polynomial or not by polynomial functions of lower degree is of significant importance in many other areas. For example in Computer Aided Geometric Design this problem arises because different systems use different representations for curves and surfaces. Therefore we need to convert from one representation to the other. For example Patrikalakis [27], presents an algorithm which approximates a NURBS curve by a lower degree polynomial curve. Bardis and Patrikalakis [26] generalized the result in [27] for surfaces also. Other related results on approximating curves and surfaces by lower degree curves and its importance in Computer Aided Geometric Design applications appears in [24], [25], [26].

The paper is organized as follows: In section 2 we present some background information, in section 3 we show why Rippa's technique [7] does not generalize in more than two dimensions. In section 4 we present our main result which is a proof that the optimum triangulation in any finite dimension $d \geq 2$ is obtained from a Delaunay triangulation after applying a linear transformation and thus our triangulation may have "long" and "thin" simplices.

2 Background

In this section we introduce the basic definitions and notation as well as well known results which are necessary for our proofs.

2.1 Definitions and Notation

Let x be a point in R^d, then $x^1, x^2, ..., x^d$ represent the coordinates of x. Let X be a set of points in R^d (either finite or infinite), $\partial(X)$ represents the boundary of X and $CH(X)$ represents the convex hull of X.

Clearly $\partial(CH(X))$ represents the boundary of the convex hull of X. For any point $p \in \partial(CH(X))$ there exists at least one hyperplane $h : x^d = \sum_{i=1}^{d-1} {}_i x^i$ such that either $p^d \leq \sum_{i=1}^{d-1} {}_i p^i$ or $p^d > \sum_{i=1}^{d-1} {}_i p^i$. In the first (second resp.) case we say that p lies in the *lower hull* (*upper hull* resp.) of X .

If A represents a matrix then A^τ represents the *transpose* of A and $det(A)$ the determinant of A and I represents the identity matrix. A function $f : R^d \longrightarrow R$ is called *quadratic* iff $f(x) = x^\tau A x + b^\tau x + c$ where matrix A symmetric and positive definite, $b \in R^d$ and $c \in R$. If $A = I$ and $b = O$ and $c = 0$ then function f is called the *unit quadratic* function or *unit paraboloid.*

Let $f : R^d \longrightarrow R$ be a function. Let $X = \{x_1, x_2, ..., x_n\}$ be a set of n points in R^d and let $F = \{f_1, f_2, ..., f_n\}$ the corresponding values of function f at each of the points of set X (i.e $f(x_i) = f_i$ for $i = 1, 2, ..., n$). We call F as the *sampling vector.* Let $X = \{x_1, x_2, ..., x_n\}$ be a set of n points in R^d and let x be another point in R^d. If there exist $\kappa_i \in [0, 1]$ for $i = 1, ..n$ such that $\sum_{i=1}^{n} \kappa_i = 1$ and $x = \sum_{i=1}^{n} \kappa_i x_i$ then x is called *convex combination* of the $x_1, x_2, ..., x_n$.

It is clear that given the point set X and the *sampling vector* F then for each triangulation T of the convex hull $CH(X)$ of X there exists a unique piecewise linear function $\pi_F^T : R^d \longrightarrow R$ which approximates f.

The L_p error of such an approximation is defined as

$$||f(x) - \pi_F^T(x)||_{L_p} = (\int_{CH(X)} |f(x^1, x^2, ..., x^d) - \pi_F^T(x^1, x^2, ..., x^d)|^p \, dx^1 \, dx^2 ... dx^d)^{\frac{1}{p}} \quad (1)$$

For notational simplicity from now on the integral on the right hand side of (1) will be represented as

$$(\int_{CH(X)} |f(x) - \pi_F^T(x)|^p \, dx)^{\frac{1}{p}}$$

where $dx = dx^1 dx^2 ... dx^d$.

In our proofs we are going to use the well known relationship between Delaunay triangulation in dimension d and convex hulls in dimension $d+1$. This fundamental relationship allows us to compute Delaunay triangulations using convex hull algorithms. (see for example [3],[29],[30],[4]).

Let X be a set of n points in R^d. Let K be $k+1$ affinely independent points of X. The convex hull $CH(K)$ defines a *k-simplex* in R^d.

Consequently the convex hull of any $d+1$ *affinelly independent* points in R^d define a *d-simplex* in R^d. In R^2 for example, a *2-simplex* is a triangle and in R^3 a *3-simplex* is a tetrahedron. A *0-simplex* is always a vertex and an *1-simplex* is an edge. A *triangulation* P is a set of *k-simplices* $k = 1,..,d$ such that a) for any two simplices t_i, t_j their intersection is either a *k-simplex* or the empty set and b) their union is the convex hull $CH(X)$ of X. For example in 2-D means that two triangles either intersect at a vertex *0-simplex* or at an edge *1-simplex* or they have empty intersection.

2.2 Lifting Transformation

A point $x = (x^1, x^2, ..., x^d) \in R^d$ is *lifted* to the point $x^* = (x^1, x^2, ..., x^d, \sum_{i=1}^d (x^i)^2) \in R^{d+1}$. (i.e. point $x \in R^d$ is vertically projected onto the unit paraboloid in R^{d+1}.)

Let t be a *d-simplex* in R^d with vertices $p_1, p_2, ..., p_{d+1}$ and let s_t be the hypersphere in R^d which passes through $p_1, p_2, ..., p_{d+1}$. Let q be a point in R^d and let $p_1^*, p_2^*, ..., p_{d+1}^*$ and q^* the corresponding lifted points on the unit paraboloid in R^{d+1}.

Then q lies in the interior, boundary, exterior of s_t, iff q^* lies below, on, above h where h is the hyperplane in R^{d+1} which passes through $p_1^*, p_2^*, ..., p_{d+1}^*$.

The *lifting transformation* gives a direct relationship between Delaunay triangulation in R^d and convex hull in R^{d+1} [5], [2].

In other words a *d-simplex* s in R^d is a Delaunay simplex iff the $d+1$ vertices of s lifted onto the unit paraboloid define a $d-face$ of the lower hull of the lifted points.

For example in two dimensions we have the following:

Let A, B, C, D four points in R^2 and let C_{ABC} the circle through points A, B, C. Let A^*, B^*, C^*, D^* the lifted versions of A, B, C, D on the unit paraboloid. Then D lies inside, on, or outside of C_{ABC} iff D^* lies below, on, or above respectively the plane defined by points A^*, B^*, C^*. (see fig. 1).

2.3 Flipping in Higher Dimensions

This subsection is not necessary for the proof of our result since as we will point out in the next section flipping does not in general produce *globally* optimal triangulations in more than two dimensions. Thus the reader may completely skip this section. However as we prove in lemma 4.4 flipping in d dimensions produces *locally* optimal triangulations.

The concept of *flipping* or *diagonal swapping* goes back to the work of Lawson in 1977, [16], [17].

Assume that you are given a set P of n points in R^2 and you want to construct the Delaunay triangulation of P. You can start with any triangulation T of P and apply repeatedly the following transformation. Let BC be an edge of T and let $\triangle ABC, \triangle DBC$

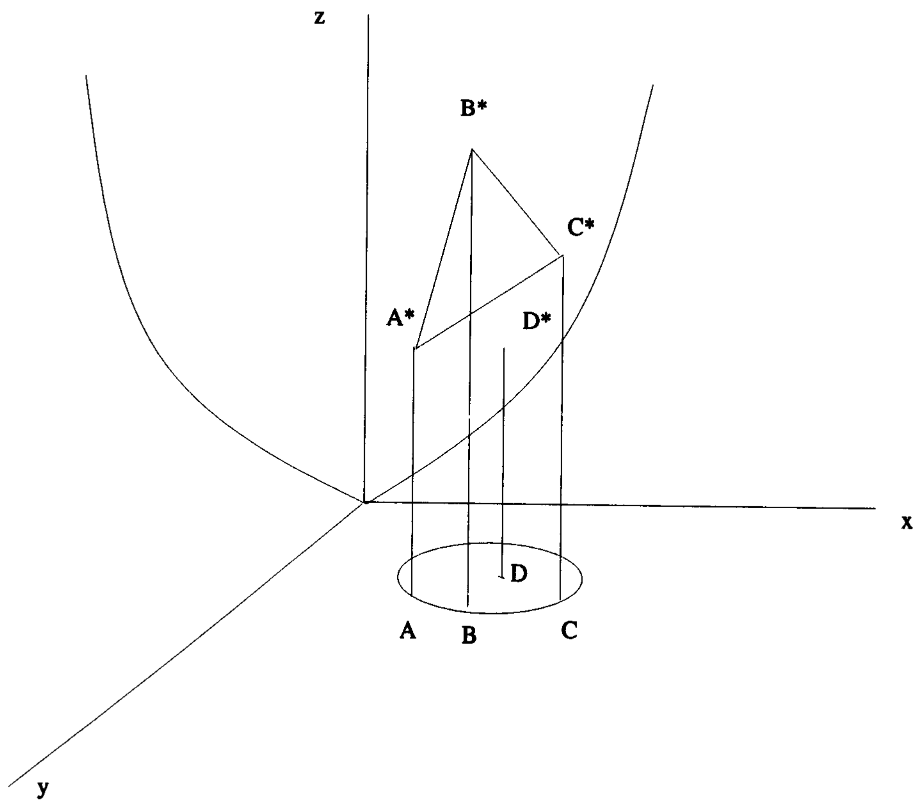

Figure 1: The Lifting Transformation

be the two adjacent triangles of BC. Let C_{ABC} be the circle through points A, B, C. If the quadrilateral $ABCD$ is not convex or point D does not lie in the interior of C_{ABC} then do nothing. If D lies in the interior of C_{ABC} then delete edge BC from the triangulation and insert edge AD.

The above transformation is called *flipping*. Lawson [16], [17] used this transformation to construct 2-D Delaunay triangulations.

However the concept of flipping goes beyond the concept of diagonal swapping. In 2-D has to do to the number of possible triangulations of the convex hull of four points.

Let A, B, C, D be four points in the plane. Then a) either A, B, C, D are all in convex position or b) one of them, say D, lies in the convex hull of the other three. In case of a) we get two possible triangulations namely $\triangle ABC, \triangle DBC$ or $\triangle ABD, \triangle DAC$. In case of b) there are again two possible triangulations namely $\triangle ABC$ and the other consists of three triangles $\triangle DAB, \triangle DBC, \triangle DAC$. However the case where the four points are not in convex position is not of interest since point D is ignored something unacceptable for a triangulation algorithm. (see fig. 2).

Thus in two dimensions there are exactly two ways to triangulate the convex hull of these 4 points.

Lawson [18], generalized the idea of flipping in any fixed dimension d. Other authors like Joe [9], Rajan [6], Edelsbruner and Shah [1] used flipping in higher dimensions in order to construct d- dimensional Delaunay triangulations.

According to [18], given $d+2$ points in R^d, there are exactly two ways that you can triangulate the convex hull of these $d+2$ points. These two triangulations correspond to the lower and upper convex hull of the lifted $d+2$ points on the unit paraboloid in R^{d+1}.

For example in 2-D the four points A, B, C, D are lifted to the four points A^*, B^*, C^*, D^* on the unit paraboloid $z = x^2 + y^2$. Points A^*, B^*, C^*, D^* correspond to the vertices of a tetrahedron. Assume that A, B, C, D are in convex position. The two possible triangulations of the convex quadrilateral $ABCD$ are the projections of the upper and lower faces of the tetrahedron A^*, B^*, C^*, D^*.

The transformation between these two triangulations is defined as *flipping*.

3 Why Rippa's approach does not generalize to more than two dimensions

Rippa's approach [7] can be summarized as follows:

a) First proves that the 2-D Delaunay triangulation is the optimum triangulation for the quadratic function $z = x^2 + y^2$.

b) Second uses coordinate transformation to prove optimal triangulations for convex quadratic functions in two variables x, y.

The part that makes his technique not applicable to more than two dimensions is the way he treats part a). Thus we are going to comment on part a) of his technique. (i.e the optimality proof of the 2-D Delaunay triangulation).

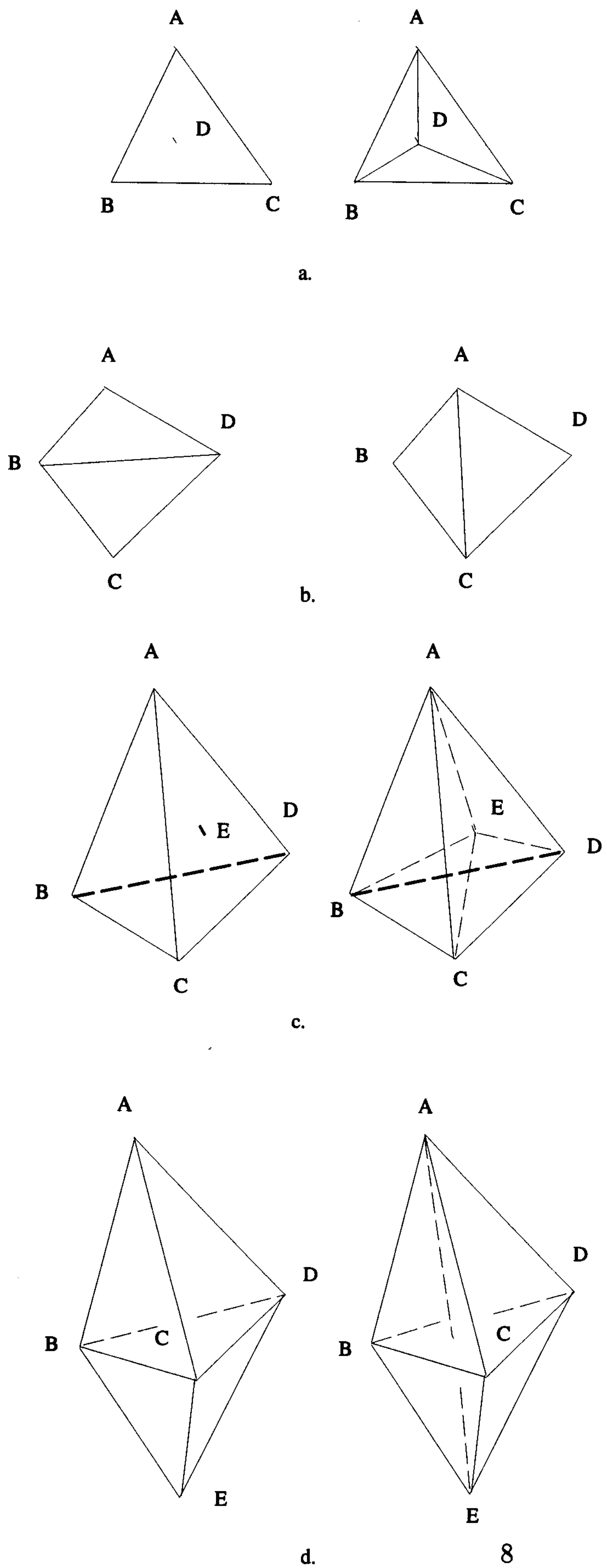

Figure 2: In a, and b the two possible cases of flipping in 2-d. In c and d the two possible cases of flipping in 3-d.

The proof of [7] about the optimality of 2-D Delaunay triangulation can be sketched as follows:

a) First proves that diagonal flipping, according to the empty circle criterion, reduces the L_p error.

b) Secondly uses Lawson's results [16], [17] about the following property of the 2-D Delaunay triangulation: A triangulation $\mathcal{T}$ of a point set $P \subset R^2$ is Delaunay *if and only if* flipping is not applicable.

c) Then combines a) and b). In other words if the optimum triangulation with respect to L_p norm was not Delaunay then according to b) flipping is applicable. But then according to a) flipping reduces L_p norm thus a contradiction.

First the technique which uses to prove that flipping reduces the L_p norm relies on a property of the 2-D Delaunay triangulation and there is no corresponding 3-D counterpart of this property. (see lemma 2.3 of [7]).

Second and most important, the proof of [7] relies on the following fact about two-dimensional Delaunay triangulation: Starting with *any* triangulation of a set of points in R^2 we can construct the Delaunay triangulation of the point set using diagonal flipping.

Unfortunately the above is not true for more than two dimensions. Although flipping generalizes in higher dimensions (see [18], [8],[1], [6]) there is no guarantee that starting with *any* triangulation in R^d $d \geq 3$ and applying flipping as necessary we will eventually construct a Delaunay triangulation.

B. Joe [9], shows cases where the flipping in 3-D stucks although the current tetrahedrization is not Delaunay. In other words if you start with *any* tetrahedrization of a point set in R^3 and you apply flipping you may stuck before you reach the Delaunay stage. That in turn means that if flipping is not applicable we cannot imply that the triangulation is Delaunay. (In order to construct a higher order Delaunay triangulation using flipping this can be done only incrementally. We have to start with a Delaunay triangulation of k points, then add the $k+1$ th point and after the insertion apply flipping. (see [8], [6], [1]). The conclusion is that using flipping the L_p error may stuck at local optima.

In the following section we provide a proof that the Delaunay triangulation is optimum, for unit quadratic functions, for any finite dimension $d \geq 2$. The proof technique is "global" in the sense that does not make use of any local transformations like flipping. It is based on the well known relationship between Delaunay triangulations and Convex Hulls, [5], [2]. Based on this optimality property of the higher dimensional Delaunay triangulation we use coordinate transformation as in [7], [13] to prove optimality results for general quadratic functions in any fixed dimension d. This result casts more light in the new research area of data dependent triangulations.

4 Main Result

Theorem 4.1 *Given the unit quadratic function $f : R^d \longrightarrow R$, $d \geq 2$, a set of $X = \{x_1, x_2, ..., x_n\}$ of n points in R^d, and a set $F = \{f_1, f_2, ..., f_n\}$ of n values of f such* *that $f(x_i) = f_i$ for $i = 1, ..., n$, then the piecewise linear function which corresponds to the Delaunay triangulation of set X minimizes the L_p error.*

Proof:

Let $\pi_F^T(x)$ ($\pi_F^{DT}(x)$ resp.) be the piecewise linear function defined over the data values F and a triangulation T (the Delaunay triangulation resp.) of the point set X.

By definition the L_p error is

$$\|f(x) - \pi_F^T(x)\|_{Lp} = \left(\int_{CH(X)} |f(x) - \pi_F^T(x)|^p \, dx\right)^{\frac{1}{p}}$$

where $x = (x^1, x^2, ..., x^d)$ and $dx = dx^1 \, dx^2 .. dx^d$

Let

$$E_F^T = (\|f(x) - \pi_F^T(x)\|_{Lp})^p$$

Then we get

$$E_F^T = \int_{CH(X)} |f(x) - \pi_F^T(x)|^p \, dx \ (1)$$

$$E_F^{DT} = \int_{CH(X)} |f(x) - \pi_F^{DT}(x)|^p \, dx \ (2)$$

From (1) and (2) we get

$$E_F^{DT} - E_F^T = \int_{CH(X)} |f(x) - \pi_f^{DT}(x)|^p \, dx - \int_{CH(X)} |f(x) - \pi_F^T(x)|^p \, dx =$$

$$= \int_{CH(X)} (|f(x) - \pi_F^{DT}(x)|^p - |f(x) - \pi_F^T(x)|^p) \, dx \ (3)$$

Let $a = \pi_F^{DT}(x) - f(x)$ and $b = \pi_F^T(x) - f(x)$

Because of the convexity of f we have that $a, b \geq 0$.

According to the mean value theorem we have

$$a^p - b^p = (a - b) \, p \, \xi^{p-1}$$

for some $\xi \in (a, b)$ then (3) becomes

$$E_F^{DT} - E_F^T = \int_{CH(X)} (a - b) p \, \xi^{p-1} \, dx = \int_{CH(X)} (\pi_F^{DT}(x) - \pi_F^T(x)) \, p \, \xi^{p-1} \, dx \ (4)$$

But according to the lemma below 4.2

$$\pi_F^{DT}(x) - \pi_F^T(x) \leq 0$$

and thus (4) is always non-positive. Q.E.D. □

Lemma 4.2 *Let $X = \{x_1, x_2, ...x_n\}$ be a set of n points in R^d, let $f : R^d \longrightarrow R$ be the unit quadratic function $F = \{f_1, f_2, .., f_n\}$ be a set of values such that $f(x_i) = f_i$. Then for any triangulation T of X we have $\pi_F^{DT}(x) \leq \pi_F^T(x)$ $\forall x \in CH(X)$.*

Proof:

Assume we *lift* the points in X onto f and let $Q = \{q_1, q_2, ..., q_n\}$ be the set of the *lifted* points.

Let $x \in CH(X)$, with coordinates $(x^1, x^2, ..., x^d)$ and let t be the d-simplex of T such that $x \in t$ and let dt be the d-simplex of DT such that $x \in dt$.

Let $a = \pi_F^{DT}(x)$ and $b = \pi_F^T(x)$.

Consider the points $q_a = (x^1, x^2, ..., x^d, a)$ and $q_b = (x^1, x^2, ..., x^d, b)$ in R^{d+1}.

Let e_{dt} (e_t resp.) be the $d - simplex$ of π_F^{DT} (π_F^T resp.) such that $q_a \in e_{dt}$ and $q_b \in e_t$.

According to the lifting transformation, q_a lies on the lower hull of set Q which means that for every point $q = (x_1, x_2, ..., x_d, c) \in CH(Q)$ we have $c \geq a$.

Since $q_b \in e_t$ implies that q_b can be expressed as a convex combination of the vertices of e_t. But since the vertices of e_{dt} are in Q then q_b can be expressed as a convex combination of the vertices in Q which implies that $q_b \in CH(Q)$ which implies $b \geq a$.

□

Theorem 4.3 *Let $f : R^d \longrightarrow R$ be a quadratic function $f(x) = x^\tau A x + b^\tau x + c$ such that A is a $d * d$ symmetric and positive definite matrix and b is a $d * 1$ vector and $c \in R$. Let $X = \{x_1, x_2, ..., x_n\}$ be a set of n points in R^d and let $F = \{f_1, f_2, ..., f_n\}$ the corresponding values of function f at each of the points of set X (i.e $f(x_i) = f_i$ for $i = 1, 2, ..., n$). Then there exists a linear transformation L which maps each point $x_i \in X$ to a point $y_i \in R^d$ such that the Delaunay triangulation of $y_1, y_2, ..., y_n$ corresponds to the optimal triangulation of X with respect to the L_p error.*

Proof: In our proof we will assume that $b = O$ and $c = 0$ since $b^\tau x + c$ represents the linear term of the quadratic function which clearly can be approximated exactly in a piecewise linear approximation.

Since matrix A is symmetric then there exists an orthogonal matrix S (i.e $SS^\tau = I$) such that

$$A = S^\tau D S$$

where D is the diagonal matrix of the eigenvalues of A. (i.e $D = diag(\lambda_1\ \lambda_2\ ...\lambda_d)$). Since A is also positive definite implies that $\lambda_i > 0$ $i = 1, ..., d$ thus $D = E^\tau E$ where $E = diag(\sqrt{\lambda_1}\ \sqrt{\lambda_2}\ ..\sqrt{\lambda_d})$ and thus

$$A = S^\tau E^\tau E S$$

Thus we get

$$f(x) = x^\tau A x = x^\tau S^\tau E^\tau E S x = y^\tau y = g(y) \ (1)$$

Let T be any triangulation of the point set X. Consider the linear transformation $y = E\,Sx$ which maps each point x_i in X to a point y_i such that $y_i = E\,Sx_i$. Then a triangulation T of X is mapped to a triangulation T^* of Y and the convex hull $CH(X)$ of X is mapped to the convex hull $CH(Y)$ of Y. Let also G be the sampling vector of function $g(y)$ at points y_i $i = 1, ..., n$ (i.e $G = (g(y_1), g(y_2), ..., g(y_n))$.

Let $\pi_F^T(x)$ ($\pi_G^{T^*}(y)$ resp.) defined as usual.

Then $\pi_F^T(x_i) = f(x_i)$ and $\pi_G^{T^*}(y_i) = g(y_i)$ by definition. But $f(x_i) = g(y_i)$ because of (1). Thus $\pi_F^T(x_i) = \pi_G^{T^*}(y_i)$ for $i = 1, ..n$ which implies $\pi_F^T(x) = \pi_G^{T^*}(y)$ $\forall\, y \in CH(Y)$ such that $x = S^\tau E^\tau y$.

Then

$$(||f(x) - \pi_F^T(x)||_{Lp})^p = \int_{CH(X)} |f(x) - \pi_F^T(x)|^p\, dx =$$

$$= det(S^\tau E^\tau) \int_{CH(Y)} |g(y) - \pi_G^{T^*}(y)|^p\, dy = det(S^\tau E^\tau)(||g(y) - \pi_G^{T^*}(y)||_{Lp})^p$$

Thus

$$||f(x) - \pi_F^T(x)||_{Lp} = (det(S^\tau E^\tau))^{\frac{1}{p}} ||g(y) - \pi_G^{T^*}(y)||_{Lp} \quad (2)$$

But according to theorem 4.1, the norm in the right-hand side of (2) is minimized if the triangulation T^* is the Delaunay triangulation of Y.

Thus in order to compute the optimal triangulation for the point-set X

a) apply the transformation $y = Lx$ with $L = E\,S$ on the point-set X, thus obtaining a transformed point-set Y.

b) Compute the Delaunay triangulation of Y. The triangulation of Y induces a corresponding triangulation for X which according to (2) is the optimal one.

□

The following lemma 4.4 is not necessary for the proof of our result since as we pointed out we cannot use the flipping to prove global optimality results for more than two dimensions. Thus the reader may skip it. Lemma 4.4 justs says that flipping in d dimensions *locally* improves the L_p error.

Lemma 4.4 *Let T be a triangulation of n points in d-space. Let f be a d-1 simplex of T. If f is flippable with respect the empty sphere criterion then the L_p norm decreases. Let s_1 and s_2 be two $d-$ simplices which share a $d-1$face. If flipping is applicable between s_1 and s_2*

Proof: Let t_1 and t_2 be the two *d-simplices* adjacent to f. If $v_1, v_2, ...v_d$ are the d vertices of f then $a, v_1, v_2, ...v_d$ are the vertices of t_1 and $b, v_1, v_2, ...v_d$ are the vertices of t_2. Let D be the set $a, b, v_1, v_2, ...v_d$.

For each triangulation of $CH(D)$ there corresponds a piecewise linear. According to Lawson [18], there are two possible triangulations of the convex hull $CH(D)$ of above $d+2$ points of set D. According again to [18], these two possible triangulations are the projections, onto R^d space, of the upper and lower hull of the convex hull of the *lifted* $d+2$ points onto the unit paraboloid. Let T_1 and T_2 be the above mentioned two possible triangulations of $CH(D)$ and let $\pi_F{}^{T_1}$ and $\pi_F{}^{T_2}$ be the corresponding piecewise linear functions. Without loss of generality, let T_1 (T_2 resp.) corresponds to the lower (upper resp.) hull of $CH(D^*)$. Since f is flippable implies that $b \in sphere(t_1)$ implies that the lifted point b^* lies below the hyperplane of the lifted vertices of t_1. This consequently means that simplices t_1 and t_2 belong to triangulation T_2. Thus the flipping transformation converts t_2 to T_1. But by definition of the upper and lower hulls we have that

$\forall\, x \in CH(D)\ \pi_F{}^{T_1}(x) \leq \pi_F{}^{T_2}(x)$. Then using Theorem 4.1 we get that the L_p norm decreases. □

Corollary 4.5 *The optimum L_p triangulation of n points in R^d can be computed in polynomial time.*

Proof: Since it reduces to the computation of a Delaunay triangulation any algorithm which computes higher dimensional Delaunay triangulations or higher dimensional convex hulls can be used. (see [29],[30], [1], [3], [8], [4]). □

Corollary 4.6 *The optimality result described in this paper holds for the L_∞ error.*

Proof: Can be derived from lemma 4.2. □

5 Conclusions and Future research

We presented a method for computing L_p optimal piecewise linear approximation of quadratic functions in R^d. It would be interesting and of significant practical importance to extend the above results to other classes of functions.

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