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Abstract

In this paper we consider the following type of game: two players must color the vertices of a given graph $G = (V, E)$, in a predescribed order, such that no two adjacent vertices are colored with the same color. In one variant, the first player which is unable to move loses the game. In another variant, player 1 wins the game, if and only if the game ends with all vertices colored. In this paper, we obtain several results on the complexity of the problem to decide whether there is a winning strategy for player 1 in a given game instance, when G is restricted to split graphs, interval graphs, or bipartite graphs.

1 Introduction

Much research has recently been done on restrictions of NP-complete graph problems to special classes of graphs (see e.g., [7]). In contrast, only little research has been done on restricting graph problems that are e.g. PSPACE-complete to special classes of graphs. In this paper we consider two PSPACE-complete graph problems, and look at their complexity when restricted to a number of important classes of perfect graphs, namely the split graphs, the interval graphs, and the bipartite graphs.

The problems we consider come from two games, where two players alternately color the vertices of a graph. To be precise, we actually consider classes of games, where game-instances differ in details like: the number of colors, and the graph where the game is played on, but not in the 'basic rules'.

The first game we consider is the SEQUENTIAL COLORING GAME. In this game, a graph $G = (V, E)$, a linear ordering of G (i.e., a bijection $f : V \rightarrow \{1, 2, \cdots, |V|\}$), an ownership function of G (i.e., a function $owner : V \rightarrow \{1, 2\}$), and a finite set of

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colors C are given. The game is played with two players, 1 and 2, that always have full information. Until the game ends, the player that is the owner of the smallest numbered yet uncolored vertex v must color v with a color from C that has not been given before to any vertex adjacent to v. I.e., player *owner*(1) must color $f^{-1}(1)$, then player *owner*(2) must color $f^{-1}(2)$, etc. The game ends when a player is unable to color a vertex that he must color, or when all vertices are colored. In the former case, the player that must color a vertex v but is unable to do so, loses the game. In the latter case, the player owning the last vertex $f^{-1}(|V|)$ wins the game.

The SEQUENTIAL COLORING CONSTRUCTION GAME is similar, with the difference that now player 1 wins, if and only if the game ends when all vertices are colored. This game arises in the following, more or less practical situation. Suppose a number of jobs must be sceduled on a number of machines, in a fixed order. There are a number of constraints of the type: 'job i and job j may not be sceduled on the same machine'. The order, and the constraints are known in advance. Not all jobs however will be sceduled by the same algorithm: e.g. another machine or users will make sceduling decisions for some of the jobs. This situation can be modeled in a straightforward way as an instance of the SEQUENTIAL COLORING CONSTRUCTION GAME (see [3]), and there exists an algorithm that makes sure that all jobs can be sceduled, if and only if there is a winning strategy for player 1 in this game instance.

In this paper we consider the computational complexity of the following type of problems: given a game (instance) from a game (class), determine whether there is a winning strategy for player 1. We use the name of the game (class) to denote this problem.

In [3] it was proved that SEQUENTIAL COLORING GAME and SEQUENTIAL COLORING CONSTRUCTION GAME are PSPACE-complete, even when there are exactly three colors. If there are two colors, then efficient polynomial time algorithms exist for the problems. Also, if the linear ordering has bounded separation number, then the problems can be solved in polynomial time. (See [3] for details.)

In [3] it was also observed that — when the restrictions on the problem remain true under insertion of isolated vertices — we equivalently may assume that players alternately color the vertices (i.e., $owner(i) = i \bmod 2$): insert between every pair of vertices that is owned by the same player a new isolated vertex owned by the other player.

In this paper we consider the complexity of the problems, when the graph G is restricted to certain classes of perfect graphs. After some necessary definitions in section 2, we consider split graphs in section 3, interval graphs in section 4, and bipartite graphs in section 5. Some open problems are mentioned in section 6.

2 Definitions

In this section we give most of the definitions and notations used in this paper.

Let graph $G = (V, E)$, linear ordering $f : V \to \{1, 2, \cdots, |V|\}$, set of colors C, and ownership function $owner : V \to \{1, 2\}$ be given. A *partial coloring* is a function $col : \{1, \cdots, i\} \to C$ for some i, $0 \le i \le |V|$, and for all $(v, w) \in E$, $f(v), f(w) \in \{1, \cdots, i\} : col(f(v)) \ne col(f(w))$. A partial coloring $col : \{1, \cdots, i\} \to C$ is a *total coloring*, if $i = |V|$. We say that a partial coloring $col : \{1, \cdots, i\} \to C$ is a *winning position* (or *winning*) *for player* α ($\alpha = 1, 2$), if player α has a winning strategy in the considered game, when started with all vertices $j \in \{1, \cdots, i\}$ colored with color $col(j)$, and all vertices $i + 1, \cdots, |V|$ uncolored. A color c is called a *legitimate move* from partial coloring $col : \{1, \cdots, i\} \to C$, if $i + 1 \le |V|$, and there does not exist an edge $(v, w) \in E$, with $f(v) \in \{1, \cdots, i\}$, $f(w) = i + 1$, and $col(f^{-1}(v)) = c$. The *color classes* of a partial coloring $col : \{1, \cdots, i\} \to C$, are the sets $\{f^{-1}(j) \mid col\ (j) = c\} \subseteq V$ for all colors $c \in C$.

For $G = (V, E)$, $W \subseteq V$, $N_G(W)$ denotes the set of neighbors of W: $\{v \in V \mid \exists w \in W : (v, w) \in E\}$.

A *perfect elimination ordering* of a graph $G = (V, E)$ is a linear ordering $f : V \to \{1, \cdots, |V|\}$, such that for all v, w, x: if $f(v) < f(w) < f(x)$, and $(v, w) \in E$, $(v, x) \in E$, then $(w, x) \in E$.

A graph $G = (V, E)$ is called a *split graph*, if V can be partitioned into two sets V_1, V_2, with V_1 an independent set ($\forall v, w \in V_1 : (v, w) \notin E$), and V_2 a clique ($\forall v, w \in V_2 : (v, w) \in E$). We call V_1 *the independent set of* G, and V_2 the *clique of* G. Split graphs have perfect elimination orderings.

A graph $G = (V, E)$ is bipartite, if V can be partitioned into two independent sets V_1, V_2, called the *color classes* of G.

A graph $G = (V, E)$ is an interval graph, if one can associate with each vertex $v \in V$ an interval $[b_v, e_v] \subseteq R$, such that for all $v, w \in V$, $v \ne w$: $(v, w) \in E \Leftrightarrow [b_v, e_v] \cap [b_w, e_w] \ne \emptyset$. Interval graphs have perfect elimination orderings. A very natural one is to order the vertices with respect to the right endpoints of the associated intervals.

More on split graphs, interval graphs, and other classes of perfect graphs can be found in [4, 6].

3 Split graphs

In this section we consider the SEQUENTIAL COLORING GAME and the SEQUENTIAL COLORING CONSTRUCTION GAME on split graphs. We show NP-hardness and co-NP-hardness of the problems, even when the linear ordering is such that first all vertices in the 'independent set' must be colored, and then all vertices in the 'clique'. In all but one cases, we may assume that one player owns all vertices in the independent set and the other player owns all vertices in the clique.

We use the following problem to transform from:

> PERFECT COLORING OF 3-REGULAR GRAPHS
> **Instance**: 3-Regular graph $G = (V, E)$.
> **Question**: Is there a coloring $f : V \to \{1, 2, 3, 4\}$ of G with four colors, such that for all $v \in V$: $f(\{v\} \cup N_G(v)) = \{1, 2, 3, 4\}$?

In other words, in a perfect coloring, for each vertex v, each color appears exactly once in the set of colors given to v and its neighbors. Bakker et. al. [1] showed that PERFECT COLORING OF 3-REGULAR GRAPHS is NP-complete.

Theorem 3.1
The SEQUENTIAL COLORING GAME on split graphs is NP-hard, even when first every vertex in the independent set and then every vertex in the clique appears in the linear ordering, and player 1 owns every vertex in the independent set, and player 2 owns every vertex in the clique.

Proof.
To show NP-hardness, we use a transformation from the PERFECT COLORING FOR 3-REGULAR GRAPHS problem. Let $G = (V, E)$ be a 3-regular graph with $|V| = n$. W.l.o.g. suppose that $n \geq 6$.

Now construct a split graph $G' = (V', E')$ as follows. Let $V' = V_1 \cup V_2$, with $V_i = \{v_i \mid v \in V\}$, $i = 1, 2$. Let $E' = \{(v_1, w_2) \mid (v, w) \in E \text{ or } v = w\} \cup \{(v_2, w_2) \mid v_2, w_2 \in V_2,\ v_2 \neq w_2\}$. Let f be an arbitrary linear ordering on G. Now let f' be the linear ordering on G', with for all $v \in V$:

$$\begin{aligned} f'(v_1) &= f(v) \\ f'(v_2) &= n + f(v) \end{aligned}$$

Let $owner(v_i) = i$, for all $v \in V$, $i = 1, 2$.

We now claim that there is a winning strategy for player 1 in the SEQUENTIAL COLORING GAME played on G' with linear ordering f', ownership function as defined above, and with $n + 3$ colors, if and only if G has a perfect coloring (with four colors).

Note that player 1 has a winning strategy, if and only if there is a partial coloring of V_1, such that there is no proper coloring of G' with $n + 3$ colors, containing that coloring. For when such a coloring exists, then player 1 colors V_1 with that coloring, and player 2 will be unable to color all vertices in V_2 and loses, otherwise player 2 is able to color all vertices in V_2 and wins.

Suppose G has a perfect coloring $col : V \to \{1, 2, 3, 4\}$. Then player 1 colors V_1 in the same way: color v_1 with $col(v)$ for all $v_1 \in V_1$. Now every vertex $v_2 \in V_2$ is adjacent to a vertex with color i, for each $i \in \{1, 2, 3, 4\}$. So player 2 has only $n + 3 - 4$ colors available to color the clique V_2. This is impossible because $|V_2| = n$, hence player 1 wins.

Now suppose G has no perfect coloring. We consider four different cases.

Case 1. Player 1 used at most three different colors to color the vertices of V_1. There are at least n other colors, so player 2 can use these to color the vertices of V_2.

Case 2. Player 1 used exactly four different colors to color the vertices in V_1, say c_1, c_2, c_3, c_4. Then at least one vertex $v_2 \in V_2$ is not adjacent to a color $c \in \{c_1, c_2, c_3, c_4\}$, because otherwise the partial coloring of V_1 would imply a perfect coloring of G. The partial coloring of V_1 can now be extended to a total coloring of G', by giving color c to v_2, and using the other $n-1$ colors to color the other vertices of V_2.

Case 3. Player 1 used $4 < k < n$ colors to color the vertices of V_1. We prove by induction to k, starting with $k = 4$, that any k-coloring of V_1 is contained in a $(n+3)$-coloring of G', if $4 \leq k < n$ holds. The case that $k = 4$ is dealt with in Case 2.

Suppose the assertion is true for $k \geq 4$, $k < n-1$. Let $\tilde{c}'$ be a partial $(k+1)$-coloring of V_1 with color classes $I_1, I_2, \cdots, I_{k+1}$. W.l.o.g. suppose that $|I_1| \leq |I_2| \leq \cdots \leq |I_{k+1}|$. Hence $|I_1| \leq \frac{n}{k+1}$. Let $c' : V_1 \to \{1, \cdots, k\}$ be the partial k-coloring of V_1, defined by $c'(v) = 1$, if $\tilde{c}'(v) = 1$ or $\tilde{c}'(v) = 2$; and $c'(v) = \tilde{c}'(v) - 1$ otherwise. By induction hypothesis, there is a $(n+3)$-coloring $c : V_1 \cup V_2 \to \{1, \cdots, n+3\}$, containing c'.

Now note that $|I_1| \leq n/(k+1)$ implies that $|N_{G'}(I_1)| \leq \frac{4n}{k+1}$, since any vertex of V_1 has degree four. As $\frac{4n}{k+1} < n + 3 - k$ for $4 \leq k < n-1$, it follows that there is at least one color α that does not belong to $c(V_1) \cup c(N_{G'}(I_1))$. Hence, if we change c such that all vertices in I_1 get color α, then we obtain again a coloring c'' of G'. Note that the color classes of the partial coloring, obtained by restricting c'' to V_1 are exactly $I_1, I_2, \cdots, I_{k+1}$. So $\tilde{c}'$ is contained in a coloring $\tilde{c}$ of G', that can be obtained from c'' by renaming some colors.

Case 4. Player 1 used n different colors to color the vertices in V_1, i.e., every vertex in V_1 received a different color. Consider the bipartite $(n-4)$-regular graph $G'' = (V_1 \cup V_2, E'')$ with $E'' = \{(v_1, w_1) \mid (v_1, w_2) \notin E,\ v_1 \in V_1,\ w_2 \in V_2\}$. Since every r-regular bipartite graph contains a perfect matching (see e.g., [2], p. 133), G'' has a perfect matching. Player 2 colors each vertex v_2 with the color r, such that there is a matching edge (w_1, v_2) in G'', and w_1 was colored with color r by player 1. This yields a total coloring of G'.

We have shown that the graph G has a perfect coloring, if and only if player 1 has a winning strategy in the corresponding instance of SEQUENTIAL COLORING GAME. As the transformation can be carried out in polynomial time, the theorem now follows. □

By just changing the roles of player 1 and player 2, we directly obtain the following result.

Theorem 3.2
The SEQUENTIAL COLORING GAME and the SEQUENTIAL COLORING CONSTRUCTION GAME on split graphs are coNP-hard, even when first every vertex in the independent set and then every vertex in the clique appears in the linear ordering, and player 2 owns every vertex in the independent set, and player 1 owns every vertex in the clique.

We also have the following result.

Theorem 3.3
The SEQUENTIAL COLORING CONSTRUCTION GAME on split graphs is NP-hard, even when first every vertex in the independent set and then every vertex in the clique appears in the linear ordering.

Proof.
Again we transform from PERFECT COLORING FOR 3-REGULAR GRAPHS. (One can also use the DOMATIC NUMBER problem (cf. [5]) instead.)

Let G be a 3-regular graph. Let an arbitrary linear ordering f of G be given.

Construct a graph $G' = (V', E')$ as follows: let $V' = V_1 \cup V_2 \cup V_3 \cup V_4$, with $V_1 = \{x^i \mid 1 \leq i \leq n\}$, $V_2 = \{v_2 \mid v \in V\}$, $V_3 = \{v_3 \mid v \in V\}$, $V_4 = \{y^i \mid 1 \leq i \leq 4\}$, and $E' = \{(x^i, y^j) \mid x^i \in V_1,\ y^j \in V_4\} \cup \{(v_3, w_3) \mid v_3, w_3 \in V_3,\ v \neq w\} \cup \{(y^i, y^j) \mid y^i, y^j \in V_4,\ i \neq j\} \cup \{(v^3, y^j) \mid v^3 \in V_3,\ y^j \in V_4\} \cup \{(v_2, w_3) \mid v_2 \in V_2,\ w_3 \in V_3,\ v = w \vee (v, w) \in E\}$. (So, $V_1 \cup V_2$ form the independent set, and $V_3 \cup V_4$ form the clique.)

Player 1 owns every vertex in V_2; player 2 owns every other vertex. Let $f' : V' \to \{1, 2, \cdots, 3n+4\}$ be the linear ordering of G', defined by

$$\begin{array}{rcll} f'(x^i) & = & i & (x^i \in V_1) \\ f'(v_2) & = & n + f(v) & (v_2 \in V_1) \\ f'(v_3) & = & 2n + f(v) & (v_3 \in V_1) \\ f'(y^i) & = & 3n + i & (y^i \in V_1) \end{array}$$

We now claim that there is a winning strategy for player 1 for the SEQUENTIAL COLORING CONSTRUCTION GAME, played on G' with linear ordering f', $n+4$ colors, and ownership, as described above, if and only if G has a perfect coloring.

Suppose G has a perfect coloring $col : V \to \{1, 2, 3, 4\}$. When coloring V_2, player 1 selects four colors that were not used by player 2 when coloring V_1. (There are at least four such colors.) Let these colors be c_1, c_2, c_3, c_4. Now player 1 colors each vertex $v_2 \in V_2$ with color $c_{c(v)}$, i.e. with color c_i, if v is colored with i in the perfect coloring of G. Now player 2 cannot use colors c_1, c_2, c_3, c_4 when coloring V_3. So player 2 must color V_3 with all n other colors, and then he can and must color V_4 with colors $c_1, \cdots, v_4$. So player 1 wins the game.

Next suppose that G has no perfect coloring. Player 2 starts by giving each vertex x^i color i. After player 1 has colored V_2, there must be at least one vertex $v_3 \in V_3$

and a color $\alpha \in \{n+1, n+2, n+3, n+4\}$ such that v_3 is not adjacent to a vertex in V_2 with color α. (Otherwise, the coloring of V_2 would imply a perfect coloring of G: give $v \in V$ color $r - n$, where r is the color of v_2.) So player 2 can color V_3, such that at least one vertex in V_3 receives a color $\alpha \in \{n+1, n+2, n+3, n+4\}$. But now V_4 cannot entirely be colored, as V_4 contains four vertices that are adjacent to vertices in all the colors in $\{1, 2, \cdots, n, \alpha\}$. So player 2 wins the game. This proves the claim, and, because the transformation can be done in polynomial time, also the theorem. □

By noting that inserting isolated vertices in a perfect elimination ordering yields again a perfect elimination ordering, we have the following corollary:

Corollary 3.4
SEQUENTIAL COLORING GAME, and SEQUENTIAL COLORING CONSTRUCTION GAME are NP-hard and coNP-hard, when restricted to split graphs, with the linear ordering a perfect elimination ordering, and players alternately coloring one vertex.

4 Interval graphs

In this section we give a polynomial time algorithm for the SEQUENTIAL COLORING CONSTRUCTION GAME for interval graphs for a certain type of linear orderings. This type includes the perfect elimination orderings and the reversals of these as special cases.

Definition. Let $G = (V, E)$ be an interval graph, and let for each $v \in V$ two numbers be given $b_v, e_v \in R$, such that

- $\forall v \in V : b_v \leq e_v$
- $\forall v, w \in V : (v, w) \in E \Leftrightarrow [b_v, e_v] \cap [b_w, e_w] \neq \emptyset$.

A linear ordering $f : V \rightarrow \{1, 2, \cdots, |V|\}$ of G is called an *interval representation compatible linear ordering* (or: *irclo*), if and only if there exists a function $g : V \rightarrow R$, such that

- $\forall v \in V : b_v \leq g(v) \leq e_v$
- $\forall v, w \in V : f(v) \leq f(w) \Leftrightarrow g(v) \leq g(w)$.

g is called the *stitch-function* of irclo f.

In other words, an irclo f is obtained by choosing a point in each interval with a stitch-function g, and then ordering the vertices in the order on the real line of the corresponding stitch-points.

Clearly, the perfect elimination ordering of G, obtained by ordering vertices with respect to the right endpoints of the corresponding intervals is an irclo. The 'reversal' of this ordering is also an irclo, as is shown by the following lemma.

Lemma 4.1
If $f : V \to \{1, \cdots, n\}$ is an irclo, then the reversal of f, $f^{rev}(i) = n + 1 - f(i)$ is also an irclo.

Proof.
Suppose f is an irclo with respect to interval representation $[b_v, e_v] \subseteq R$ for all $v \in V$, and with stitch function g. Now f^{rev} is an irclo with respect to interval representation $[-e_v, -b_v]$ for all $v \in V$, and stitch function $-g$. □

In our algorithm, the following function plays a vital role. Let C denote the set of colors.

Definition. *last* is a function, mapping partial colorings $col : \{1, \cdots, i\} \to C$ and colors $c \in C$ to real numbers:

$$last(col, c) = \max\{e_v \mid 1 \leq f(v) \leq i \wedge col(i) = c\}$$

Definition. A partial coloring $col_1 : \{1, \cdots, i\} \to C$ dominates a partial coloring $col_2 : \{1, \cdots, i\} \to C$, if and only if there exists a bijection $\psi : C \to C$, such that for all $c \in C$: $last(col_1, c) \geq last(col_2, \psi(c))$.

We may and will assume that the bijection ψ in the definition above is order-preserving, in the sense that for all pairs of colors c_1, $c_2 \in C$: if $last(col_1, c_1) \geq last(col_1, c_2)$, then if $last(col_2, \psi(c_1)) \geq last(col_2, \psi(c_2))$. (If col_1 dominates col_2, then one can use for ψ the bijection, obtained by first ordering C with respect to $last(col_1, c)$, then ordering C with respect to $last(col_2, c)$, and mapping the i'th color in the first ordering to the i'th color in the second ordering, for all i, $1 \leq i \leq |C|$.)

We assume that $G = (V, E)$, and b_v, e_v, $g(v)$ (for all $v \in V$) are given. To ease presentation, we assume that $V = \{1, 2, 3, \cdots, n\}$, and $f(i) = i$ for all $i \in V$; f is an irclo.

Lemma 4.2
Suppose $col_1 : \{1, \cdots, i\} \to C$ dominates $col_2 : \{1, \cdots, i\} \to C$, with order preserving bijection ψ, and c is a legitimate move for $i + 1$ from col_1, then $\psi(c)$ is a legitimate move for $i + 1$ from col_2.

Proof.
Suppose $\psi(c)$ is not a legitimate move for $i + 1$ from col_2. Then there exists a $j \leq i$ with $(j, i) \in E$ and $col_2(j) = \psi(c)$. So $last(col_2, \psi(c)) \geq e_j$, hence $last(col_1, c) \geq e_j$. Write $e_k = last(col_1, c)$. From $(j, i) \in E$, it follows that $e_j \geq b_i$. Also, $k \leq i$, hence $g(k) \leq g(i)$. So $e_k \geq e_j \geq b_i$, and $b_k \leq g(k) \leq g(i) \leq e_i$. Now $[b_k, e_k] \cap [b_i, e_i] \neq \emptyset$, hence $(i, k) \in E$. But then c was no legitimate move for $i+1$ from col_1, as $col_1(k) = c$. Contradiction. □

Lemma 4.3
Color c is allowed for $i+1$ from $col : \{1, \cdots, i\} \to C$, if and only if $last(col, c) < b_{i+1}$.

Proof.
If $last(col, c) \geq b_{i+1}$, then there exists a j, $1 \leq j \leq i$, with $col(j) = c$ and $e_j \geq b_{i+1}$. We now have: $b_j \leq g(i) \leq g(i+1) \leq e_{j+1}$. Hence $[b_j, e_j] \cap [b_{i+1}, e_{i+1}] \neq \emptyset$, i.e., $(j, i+1) \in E$. Hence c is not allowed for $i+1$.

Alternately, suppose that c is not allowed for $i+1$ from col. Then there must be a j, $1 \leq j \leq i$, with $col(j) = c$, and $(j, i+1) \in E$. As $b_j \leq g(j) \leq g(i+1) \leq e_{j+1}$ and $[b_j, e_j] \cap [b_{i+1}, e_{i+1}] \neq \emptyset$, we must have that $e_j \leq b_{i+1}$. Now $last(col, c) \geq e_j \geq b_{i+1}$.
□

Lemma 4.4
If a partial coloring $col_1 : \{1, \cdots, i\} \to C$ is winning for player 1, and col_1 dominates partial coloring $col_2 : \{1, \cdots, i\} \to C$, then col_2 is winning for player 1.

Proof.
We use the following notation in this proof: for a partial coloring $col : \{1, \cdots, i\} \to C$, $i < n$, and color $c \in C$, we denote with $(col + c)$ the partial coloring $\{1, \cdots, i, i+1\} \to C$, defined by $(col + c)(j) = col(j)$ for $1 \leq j \leq i$, and $(col + c)(i+1) = c$.

To prove the lemma, we use downward induction to i, starting with $i = n$.

Clearly, if $i = n$, then both $col_1 : \{1, \cdots, i\} \to C$ and $col_2 : \{1, \cdots, i\} \to C$ are total colorings, and hence winning positions for player 1.

Suppose that for certain $i < n$, the lemma is true for all partial colorings $\{1, \cdots, i+1\} \to C$. We consider the case that player 1 owns $i+1$, and the case that player 2 owns $i+1$.

First suppose that player 1 owns $i+1$, col_1 is a winning position for player 1 and dominates col_2; $col_1, col_2 : \{1, \cdots i\} \to C$. Suppose that coloring $i+1$ with color c is a winning move for player 1 from col_1, i.e., the partial coloring $(col_1 + c)$ is winning for player 1. Then by lemma 4.2, $\psi(c)$ is a legitimate move for $i+1$ from col_2. It is also a winning move. To prove this, it is, by induction hypothesis, sufficient to prove that $(col_1 + c)$ dominates $(col_2 + \psi(c))$. This follows, because for all colors $c' \neq c$, $last((col_1 + c), c') = last(col_1, c') \geq last(col_2, \psi(c')) = last((col_2 + \psi(c)), \psi(c'))$, and $last((col_1 + c), c) = e_{i+1} = last((col_2 + \psi(c)), \psi(c))$.

Next suppose that player 2 owns $i+1$. We now show that if $col_1 : \{1, \cdots, i\} \to C$ dominates $col_2\{1, \cdots, i\} \to C$, and col_2 is a winning position for player 2, then col_1 is also a winning position for player 2. We consider several cases.

Case 1. Player 2 cannot move from col_2. Then, by lemma 4.2 he also cannot move from col_1. Hence col_1 is a winning position for player 2.

Case 2. Player 2 has a winning move from col_2, say this move consists of coloring $i+1$ with c.

Case 2.1. $\psi^{-1}(c)$ is a legitimate move for $i+1$ from col_1. Note that $(col_1+\psi^{-1}(c))$ dominates $(col_2 + c)$. As $(col_2 + c)$ corresponds to a winning position for player 2, $(col_1 + \psi^{-1}(c))$ cannot correspond to a winning position for player 1, hence $\psi^{-1}(c)$ is a winning move for player 2 from col_2.

Case 2.2. Player 2 has no legitimate move for $i+1$ from col_1. Clearly then col_1 corresponds to a position that is winning for player 2.

Case 2.3. $\psi^{-1}(c)$ is not a legitimate move for $i + 1$ from col_1, but c' is a legitimate move for $i+1$ from col_1. By lemma 4.3 we have that $last(col_1, c') < b_{i+1} \leq last(col_1, \psi^{-1}(c))$. Now $(col_1 + c')$ dominates $(col_2 + c)$: use bijection $\psi' : C \rightarrow C$, defined by $\psi'(c'') = \psi(c'')$ for all $c'' \in C$, $c'' \neq c'$, $c'' \neq \psi^{-1}(c)$, and $\psi'(c') = c$, $\psi'(\psi^{-1}(c)) = \psi(c')$. One can easily check the domination requirements: for $c'' \neq c', \psi^{-1}(c)$, $last((col_1 + c'), c'') = last(col_1, c'') \geq last(col_2, \psi(c'')) = last((col_2 + c), \psi(c''))$; $last((col_1 + c'), c') = e_{i+1} = last((col_2 + c), c)$; and $last((col_1 + c), \psi^{-1}(c)) = last(col_1, \psi^{-1}(c)) > last(col_1, c') \geq last(col_2, \psi(c')) \geq last((col_2 + c), \psi'(\psi^{-1}(c)))$. We have shown that $(col_1 + c')$ dominates $(col_2 + c)$, hence, by induction, c' is a winning move for player 2 for $i + 1$ from col_1.

This ends our cases analysis, and the inductive proof of our theorem. □

Lemmas 4.3 and 4.4 enable us to identify the 'best possible moves' for players 1 and 2.

Lemma 4.5
Let $col : \{1, \cdots, i\} \rightarrow C$ be a partial coloring, $i < n$, and suppose there exists at least one color $c \in C$ with $last(col, c) < b_{i+1}$.
(i) Suppose player 1 owns $i + 1$. Player 1 has a winning move for $i + 1$ from col, if and only if coloring $i + 1$ with the color c fulfilling the following condition is a winning move for $i + 1$ from col for player 1: $last(col, c) < b_{i+1}$ and there does not exist a color c' with $last(col, c) < last(col, c') < b_{i+1}$.
(ii) Suppose player 2 owns $i + 1$. Player 2 has a winning move for $i + 1$ from col, if and only if coloring $i + 1$ with the color c fulfilling the following condition is a winning move for $i + 1$ from col for player 2: $last(col, c)$ is minimal over all $c \in C$.

Proof.
From the assumption $\exists c \in C : last(col, c) < b_{i+1}$ it follows that at least one color is possible for $i + 1$ from col.

(i) Suppose $last(col, c_1) \leq last(col, c_2) < b_{i+1}$. Then c_1, c_2 are allowed colors for $i+1$ from col. The coloring $(col+c_1)$ dominates the coloring $(col+c_2)$: take $\psi(c') = c'$ for $c' \neq c_1,\ c_2$; $\psi(c_1) = c_2$, $\psi(c_2) = c_1$. One easily checks the domination conditions. So, if coloring $i + 1$ with c_1 is winning for player 1, then coloring $i + 1$ with c_2 is also winning for player 1. It follows that 'the best possible move' for player 1 is the color, that is still allowed (i.e., has $last(col, c) < b_{i+1}$), and has maximal $last(col, c)$ among all colors that are allowed.

(ii) Similar to (i). □

From lemma 4.5 we obtain the following algorithm to solve SEQUENTIAL COLORING CONSTRUCTION GAME on interval graphs with an irclo. Each move, the 'best' move is selected according to lemma 4.5. From this lemma it follows that the algorithm indeed outputs the correct player that has a winning strategy.

```
      for all c ∈ C do last(c) := −∞;
      game_end := false;
      i := 1;
      while i ≤ n and not(game_end)
      do begin check whether there exists c ∈ C with last(c) < b_{i+1}.
               if  such c does not exist
                  then  begin game_end := true;
                              go to exit
                        end;
               if owner(i) = 1
               then begin find color c ∈ C with last(c) < b_{i+1} and last(c)
                          is maximal among all colors c' with last(c') < b_{i+1};
                          last(c) := e_i;
                    end
               else begin find color c ∈ C with last(c) minimal;
                         last(c) := e_i
                    end
         end;
exit: if game_end then output("winning strategy for player 2")
                  else output("winning strategy for player 1")
```

To speed up the algorithm, we do not use an array to store the numbers *last*(c) for $c \in C$, but use a balanced search tree (e.g., an AVL-tree, or a 2-3-tree.) It is easy to see that with such a data structure, the 'find'-operations, and the updates of the values *last*(c) can be implemented to take $O(\log n)$ time per operation or update. As we have $O(n)$ such operations and updates, the total time of the algorithm is $O(n \log n)$.

Theorem 4.6
SEQUENTIAL COLORING CONSTRUCTION GAME, when restricted to interval graphs and the ordering an interval respresentation compatible linear ordering, is solvable in $O(n \log n)$ time.

5 Bipartite graphs

In this section we show that SEQUENTIAL COLORING GAME and SEQUENTIAL COLORING CONSTRUCTION GAME are PSPACE-complete for bipartite graphs.

Theorem 5.1
SEQUENTIAL COLORING GAME, with three colors, is PSPACE-complete, when restricted to bipartite graphs.

Proof.
We use a transformation from SEQUENTIAL COLORING GAME (without restrictions on the graphs) with three colors.

Let a graph $G = (V, E)$ be given, $n = |V|$, and let a linear ordering $f : \{1, \cdots, n\}$ and a function $owner : V \to \{1, 2\}$ be given.

Now a graph $G' = (V', E')$, and linear ordering f' of G', and ownership function $ow' : V' \to \{1, 2\}$ are made in the following way. For each edge $e = (v, w) \in E$, four extra vertices $v_{e,1}$, $v_{e,2}$, $v_{e,3}$, $v_{e,4}$ are added to G. The edge $e = (v, w)$ does not appear in G', but instead there are edges $(v, v_{e,1})$, $(v, v_{e,2})$, $(w, v_{e,1})$, $(w, v_{e,2})$, $(v_{e,1}, v_{e,4})$, $(v_{e,2}, v_{e,4})$, and $(v_{e,3}, v_{e,4})$. Suppose $f(v) < f(w)$. Then $v_{e,1}, \cdots, v_{e,4}$ are placed in the linear ordering f' after w but before the next vertex in V, and $f'(v_{e,1}) < f'(v_{e,2}) < f'(v_{e,3}) < f'(v_{e,4})$. The player that owns w also owns $v_{e,4}$. The other player owns $v_{e,1}$, $v_{e,2}$, $v_{e,3}$.

See figure 1 for a graphical illustration of the construction. Note that G' is bipartite.

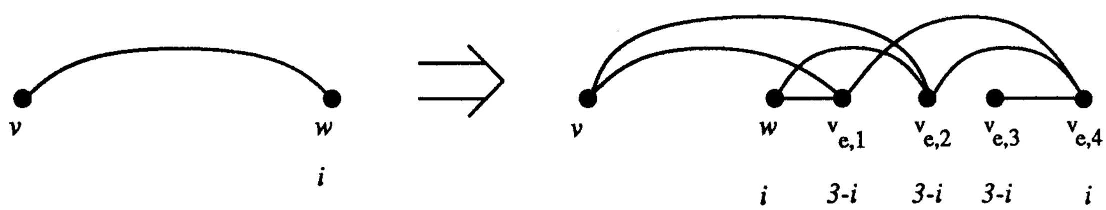

Figure 1: Construction in proof of theorem 5.1

Now note the following: if a player i colors a vertex w with $\exists v \in V : (v, w) \in E$, $f(v) < f(w)$, then he must give w a color different from v in the game, played on G' with f', ow', or else player $3 - i$ can win the game before the game reaches the coloring of another vertex in V: if player i colors v and w with the same color, then player $3 - i$ gives vertices $v_{(v,w),1}$, $v_{(v,w),2}$, different colors. He can do this, because these vertices have only incoming edges from v and w. He colors then $v_{(v,w),3}$ different from $v_{(v,w),1}$ and $v_{(v,w),2}$. Player i then loses as he cannot color $v_{(v,w),4}$. Note that if player i colors w different from v, then player $3 - i$ must give $v_{(v,w),1}$ and $v_{(v,w),2}$ the same color, hence player 1 has at least one possible color to give to $v_{(v,w),4}$.

It follows that there is a winning strategy for player 1 for the Sequential Coloring Game played on G with f and $owner$, if and only if there is a winning strategy for player 1 for the Sequential Coloring Game played on G' with f' and ow'.

The theorem now follows by noting that the problem is in PSPACE, and that the transformation can be carried out with logarithmic work space. □

Theorem 5.2
SEQUENTIAL COLORING GAME, with three colors, is PSPACE-complete, when restricted to bipartite graphs, where vertices owned by the same player form an independent set.

Proof.
Modify the construction in the proof of theorem 5.1 as follows: for $(v, w) \in E$, $f(v) < f(w)$, if v and w are owned by the same player, then add $v_{(v,w),1}, \cdots, v_{(v,w),4}$ as in the proof of theorem 5.1; if v and w are owned by different players, then just keep the edge between v and w, without a local replacement. □

Next we consider SEQUENTIAL COLORING CONSTRUCTION GAME on bipartite graphs.

Lemma 5.3
SEQUENTIAL COLORING CONSTRUCTION GAME with three colors is PSPACE-complete, even if there are no edges between vertices owned both by player 2.

Proof.
We use a transformation from the standard SEQUENTIAL COLORING CONSTRUCTION GAME problem with three colors.

Given an instance of the latter problem, replace every vertex $v \in V$ that is owned by player 2 by four vertices v_1, v_2, v_3, v_4, with edges (v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4), and (v_3, v_4). All edges to v from lower numbered vertices now go to v_1; all edges from v to higher numbered vertices now go from v_4. Player 2 owns v_1; player 1 owns v_2, v_3, and v_4. The linear ordering is modified, by replacing v by the sequence v_1, v_2, v_3, v_4. (See figure 2.)

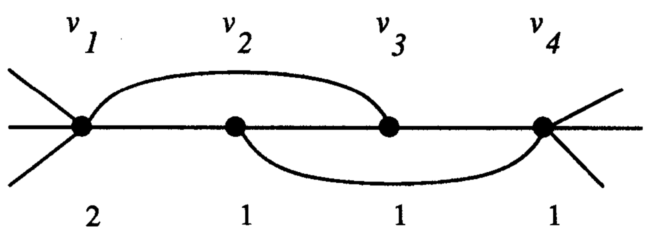

Figure 2: Construction in proof of lemma 5.3

Now note that player 1 must color v_4 with the same color as v_1. It follows that the same player has a winning strategy in the modified graph. □

Theorem 5.4
SEQUENTIAL COLORING CONSTRUCTION GAME with three colors is PSPACE-complete, when restricted to bipartite graphs, where vertices owned by the same player form an independent set.

Proof.
Transform from the problem, proved to be PSPACE-complete in lemma 5.3, and replace every edges between two vertices that are both owned by player 1 in the same way as in the proof of theorem 5.1. □

6 Open problems

Although we obtained several results on the complexity of the SEQUENTIAL COLORING GAME and the SEQUENTIAL COLORING CONSTRUCTION GAME for special classes of graphs, there are still a large number of interesting cases open. Below we mention some of the questions that we think are interesting, and are still unresolved.

- For split graphs, we were only able to obtain NP-hardness and coNP-hardness. Are the problems, e.g., for arbitrary orderings of G, PSPACE-complete on split graphs?
- What is the complexity of SEQUENTIAL COLORING GAME on interval graphs?
- What happens with the SEQUENTIAL COLORING CONSTRUCTION GAME on interval graphs, when we allow arbitrary orderings?
- What are the complexities of the problems, when restricted to trees?

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