Linear logic, domain theory and semi-functors

R. Hoofman

RUU-CS-90-34
November 1990

Utrecht University
Department of Computer Science
Padualaan 14, P.O. Box 80.089,
3508 TB Utrecht, The Netherlands,
Tel. : ... + 31 - 30 - 531454

Linear logic, domain theory and semi-functors

R. Hoofman

Technical Report RUU-CS-90-34
November 1990

Department of Computer Science
Utrecht University
P.O.Box 80.089
3508 TB Utrecht
The Netherlands

ISSN:0924-3275

Linear Logic, Domain Theory and Semi-functors

R. Hoofman
Department of Computer Science, Utrecht University
P.O. Box 80.089, 3508 TB Utrecht, the Netherlands

November 19, 1990

Abstract

Girard categories (GC's) were defined in [14] as categorical models for linear logic. It was shown that the Kleisli category of a GC is Cartesian closed.

We show that the category of sets and relations **Rel** nearly is a GC, but not quite. This situation is studied in more abstract terms: using the *semi*-category theoretical notions of [3], *weak* GC's are defined. It is shown that the *semi*-Kleisli category of a weak GC is *weak* Cartesian closed.

The Karoubi envelope construction can be used to transform various semi-notions to normal notions [3]. We show that the Karoubi envelope of a weak GC is a GC.

By applying these constructions to **Rel** we find that the Karoubi envelope of **Rel** is a simple example of a *cancellative* GC, and that the Karoubi envelope of the Kleisli category of **Rel** is equivalent to the category of *continuous lattices* and continuous functions.

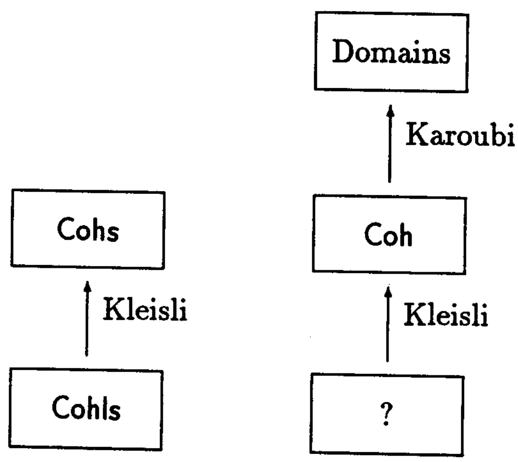
Another example of a weak GC is the category of *weak coherence spaces* and linear mappings. The structure of this category is similar to, but more natural than the structure on the well-known GC of coherence spaces and linear stable mappings [2]. In particular the units of *tensor product* and *tensor sum* are not collapsed into a single object.

One idea behind this paper is that semi-notions are perhaps more important in domain theory than is usually realised.

1 Introduction

In this paper *Linear Type Theory (LTT)* [2] will be considered as being a *decomposition* of the usual type theory belonging to the typed lambda calculus. The main feature of this decomposition is the decomposition of the exponential type constructor $\Rightarrow$ into two new type constructors $\multimap$ and $!$. The type $A \Rightarrow B$ may then be

written as $!(A) \multimap B$. Following the ***propositions-as-types*** rule [7], there is a corresponding decomposition of logic into *linear logic.*
Of course this is only a very rough explanation of LTT. For example, there is also a linear type operator $\neg$ which behaves as a sort of negation in the sense that $\neg\neg A \cong A$. Clearly this operator pertains more to the logical side.
As is well-known, typed lambda calculi correspond to *Cartesian closed categories (CCC's).* In [14] categorical models of LTT, called *Girard categories (GC's)*, are defined. Corresponding to the decomposition of type theory into LTT, each GC C is a decomposition of a CCC, which may be regained by taking the *Kleisli category* of C. Roughly, the Kleisli category Kl(C) of C has the same objects, but an arrow $A \rightarrow B$ in Kl(C) is an arrow $!A \rightarrow B$ in C. For example, the category Cohs of coherence spaces, which alternatively can be viewed as graphs or as a special kind of domains, and stable continuous functions is a CCC [2]. Girard found that Cohs may be constructed out of the category Cohls via the Kleisli category construction. The category Cohls has the same objects as Cohs, but *linear* stable continuous functions as arrows. It is a GC and was one of the first models of LTT.
In domain theory it is more common to take continuous functions as arrows, instead



of *stable* continuous functions. However, the category Coh of coherence spaces and continuous functions is not a CCC. Yet it seems to be an important category. For example, by results of [5] it follows that, in a certain sense, Coh underlies the category of coherent Scott domains and continuous functions. Technically, the Karoubi envelope of Coh is equivalent to this category of Scott domains. Various datatypes and constructions such as universal domains may be defined in the "easy" category Coh, and may then be "translated" to the more complicated category of coherent Scott domains.
Although Coh is not a CCC, it turns out that it has a somewhat weaker property: it is a *weak* Cartesian closed category (*wCCC*) [3, 11]. Like there is a correspondence between CCC's and typed lambda calculi, there is a correspondence between wCCC's and *non-extensional*, typed lambda calculi (i.e. lambda calculi without η-

rule).
In this paper we shall try to decompose the type theory of non-extensional lambda calculi. Working entirely on the semantical level, we weaken the definition of GC to that of *weak* GC (*wGC*). The main difference between GC's and wGC's is that the linear type constructor ! must be a functor on a GC, whereas it need only be a *semi*-functor on a wGC. Semi-functors are defined like functors, except that they need not preserve identities. By means of the *semi*-Kleisli category construction we may construct wCCC's out of wGC's. In particular, the decomposition of Coh is a simple and natural category.
The remainder of this paper is organised as follows. In section 2 the very simple wGC Rel of sets and relations is described to motivate the abstract definitions in the next section. In section 3 *wGC's* are defined, and in section 4 some constructions on wGC's are described, viz. the *Karoubi envelope* and the *semi-Kleisli category*. It is shown that the Karoubi envelope of a wGC is a GC, and that the semi-Kleisli category of a wGC is weak Cartesian closed. In section 5 these constructions are applied to Rel. The Karoubi envelope of Rel is a particular simple example of a *cancellative* GC, and the semi-Kleisli category of Rel underlies the category of *continuous lattices*. In section 6 we decompose the weak Cartesian closed structure of Coh, i.e. we find a category which relates to Coh like Cohls relates to Cohs. It turns out that we not only have to take different morphisms, but also to relax the definition of coherence spaces. The category WCohl of *weak* coherence spaces and linear mappings is defined and proved to be a wGC. The relevant structure on WCohl is similar to, but simpler and more natural than the structure on Cohls. In particular WCohl does not collapse the units of "tensor product" and "tensor sum" into a single object, just as in the more complex model of [15], but unlike Cohls. The semi-Kleisli category of WCohl is equivalent to Coh. In section 7 *qualitative domains* are considered. They nearly form a wGC, but not quite: the negation in this category is *intuitionistic* rather than *classical*, i.e. it is not always the case that $\neg\neg A \cong A$.
Finally, we have the following table:

(semi) GC	(semi) Kleisli category	Karoubi envelope of Kleisli category
Cohls	Cohs	
Rel		continuous lattices
WCohl	Coh	continuous coherent dcpo's
WQdl	Qd	continuous Scott domains

If we take the *Closure* Karoubi envelope [5] instead of the Karoubi envelope, then we get various categories of *algebraic* domains. It is the theme of this paper that perhaps *semi*-notions rather than *normal* notions underlie the structures worked with in domain theory.

2 The Category of Relations

In this section the category Rel of sets and relations is described, and some properties of it are stated. In the next section we shall see that by these properties Rel is an example of a *wGC*.

2.1 Rel

Definition 1 Rel *is the category with as objects sets* A *and as arrows* $R : A \to B$ *relations* $R \subseteq A \times B$.

The identity $id_A : A \to A$ in Rel is the identity relation: $a\ id_A a' \Leftrightarrow a = a'$. The composition $S \circ R : A \to C$ of relations $R : A \to B$ and $S : B \to C$ is as usual: $aS \circ Rc \Leftrightarrow \exists b(aRb \& bSc)$.
The empty set is both an initial and a terminal object in Rel (the unique arrows from and to it are the empty relations). The (categorical) product in Rel of two sets is not the Cartesian product, but the disjoint union of the sets. The coproduct in Rel of two sets also is the disjoint union, so product and coproduct have the same underlying object.
We can reintroduce the Cartesian product in Rel as the functor $\otimes$, called *tensor product.*

Definition 2 *The* tensor product $\otimes$: Rel $\times$ Rel $\to$ Rel *is defined on objects as* $A \otimes B = \{< a, b > | a \in A, b \in B\}$, *and on relations* $R : A \to A'$ *and* $S : B \to B'$ *as* $< a, b > R \otimes S < a', b' > \Leftrightarrow aRa' \& bSb'$.

The tensor product is commutative and associative (up to isomorphism), and has as unit the one-point set $I = \{*\}$.
The category Rel is not Cartesian closed, but it is *monoidal closed* with respect to the tensor product $\otimes$. A monoidal closed category is defined just as a Cartesian closed category, except that in the definition the categorical product is replaced by a tensor product. Indeed there is an "exponent functor" $\multimap$: $\mathsf{Rel}^{op} \times \mathsf{Rel} \to \mathsf{Rel}$ provided with natural isomorphisms between the Hom-sets $\mathsf{Rel}(A \otimes B, C)$ and $\mathsf{Rel}(A, B \multimap C)$.

Definition 3 *The functor* $\multimap$: $\mathsf{Rel}^{op} \times \mathsf{Rel} \to \mathsf{Rel}$ *is defined on objects as* $A \multimap B = \{< a, b > | a \in A, b \in B\}$, *and on relations* $R : A' \to A$ *and* $S : B \to B'$ *as* $< a, b > R \multimap S < a', b' > \Leftrightarrow a'Ra \& bSb'$.

Note that on objects the functor $\multimap$ is the same as the tensor product, but on arrows it differs, being contravariant in its first argument. The isomorphism between the Hom-sets is easy to describe: given $R : A \otimes B \to C$ we have $R' : A \to B \multimap C$ defined by $aR' < b, c > \Leftrightarrow < a, b > Rc$. The other way round one uses "evaluation arrows" $\varepsilon_{B,C} : (B \multimap C) \otimes B \to C$ defined by $<< b_1, c_1 >, b_2 > \varepsilon c_2 \Leftrightarrow b_1 = b_2 \& c_1 = c_2$.
As a special case, one obtains the (contravariant) functor $(-)\multimap I$: $\mathsf{Rel}^{op} \to$ Rel by

fixing $I = \{*\}$ as second argument. On objects, one has $A \multimap I \cong A$, but on arrows $R \multimap id_I = R^{-1}$, the inverse relation. Hence this functor is - up to isomorphism - its own inverse.

2.2 Powersets

A semi-functor is defined just as a functor, except that it need not preserve identities.

Definition 4 *The semi-functor* $! : \mathsf{Rel} \rightarrow \mathsf{Rel}$ *is defined on objects as* $!A = \{X | X \subseteq A$ *and* X *finite* $\}$, *and on relations* $R : A \rightarrow B$ *as* $X!RY \Leftrightarrow \forall b \in Y \exists a \in X(aRb)$.

Note that ! applied to the identity relation gives us subset inclusion: $X!(id_A)X' \Leftrightarrow X' \subseteq X$. This is not equal to the identity on $!A$: $X\ id_{!A}X' \Leftrightarrow X = X'$ [1].
There is some more structure related to the semi-functor !. For example, the membership relation is a natural transformation.

Definition 5 *The natural transformation* $\eta :! \rightarrow Id_{\mathsf{Rel}}$ *has components* $\eta_A :!A \rightarrow A$ *defined by* $X\eta_A a \Leftrightarrow a \in X$.

Set-union can be considered as a natural transformation.

Definition 6 *The natural transformation* $\mu :! \rightarrow !!$ *has components* $\mu_A :!A \rightarrow !!A$ *defined by* $X\mu_A\alpha \Leftrightarrow \bigcup \alpha \subseteq X$.

It is easy to see that $!(\eta_A) \circ \mu_A$ and $\eta_{!A} \circ \mu_A$ are equal to set inclusion. The equality $\mu_{!A} \circ \mu_A = !(\mu_A) \circ \mu_A$ also holds.
The semi-functor ! relates the product-structure and the tensor product-structure in a certain way. First ! applied to the unit of $\times$ (the terminal object) is isomorphic to the unit of $\otimes$, i.e. $!(\emptyset) \cong I$. Furthermore the elements of $!(A \times B)$ can be considered as disjoint unions $X \uplus Y$ of subsets $X \subseteq A$ and $Y \subseteq B$. Hence there is an obvious natural isomorphism between $!(A \times B)$ and $!A \otimes !B$, this last set consisting of pairs $< X, Y >$.

Definition 7 *The natural isomorphism* $\sim :!(-) \otimes !(-) \cong !(- \times -)$ *has components* $\sim_{A,B} :!A \otimes !B \cong !(A \times B)$ *defined by* $< X, Y > \sim_{A,B} X' \uplus Y' \Leftrightarrow X' = X \& Y' = Y$.

Finally arrows $\varepsilon'_A :!A \rightarrow 1$ and $\delta'_A :!A \rightarrow !A \otimes !A$ can be defined as $\varepsilon'_A = i \circ !(t_A)$ (where $i :!\emptyset \cong I$ and $t_A : A \rightarrow \emptyset$) and $\delta'_A = \sim^{-1}_{A,A} \circ !(< id_A, id_A >)$, i.e. $X\varepsilon'_A * \Leftrightarrow true$ and $X\delta'_A < X_1, X_2 > \Leftrightarrow X_1 \cup X_2 = X$.

[1] Defining $X!RY \Leftrightarrow (\forall b \in Y \exists a \in X(aRb) \& \forall a \in X \exists b \in Y(aRb))$ would have turned ! into a functor. However, it would have been impossible to give a further (weak) comonad structure.

3 Weak GC's

Linear type theory (LTT) has eight type constructors and four constant types. The eight linear constructors are direct product $\times$, tensor product $\otimes$, direct sum $+$, tensor sum $\odot$, linear implication $\multimap$, linear negation $\neg$, of-course $!$, and why-not $?$, the four constants are $1, I, 0$ and $\perp$. Note that we use the (by category theoretical notions inspired) notation of [14] for the operators.
Categorical models of LTT (called *Girard categories (GC's)*) have been defined in [14]. The category of relations Rel, as described in the previous section, fails only to be a GC because ! is a *semi*-functor rather than a *normal* functor. This situation calls for a generalisation of GC's to *weak* GC's (wGC's). We consider such a generalisation to be successful only if the Karoubi envelope of a wGC is a GC, and the suitably adapted Kleisli category of a wGC is weak Cartesian closed, as we will see in the next section.
To begin with we define *linear categories* ([12, 14]), which have enough structure to interpret all linear operators except ! and ?.

Definition 8 *A* linear category $< \mathsf{C}, \times, 1, \otimes, I, \multimap, \perp >$ *is a category* C *such that:*

1. *The functor* $\times : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ *is a chosen product in* C, *and* 1 *is a terminal object in* C.

2. $< \mathsf{C}, \otimes, I >$ *is a symmetric monoidal category, i.e.* $\otimes$ *is a functor* $\mathsf{C} \times \mathsf{C} \to \mathsf{C}$, I *is an object in* C, *and there are natural isomorphisms* $\rho_{A,B} : A \otimes B \cong B \otimes A$, $\lambda_A : A \otimes I \cong A$ *and* $\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ *satisfying certain commutative diagrams, the MacLane-Kelly coherence conditions (for more details see [10]).*

3. $< \mathsf{C}, \otimes, I, \multimap >$ *is a symmetric monoidal closed category, i.e.* $\multimap$ *is a functor* $\mathsf{C}^{op} \times \mathsf{C} \to \mathsf{C}$ *and* $(-) \otimes B$ *is a left adjoint of* $B \multimap (-)$ *(i.e. there is a natural isomorphism* $\mathsf{C}(A \otimes B, C) \cong \mathsf{C}(A, B \multimap C)$*).*

4. $\perp$ *is a* dualising object *in* C, *i.e. for each object* A *the arrow* τ_A *given by the next derivation is an isomorphism:*
$$\dfrac{\dfrac{\dfrac{(A \multimap \perp) \xrightarrow{id} (A \multimap \perp)}{(A \multimap \perp) \otimes A \to \perp}}{A \otimes (A \multimap \perp) \to \perp}}{A \xrightarrow{\tau_A} (A \multimap \perp) \multimap \perp}$$

In each symmetric monoidal closed category there are "evaluation arrows" $\varepsilon_{B,C} : (B \multimap C) \otimes B \to C$ given by applying the natural isomorphism $\mathsf{C}(B \multimap C, B \multimap C) \cong \mathsf{C}((B \multimap C) \otimes B, C)$ to the identity on $B \multimap C$.
How are the linear operators and constants interpreted in a linear category? The operators and constants $\times, \otimes, \multimap, 1, \perp$ and I are easy. Furthermore in a linear category we can define a functor $\neg : \mathsf{C}^{op} \to \mathsf{C}$ by taking $\neg = (-) \multimap \perp$. It can be shown

([14]) that a linear category has finite coproducts, because $0 = \neg 1$ is an initial object, and $A + B = \neg(\neg A \times \neg B)$ is a coproduct of A, B. Finally, we write $A \odot B$ for $\neg(\neg A \otimes \neg B)$.
Linear categories are not rare. See [12] for examples. By the results of the previous section it follows that Rel is a linear category. Note that in Rel the following holds: $\times = +$, $\otimes = \odot$, $1 = 0$, and $I = \bot$.
Some extra structure is needed on a linear category for the interpretation of the operators of-course ! and why-not ?. Here we diverge from the usual definition of a GC to define a wGC, where ! need only be a semi-functor.

Definition 9 [2] *A* wGC $< \mathsf{C}, \times, 1, \otimes, I, \multimap, \bot, !, \eta, \mu, i, \sim>$ *is a linear category* $< \mathsf{C}, \times, 1, \otimes, I, \multimap, \bot >$ *such that*

1. $<!, \eta, \mu>$ *is a* semi-comonad *(also see appendix), i.e.* $! : \mathsf{C} \to \mathsf{C}$ *is a semi-functor, and* $\eta : ! \to Id_{\mathsf{C}}$ *and* $\mu : ! \to !!$ *are natural transformations, such that:*

   (a) $\eta_{!A} \circ \mu_A = !(\eta_A) \circ \mu_A = \sigma_A$

   (b) $\mu_{!A} \circ \mu_A = !(\mu_A) \circ \mu_A$

   (c) $\mu_A \circ \sigma_A = \mu_A$

   where the natural transformation $\sigma : ! \to !$ *has components* $\sigma_A = !(id_A) : !A \to !A$.

2. $i : !1 \cong I$ *is an isomorphism, such that:*

   (a) $i \circ \sigma_1 = i$

3. $\sim_{A,B} : !A \otimes !B \to !(A \times B)$ *is an isomorphism natural in* A, B, *such that:*

   (a) $!(<!(\pi_{A,B}), !(\pi'_{A,B})>) \circ \mu_{A\times B} = \sim_{!A,!B} \circ (\mu_A \otimes \mu_B) \circ \sim^{-1}_{A,B}$

   where $\pi_{A,B} : A \times B \to A$ *and* $\pi'_{A,B} : A \times B \to B$ *are projections, and* $< f, g >: C \to A \times B$ *is the unique arrow such that* $\pi \circ < f, g > = f$ *and* $\pi' \circ < f, g > = g$.

Note that 2 implies $\sigma_1 = id_{!1}$.
If ! is a functor, then $\sigma_A = id_{!A}$ and the definitions of wGC's and GC's coincide. In this case 1c) becomes trivial, and 1a,b) exactly say that $<!, \eta, \mu>$ is a *comonad* (see [10]).
Clause 3a) is not in the original definition of a GC ([14]), but it seems to be a necessary requirement if we want to have the (semi-)Kleisli category (weak) Cartesian closed (see next section).
The interpretation of the linear operator ! in a (weak) GC is obvious. The linear operator ? is interpreted as $\neg ! \neg$.
It is left to the reader to check that the category Rel forms a wGC.

[2] A slightly different definition of wGC might be given by replacing (some of) the required isomorphisms by *semi*-isomorphisms. For example, the isomorphism $\sim_{A,B} : !A \otimes !B \cong !(A \times B)$ might be replaced by two arrows $\sim^1_{A,B} : !A \otimes !B \to !(A \times B)$ and $\sim^2_{A,B} : !(A \times B) \to !A \otimes !B$ satisfying $\sim^1_{A,B} \circ \sim^2_{A,B} = \sigma_{A\times B}$ and $\sim^2_{A,B} \circ \sim^1_{A,B} = \sigma_A \otimes \sigma_B$.

4 Constructions on wGC's

Two constructions on wGC's are described. The first is the well-known *Karoubi envelope* of a category C. It will be shown that the Karoubi envelope of a wGC is a GC. The second construction is the *semi-Kleisli category* of a category C, a slight generalisation of the Kleisli category construction. In analogy with the fact that the Kleisli category of a GC is Cartesian closed, we find that the semi-Kleisli category of a wGC is weak Cartesian closed. This is important because taking the Karoubi envelope of a wCCC gives us a Cartesian closed category. In the next section these constructions will be applied to the category Rel.

4.1 The Karoubi envelope

The Karoubi envelope construction can be applied to categories, (semi-)functors and natural transformations.

Definition 10 *If* C *is a category, then the* Karoubi envelope $K(\mathsf{C})$ *of* C *is the category with as objects idempotent arrows* $f : A \to A$ *of* C *(i.e.* $f \circ f = f$*), and as arrows* $\phi : (f : A \to A) \to (g : B \to B)$ *arrows* $\phi : A \to B$ *of* C *such that* $g \circ \phi \circ f = \phi$*, or equivalently* $g \circ \phi = \phi$ *and* $\phi \circ f = \phi$*. The identity* id_f *in* $K(\mathsf{C})$ *on an object* f *of* $K(\mathsf{C})$ *is* f *itself, and the composition in* $K(\mathsf{C})$ *is as in* C.

Definition 11 *If* $F : \mathsf{C} \to \mathsf{D}$ *is a semi-functor, then* $K(F) : K(\mathsf{C}) \to K(\mathsf{D})$ *is defined on objects as* $K(F)(f) = F(f)$ *and on arrows as* $K(F)(\phi) = F(\phi)$.

Definition 12 *If* $\alpha : F \to G$ *is a natural transformation between (semi-)functors, then* $K(\alpha) : K(F) \to K(G)$ *is a natural transformation with components* $K(\alpha)_f : K(F)(f) \to K(G)(f)$ *defined by* $K(\alpha)_f = \alpha_{\mathsf{Dom}(f)} \circ F(f)$.

In general the Karoubi envelope construction transforms semi-notions to the corresponding strict notions.

Theorem 13 *If* $F : \mathsf{C} \to \mathsf{D}$ *is a semi-functor, then* $K(F) : K(\mathsf{C}) \to K(\mathsf{D})$ *is a functor.*

Proof: $K(F)$ preserves identities: $K(F)(id_f) = F(f) = id_{F(f)}$. ∎

Theorem 14 *If* $< \mathsf{C}, \times, 1, \otimes, I, \multimap, \bot, !, \eta, \mu, i, \sim >$ *is a wGC, then* $< K(\mathsf{C}), K(\times), id_1, K(\otimes), id_I, K(\multimap), id_\bot, K(!), K(\eta), K(\mu), i, K(\sim) >$ *is a GC.*

Proof: Straightforward. ∎

4.2 The semi-Kleisli category

In analogy with the construction of the Kleisli category of a category with a comonad, the semi-Kleisli category of a category with a semi-comonad is defined.

Definition 15 *If* C *is a category with a semi-comonad* $<!: \mathsf{C} \to \mathsf{C}, \eta, \mu>$*, then the* semi-Kleisli category $Kl(\mathsf{C})$ *of* C *is the category with as objects the objects of* C*, and as arrows* $f : A \to B$ *arrows* $f :!A \to B$ *such that* $f \circ \sigma_A = f$*. The identity* id_A *on an object* A *in* $Kl(\mathsf{C})$ *is the arrow* $\eta_A :!A \to A$*, and the composition* $g * f$ *of arrows* $f : A \to B$ *and* $g : B \to C$ *in* $Kl(\mathsf{C})$ *is defined by* $g * f = g\circ!(f) \circ \mu_A$*.*

If the semi-comonad $<!: \mathsf{C} \to \mathsf{C}, \eta, \mu>$ happens to be a monad (i.e. ! is a functor), then the semi-Kleisli category is the same as the usual Kleisli category.
The semi-Kleisli category of a wGC is not Cartesian closed (as is the Kleisli category of a GC), but it has a somewhat weaker property.

Definition 16 *A* weak Cartesian closed category (wCCC) C *([3, 11]) is a category* C *with a terminal object* 1 *and binary products* $A \times B$*, and with the following data:*

- *For each pair of objects* $A, B \in \mathsf{C}$ *an object* $A \Rightarrow B \in \mathsf{C}$*, and an arrow* $e_{A,B} \in \mathsf{C}((A \Rightarrow B) \times A, B)$*. Furthermore, for each arrow* $f \in \mathsf{C}(D \times A, B)$ *an arrow* $\Lambda(f) \in \mathsf{C}(D, A \Rightarrow B)$*.*

satisfying the following equations (omitting subscripts):

1. $e \circ (\Lambda(f) \times id) = f$
2. $\Lambda(f \circ (g \times id)) = \Lambda(f) \circ g$

We shall prove that if C is a wGC, then $Kl(\mathsf{C})$ is a wCCC.

Theorem 17 *If* $< \mathsf{C}, \times, 1, \otimes, I, \multimap, \perp, !, \eta, \mu, i, \sim>$ *is a wGC, then* $Kl(\mathsf{C})$ *has finite products.*

Proof: The terminal object 1 of C also is terminal in $Kl(\mathsf{C})$: the unique arrow in $Kl(\mathsf{C})(A, 1)$ is the unique arrow $t_A \in \mathsf{C}(!A, 1)$. This is well-defined because $t_A \circ \sigma_A = t_A$.
The product $A \times B$ in C of two objects also is a product in $Kl(\mathsf{C})$: the projections $p_{A,B} :!(A \times B) \to A$ and $p'_{A,B} :!(A \times B) \to B$ are $\eta_A\circ!(\pi_{A,B})$ and $\eta_B\circ!(\pi'_{A,B})$, and the unique arrow to the product is the same as in C. ■

Theorem 18 *If* $< \mathsf{C}, \times, 1, \otimes, I, \multimap, \perp, !, \eta, \mu, i, \sim>$ *is a wGC, then* $Kl(\mathsf{C})$ *is a wCCC.*

Proof: It has already been proved that $Kl(\mathsf{C})$ has finite products. For objects B, C define $B \Rightarrow C = !B \multimap \mathsf{C}$, and $e_{B,C} \in Kl(\mathsf{C})((B \Rightarrow C) \times B, C)$ as $\varepsilon_{!B,C} \circ (\eta_{B \multimap C} \otimes \sigma_B) \circ \sim^{-1}_{A,B}$. For $f \in Kl(\mathsf{C})(A \times B, C)$ define $\Lambda(f) = (f \circ \sim^{1}_{A,B})^* \circ \sigma_A$, where $(-)^* : \mathsf{C}(A \otimes B, C) \cong \mathsf{C}(A, B \multimap C)$.
It is easy to see that $e_{B,C}$ and $\Lambda(f)$ are arrows in $Kl(\mathsf{C})$. We have to check two equations:

$$e * < \Lambda(f) * p, p' >= f$$

$e* < \Lambda(f) * p, p' >$
$=$
$\varepsilon(\eta \otimes \sigma) \sim^{-1} !(< (f \sim)^* \sigma_A !(\eta ! \pi)\mu, \eta ! \pi' >)\mu$
$=$
$\varepsilon(\eta \otimes \sigma) \sim^{-1} !(< (f \sim)^* ! \pi ! \eta \mu, \eta ! \pi' >)\mu$
$=$
$\varepsilon(\eta \otimes \sigma) \sim^{-1} !(< (f \sim)^* ! \pi \sigma, \eta ! \pi' >)\mu$
$=$
$\varepsilon(\eta \otimes \sigma) \sim^{-1} !(< (f \sim)^* ! \pi, \eta ! \pi' >)\mu$
$=$
$\varepsilon(\eta \otimes \sigma) \sim^{-1} !((f \sim)^* \times \eta)!(<!\pi, !\pi' >)\mu$
$=$
$\varepsilon(\eta \otimes \sigma)(!((f \sim)^*) \otimes !\eta) \sim^{-1} !(<!\pi, !\pi' >)\mu$
$=$
$\varepsilon(\eta !((f \sim)^*) \otimes !\eta) \sim^{-1} !(<!\pi, !\pi' >)\mu$
$=$
$\varepsilon((f \sim)^* \eta \otimes !\eta) \sim^{-1} !(<!\pi, !\pi' >)\mu$
$=$
$\varepsilon((f \sim)^* \otimes id)(\eta \otimes !\eta) \sim^{-1} !(<!\pi, !\pi' >)\mu$
$=$
$f \sim (\eta \otimes !\eta) \sim^{-1} !(<!\pi, !\pi' >)\mu$
$=$
$f \sim (\eta \otimes !\eta) \sim^{-1} \sim (\mu \otimes \mu) \sim^{-1}$
$=$
$f \sim (\eta \otimes !\eta)(\mu \otimes \mu) \sim^{-1}$
$=$
$f \sim (\eta \mu \otimes !\eta \mu) \sim^{-1}$
$=$
$f \sim (\sigma \otimes \sigma) \sim^{-1}$
$=$
$f\sigma$
$=$
f

The second equation:

$$\Lambda(f* < g * p, p' >) = \Lambda(f) * g$$

$\Lambda(f* < g * p, p' >)$
$=$
$(f!(< g!(\eta!\pi)\mu, \eta!\pi' >)\mu \sim)^* \sigma$
$=$
$(f!(g \times \eta)!(<!\pi, !\pi' >)\mu \sim)^* \sigma$
$=$
$(f!(g \times \eta) \sim (\mu \otimes \mu) \sim^{-1} \sim)^* \sigma$
$=$
$(f!(g \times \eta) \sim (\mu \otimes \mu))^* \sigma$
$=$
$(f \sim (!g\mu \otimes !\eta\mu))^* \sigma$
$=$
$(f \sim (!g\mu \otimes \sigma))^* \sigma$
$=$
$(f \sim (\sigma \otimes \sigma)(!g\mu \otimes id))^* \sigma$
$=$
$(f\sigma \sim (!g\mu \otimes id))^* \sigma$
$=$
$(f \sim)^* !g\mu$
$=$
$(f \sim)^* \sigma !g\mu$
$=$
$\Lambda(f) * g$ ■

In general, the (semi-)Kleisli category of a wGC is only *weak* Cartesian closed, as $\Lambda(e * (f \times id)) = (\epsilon(f \otimes \sigma))^* \sigma$. However, if C is a GC, then $\sigma = id$ and it follows that $Kl(\mathsf{C})$ is a CCC.
By means of the Karoubi envelope construction wCCC's can always be transformed in Cartesian closed categories.

Theorem 19 *If* C *is a wCCC, then* $K(\mathsf{C})$ *is a CCC.*

Proof: See [3],[5]. ■

5 Constructions on Rel

The constructions described in the previous section are applied to Rel. The category $K(\mathsf{Rel})$ is a particular simple example of a (cancellative) GC. The category

$K(Kl(\mathsf{Rel}))$ is equivalent to the category with as objects *continuous lattices* and as morphisms continuous functions.

5.1 $K(\mathsf{Rel})$

A description is given of the Karoubi envelope $K(\mathsf{Rel})$ of the category Rel, which, by the results of the previous section, will be a GC. Its objects are transitive relations $R : A \to A$ with the ***interpolation property***, i.e. $a_1 R a_3 \Rightarrow \exists a_2 (a_1 R a_2 R a_3)$. An arrow $T : (R : A \to A) \to (S : B \to B)$ is a relation $T : A \to B$ such that $S \circ T \circ R = T$.
The interpretation of linear logic propositions in $K(\mathsf{Rel})$ are objects in $K(\mathsf{Rel})$, i.e. transitive relations with the interpolation property, and the various operators of linear logic are roughly interpreted as follows:

- $xR \times Sx' \Leftrightarrow (xRx'$ or $xSx')$
- $< a, b > R \otimes S < a', b' > \Leftrightarrow (aRa'$ and $bSb')$
- $< a, b > R \multimap S < a', b' > \Leftrightarrow (a'Ra$ and $bSb')$
- $a \neg R a' \Leftrightarrow a'Ra$
- $x1x' \Leftrightarrow false$
- $*I* \Leftrightarrow true$
- $X!RX' \Leftrightarrow \forall a' \in X' \exists a \in X(aRa')$
- $X?RX' \Leftrightarrow \forall a \in X \exists a' \in X'(aRa')$

It is easy to check that $+ = \times$, $0 = 1$, $\odot = \otimes$, and $\bot = I$. By these last two equations $K(\mathsf{Rel})$ is an example of a *cancellative* GC ([12]).
The full subcategory of $K(\mathsf{Rel})$ with as objects reflexive, transitive relations is closed under the above operations, and hence is a GC. The objects of this subcategory can be more simply described as *preorders*, and the arrows $R : (A, \leq_A) \to (B, \leq_B)$ are relations $R \subseteq A \times B$ satisfying $a' \leq_A aRb \leq_B b' \Rightarrow a'Rb'$.

5.2 $Kl(\mathsf{Rel})$

The semi-Kleisli category $Kl(\mathsf{Rel})$ has the same objects as Rel, viz. sets. However, the arrows are different: $f \in Kl(\mathsf{Rel})(A, B)$ iff f is a relation between $!A$ and B such that

$$X \subseteq X' \& Xfb \Rightarrow X'fb$$

This last requirement is equivalent to the requirement that $f \circ \sigma = f$. Intuitively it says that the relations should be *monotone*.
By the results of the previous section it follows that $Kl(\mathsf{Rel})$ is weak Cartesian

closed. Also, a Cartesian closed category can be constructed by taking the Karoubi envelope of $Kl(\mathsf{Rel})$. The objects of this new category are tuples $A = < Dom_A, \vdash_A >$, where Dom_A is a set and $\vdash_A$ a relation between $Dom_{!A}$ and Dom_A, such that

- $X \subseteq X' \& X \vdash_A a \Rightarrow X' \vdash_A a$
- $\exists X (X' \vdash_A X \vdash_A a) \Leftrightarrow X' \vdash_A a$

(writing $X' \vdash_A X$ for $\forall a \in X (X' \vdash_A a)$). The arrows $f \in K(Kl(\mathsf{Rel}))(A, B)$ are relations f between $Dom_{!A}$ and Dom_B such that

- $X \subseteq X' \& X f b \Rightarrow X' f b$
- $\exists X, Y (X' \vdash_A X f Y \vdash_B b) \Leftrightarrow X' f b$

(writing XfY for $\forall b \in Y (Xfb)$).
The category $K(Kl(\mathsf{Rel}))$ is in fact equivalent to a well-known category of posets, viz. the category of *continuous complete lattices* ([13]) and continuous functions. This can be proved as in [5]. Here we shall describe how an object of $K(Kl(\mathsf{Rel}))$ gives rise to a continuous lattice.

Definition 20 *An* element x *of an object* A *of* $K(Kl(\mathsf{Rel}))$ *is a subset* $x \subseteq Dom_A$, *such that*

$$a \in x \Leftrightarrow \exists X \subseteq x (X \vdash_A a)$$

where X *is finite.*

Let $Pt(A)$ denote the set of elements of A.

Theorem 21 *If* A *is an object of* $K(Kl(\mathsf{Rel}))$, *then* $Pt(A)$ *ordered by subset inclusion* $\subseteq$ *is a continuous lattice.*

Proof: See [5]. ■

6 Weak Coherence Spaces

In this section we shall decompose the category Coh of coherence spaces and continuous functions into a wGC. The linear structure on this wGC is similar to, but simpler than the linear structure on the GC Cohls.

6.1 Cohs and Cohls

First, we recollect the definitions of coherence space and (linear) stable continuous function ([2]). We shall give very concrete representations of coherence spaces as certain graphs and (linear) stable continuous functions as certain relations.

Definition 22 *A* coherence space *A is a tuple $< Dom_A, \smile_A >$, with Dom_A a set and $\smile_A \subseteq Dom_A \times Dom_A$ a symmetric, reflexive relation.*

The relation $\smile_A$ is called the *consistency relation* associated with A, and $a_1, a_2 \in Dom_A$ are *consistent* iff $a_1 \smile_A a_2$. A subset $X \subseteq Dom_A$ is *pairwise consistent* iff all pairs $a_1, a_2 \in X$ are consistent. Define $PC(Dom_A)$ as the set of all finite, pairwise consistent subsets of Dom_A.

Definition 23 *A* stable mapping *$R : A \rightarrow B$ between coherence spaces A, B is a relation $R \subseteq PC(Dom_A) \times Dom_B$ such that*

1. $XRb_1 \& XRb_2 \Rightarrow b_1 \smile_B b_2$ *(output consistency)*
2. $X \subseteq X' \& XRb \Rightarrow X'Rb$ *(monotonicity)*
3. $X_1Rb \& X_2Rb \& X_1 \cup X_2 \in PC(Dom_A) \Rightarrow (X_1 \cap X_2)Rb$ *(stability)*

The category Cohs with as objects coherence spaces and as morphisms stable mappings is a CCC.
It turns out that the Cartesian closedness of Cohs is the consequence of some deeper structure.

Definition 24 *A* linear stable mapping *$R : A \rightarrow B$ between coherence spaces A, B is a relation $R \subseteq Dom_A \times Dom_B$ such that*

1. $a_1Rb_1 \& a_2Rb_2 \& a_1 \smile_A a_2 \Rightarrow b_1 \smile_B b_2$ *(consistency preserving)*
2. $a_1Rb_1 \& a_2Rb_2 \& a_1 \smile_A a_2 \& a_1 \neq a_2 \Rightarrow b_1 \neq b_2$ *(stability)*

The category Cohls of coherence spaces and linear stable maps is a GC, and the Kleisli category of Cohls is equal to Cohs.

6.2 WCohl

Define Cohl as the category with as objects coherence spaces and as as morphisms (non-stable) linear mappings, i.e. $R \in \mathrm{Cohl}(A, B)$ iff $R \subseteq Dom_A \times Dom_B$ and

- $a_1Rb_1 \& a_2Rb_2 \& a_1 \smile_A a_2 \Rightarrow b_1 \smile_B b_2$ (consistency preserving)

The category Cohl is not a wGC because there is not a dualising object.
However, if we relax the definition of coherence spaces slightly, we can get a wGC. A *weak* coherence space is defined as a coherence space, except that the consistency relation need not be reflexive. The category of weak coherence spaces and linear mappings is a wGC. In particular the linear structure on weak coherence spaces is similar to, but simpler than the linear structure on coherence spaces.

Definition 25 *A* weak coherence space *A is a tuple* $< Dom_A, \smile_A >$, *with* Dom_A *a set and* $\smile_A$ *a symmetric relation* $\smile_A \subseteq Dom_A \times Dom_B$.

Define WCohl as the category with as objects weak coherence spaces and as morphisms linear mappings. We will show that WCohl is a wGC.

Definition 26 *The* tensor product $\otimes$: WCohl $\times$ WCohl $\rightarrow$ WCohl *is defined on objects as* $Dom_{A\otimes B} = \{< a, b > | a \in Dom_A, b \in Dom_B\}$, $< a, b > \smile_{A\otimes B} < a', b' > \Leftrightarrow a \smile_A a' \& b \smile_B b'$, *and on linear mappings* $R : A \rightarrow A'$ *and* $S : B \rightarrow B'$ *as* $< a, b > R \otimes S < a', b' > \Leftrightarrow aRa' \& bSb'$.

The tensor product has unit I with $Dom_I = \{*\}$ and $* \smile_I *$.

Definition 27 *The functor* $\multimap$: WCohl^{op} $\times$ WCohl $\rightarrow$ WCohl *is defined on objects as* $Dom_{A\multimap B} = \{< a, b > | a \in Dom_A, b \in Dom_B\}$, $< a, b > \smile_{A\multimap B} < a', b' > \Leftrightarrow (a \smile_A a' \Rightarrow b \smile_B b'$, *and on linear mappings* $R : A' \rightarrow A$ *and* $S : B \rightarrow B'$ *as* $< a, b > R\multimap S < a', b' > \Leftrightarrow a'Ra \& bSb'$.

It is not difficult to see that these functors make WCohl into a monoidal closed category. Note that the tensor product is the same as in Cohls, but that linear implication is simpler.
The dualising object $\perp$ in WCohl has $Dom_\perp = \{*\}$ and $\smile_\perp = \emptyset$. Contrary to Cohls the equation $I = \perp$ does not hold in WCohl. We can now describe negation: $Dom_{\neg A} = Dom_A$ and $a_1 \smile_{\neg A} a_2 \Leftrightarrow \neg(a_1 \smile_A a_2)$.

Definition 28 *The* product $A \times B$ *of weak coherence spaces* A, B *is defined by* $Dom_{A\times B} = Dom_A \uplus Dom_B$ *and* $x_1 \smile_{A\times B} x_2 \Leftrightarrow (x_1 \smile_A x_2 \vee x_1 \smile_B x_2 \vee (x_1 \in Dom_A \& x_2 \in Dom_B))$.

Although the operators $+$ and $\odot$ can be defined with the help of the other operators we shall give their definition directly.

Definition 29 *The* coproduct $A + B$ *of weak coherence spaces* A, B *is defined by* $Dom_{A+B} = Dom_A \uplus Dom_B$ *and* $x_1 \smile_{A+B} x_2 \Leftrightarrow (x_1 \smile_A x_2 \vee x_1 \smile_B x_2)$.

Definition 30 *The functor* $\odot$: WCohl $\times$ WCohl $\rightarrow$ WCohl *is defined on objects as* $Dom_{A\odot B} = \{< a, b > | a \in Dom_A, b \in Dom_B\}$, $< a, b > \smile_{A\odot B} < a', b' > \Leftrightarrow a \smile_A a' \vee b \smile_B b'$, *and on linear mappings* $R : A \rightarrow A'$ *and* $S : B \rightarrow B'$ *as* $< a, b > R \odot S < a', b' > \Leftrightarrow aRa' \& bSb'$.

Compare this with the more difficult definition of $\odot$ in Cohls:

$$< a, b > \smile_{A \odot B} < a', b' > \Leftrightarrow ((a \smile_A a' \& a \neq a') \vee (b \smile_B b' \& b \neq b') \vee (a = a' \& b = b'))$$

Finally, the semi-functor ! is defined.

Definition 31 *The semi-functor* ! : WCohl $\rightarrow$ WCohl *is defined on objects as* $Dom_{!A} = \{X \subseteq Dom_A | X$ *finite and pairwise consistent*$\}$, $X \smile_{!A} X' \Leftrightarrow X \cup X'$ *pairwise consistent, and on linear mappings* $R : A \rightarrow B$ *as* $X!RY \Leftrightarrow \forall b \in Y \exists a \in X(aRb)$.

It is left to the reader to define the various natural transformations (they are the same as in Rel) and to check the equations of a wGC.

Theorem 32 WCohl *is a wGC.*

6.3 Constructions on WCohl

The Karoubi envelope of WCohl is a GC which does not collapse the units of tensor product and tensor sum, just as in the model of de Paiva [15].
The semi-Kleisli category Kl(WCohl) has the same objects as WCohl, viz. weak coherence spaces. However, the arrows are different: $R \in Kl(\text{WCohl})(A, B)$ iff R is a relation between $!A$ and B such that:

- $X_1 R b_1 \& X_2 R b_2 \& X_1 \smile_{!A} X_2 \Rightarrow b_1 \smile_B b_2$
- $R \circ \sigma_A = R$

It is easy to see that these requirements are equivalent to

- $X R b_1 \& X R b_2 \Rightarrow b_1 \smile_B b_2$
- $X \subseteq X' \& X R b \Rightarrow X' R b$

i.e. R is a *mapping*. The elements of Dom_A, Dom_B that are not self-consistent (i.e. $a \smile_A a$ does not hold) are of no account: they are not in $Dom_{!A}$ and if XRb then $b \smile_B b$. Hence the proof of the following theorem is not difficult.

Theorem 33 Kl(WCohl) $\simeq$ Coh

As in [5] we may now take the Karoubi envelope of Kl(WCohl) and obtain a category equivalent to the category of continuous coherent complete dcpo's and continuous functions.

7 Qualitative domains

Qualitative domains were defined by Girard in [1]. We shall give an alternative but equivalent definition.

Definition 34 *A* qualitative domain *A is a pair* $< Dom_A, Con_A >$*, with* Dom_A *a set and* Con_A *a set of finite subsets of* Dom_A*, such that*

1. $\emptyset \in Con_A$
2. $a \in Dom_A \Rightarrow \{a\} \in Con_A$
3. $X' \subseteq X \in Con_A \Rightarrow X' \in Con_A$

It is not difficult to see that coherence spaces are in fact a special kind of qualitative domains, viz. qualitative domains which satisfy

$$X \in Con_A \Leftrightarrow \forall a_1, a_2 \in X(\{a_1, a_2\} \in Con_A)$$

The category **Qds** with as objects qualitative domains and as arrows stable mappings is Cartesian closed. However, in [5] we showed that non-stable mappings are also interesting.

Definition 35 *A* mapping $R : A \rightarrow B$ *between qualitative domains* A, B *is a relation* $R \subseteq Con_A \times Dom_B$ *such that*

1. $\forall b \in Y(XRb) \Rightarrow Y \in Con_B$
2. $X \subseteq X' \& XRb \Rightarrow X'Rb$

The category **Qd** with qualitative domains as objects and mappings as arrows is weak Cartesian closed, and underlies an important category of domains: the Karoubi envelope of **Qd** is equivalent to the category of continuous Scott domains and continuous functions ([5]).
A natural question is whether there is a wGC underlying **Qd**. The answer to this question is partially affirmative.

Definition 36 *A* weak qualitative domain *A is a pair* $< Dom_A, Con_A >$*, with* Dom_A *a set and* Con_A *a set of non-empty, finite subsets of* Dom_A*, such that*

1. $X' \subseteq X \in Con_A \Rightarrow X' \in Con_A$

Definition 37 *A* linear mapping $R : A \rightarrow B$ *between qualitative domains* A, B *is a relation* $R \subseteq Dom_A \times Dom_B$ *such that for non-empty* Y

1. $\forall b \in Y \exists a \in X(aRb) \& X \in Con_A \Rightarrow Y \in Con_B$

Define WQdl as the category with as objects weak qualitative domains and as morphisms linear mappings. The operators and constants of a wGC can be defined on WQdl analogous to WCohl. However, the dualising object $\perp =< \{*\}, \emptyset >$ of WCohl is not dualising in WQdl: $\neg\neg A \cong A$ need not be true. This means that negation in WQdl is *intuitionistic* rather than *classical*, and that WQdl is a model of intuitionistic linear logic ([15]).
Negation is not used in the semi-Kleisli category construction so it follows that Kl(WQdl) is a wCCC. In fact we have the following theorem.

Theorem 38 Kl(WQdl) $\simeq$ Qd

Acknowledgement

I would like to thank Bart Jacobs, Jan van Leeuwen, Harold Schellinx, and Nico Verwer for reading draft versions of this paper and giving useful suggestions.

References

[1] Girard, J.-Y., *The system F of variable types, fifteen years later*, Theor. Comput. Sci. 45 (1986), 159-192

[2] Girard, J.-Y., *Linear Logic*, Theor. Comput. Sci. 50 (1987),1-102

[3] Hayashi, S., *Adjunction of semi-functors: categorical structures in non-extensional lambda-calculus*, Theor. Comput. Sci. 41 (1985), 95-104

[4] Hoofman, R., *From Posets to Coherence Spaces*, Technical Report RUU-CS-90-2, Department of Computer Science, Utrecht University, January 1990

[5] Hoofman, R., *Continuous Information Systems*, Technical Report RUU-CS-90-25, Department of Computer Science, Utrecht University, July 1990

[6] Hoofman, R., *The Linear Decomposition of λ^2-Models*, manuscript 1990

[7] Howard, W.A., *The formulae-as-types notion of construction*, in: J.R. Hindley and J.P. Seldin (eds.), To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism, Academic Press, London, pp. 479-490

[8] Jacobs, B., *Personal communication*

[9] Jacobs, B., *On the Semantics of Second Order Lambda Calculus: From Bruce-Meyer-Mitchell Models to Hyperdoctrine Models and Vice-Versa*, in: D.H. Pitt, D.E. Rydeheard, P. Dybjer, A.M. Pitts and A. Poinge (eds.), Category Theory and Computer Science, Proceedings, Lect. Notes in Comp. Sci., vol. 389, Springer-Verlag, Heidelberg, 1989, pp. 198-212

[10] Maclane, S., *Categories for the Working Mathematician*, Springer-Verlag, New York,1971

[11] Martini, S., *An interval model for second order lambda calculus*, in: D.H. Pitt, A. Poigne and D. Rydeheard (eds.), Category Theory and Computer Science, Lect. Notes in Comp. Sci. 283, Springer-Verlag 1988, pp. 219-237

[12] Marti-Oliet, N., J. Meseguer, *An Algebraic Axiomatization of Linear Logic Models*, Technical Report SRI-CSL-89-11, Computer Science Laboratory, SRI International, December 1989

[13] Scott, D., *Data Types as Lattices*, SIAM J. Comput., Vol. 5, No. 3, September 1976

[14] Seely, R.A.G., *Linear Logic, *-Autonomous Categories, and Cofree Coalgebras*, Proc. AMS Conf. Categories in Computer Science and Logic, Boulder 1987

[15] de Paiva, V.C.V., *A Dialectica-like Model of Linear Logic*, in: D.H. Pitt, D.E. Rydeheard, P. Dybjer, A.M. Pitts and A. Poinge (eds.), Category Theory and Computer Science, Proceedings, Lect. Notes in Comp. Sci., vol. 389, Springer-Verlag, Heidelberg, 1989, pp. 341-356

[16] Plotkin, G.D., *T^ω as a universal domain*, Journal of Computer and System Sciences, vol. 17, 1978, pp. 209-236

Appendix

A Categories

Category	Objects	Arrows
Rel	sets	relations
Coh	coherence spaces	mappings
Cohs	coherence spaces	stable mappings
Cohl	coherence spaces	linear mappings
Cohls	coherence spaces	linear stable mappings
WCohl	weak coherence spaces	linear mappings
Qd	qualitative domains	mappings
Qds	qualitative domains	stable mappings
WQdl	weak qualitative domains	linear mappings

B Semi-comonads and Semi-adjunctions

In analogy with the connection between comonads and adjunctions, we establish a connection between semi-comonads and certain semi-adjunctions. Semi-adjunctions are defined in [3, 9].

Definition 39 *A* semi-adjunction $< F, G, \alpha, \beta >$ *consists of semi-functors* $F : \mathsf{C} \to \mathsf{D}$, $G : \mathsf{D} \to \mathsf{C}$ *and collections of arrows* $\{\alpha_{A,B}, \beta_{A,B}\}_{A \in \mathsf{C}, B \in \mathsf{D}}$ *such that the four squares in the following diagram commute.*

$$\begin{array}{ccccc} B & \mathsf{D}(FA,B) & \underset{\beta_{A,B}}{\overset{\alpha_{A,B}}{\rightleftarrows}} & \mathsf{C}(A,GB) & A \\ f\Big\downarrow & \Big\downarrow f \circ \_ \circ Fg & & \Big\downarrow Gf \circ \_ \circ g & \Big\uparrow g \\ B' & \mathsf{D}(FA',B') & \underset{\beta_{A',B'}}{\overset{\alpha_{A',B'}}{\rightleftarrows}} & \mathsf{C}(A',GB') & A' \end{array}$$

We consider semi-adjunctions $< F, G, \alpha, \beta >$ in which G is a functor. Such semi-adjunctions give rise to semi-comonads.

Theorem 40 *If* $< F, G, \alpha, \beta >$ *is a semi-adjunction and* G *is a functor, then* $< !, \eta, \mu >$ *is a semi-comonad, where*

- $! = FG$
- $\eta_B = \beta_{GB,B}(id_{GB})$
- $\mu_B = F(\alpha_{GB,FGB}(id_{FGB}))$

Proof: For example, η is a natural transformation:

$$\begin{aligned} \eta \circ !(f) &= \beta(id) \circ FG(f) \\ &= \beta(G(f)) \\ &= f \circ \beta(id) \circ FG(id) \\ &= f \circ \beta(id) \\ &= f \circ \eta \end{aligned}$$

Furthermore, the equation $!(\eta) \circ \mu = \sigma$ is satisfied:

$$\begin{aligned} !(\eta) \circ \mu &= FG(\beta(id)) \circ F(\alpha(id)) \\ &= F(G(\beta(id)) \circ \alpha(id)) \\ &= F(\alpha(\beta(id) \circ F(id)) \\ &= FG(id) \\ &= \sigma \end{aligned}$$

It is left to the reader to check the remaining details. ∎

Given a semi-comonad, there may be many different associated semi-adjunctions.

Definition 41 *Given a semi/comonad* $M =< ! : \mathsf{D} \to \mathsf{D}, \eta, \mu >$, *a* resolution of M *is a semi-adjunction* $< F, G, \alpha, \beta >$ *with* G *a functor, satisfying*

- $! = FG$
- $\eta_B = \beta_{GB,B}(id_{GB})$
- $\mu_B = F(\alpha_{GB,FGB}(id_{FGB}))$

For example, the semi-Kleisli category construction of the semi-comonad gives rise to a resolution.

Theorem 42 *If $M =<!, \eta, \mu>$ is a semi-comonad, then $< F_1 : Kl(\mathsf{D}) \to \mathsf{D}, G_1 : \mathsf{D} \to Kl(\mathsf{D}), \alpha_1, \beta_1 >$ is a resolution of M, where*

- $F_1(B) = !B$
  $F_1(f : B \to B') = !(f) \circ \mu_B$
- $G_1(B) = B$
  $G_1(f : B \to B') = f \circ \eta_B$
- $\alpha_1(u) = u \circ \sigma$
- $\beta_1(u) = u$

The resolution given by the semi-Kleisli category construction is the "least" one. To make this precise we first define the category of resolutions.

Definition 43 *Given a semi-comonad $M =<! : \mathsf{D} \to \mathsf{D}, \eta, \mu>$ the category $Res(M)$ has as objects resolutions of M, and as arrows $K :< F : \mathsf{C} \to \mathsf{D}, G, \alpha, \beta > \to < F' : \mathsf{C}' \to \mathsf{D}, G', \alpha', \beta' >$ functors $K : \mathsf{C} \to \mathsf{C}'$ such that*

- $KG = G'$
- $F'K = F$
- $K\alpha_{A,B} = \alpha'_{KA,B}$
- $\beta'_{KA,B}K = \beta_{A,B}$

Now we can state the following theorem.

Theorem 44 *If $M =<!, \eta, \mu>$ is a semi-comonad, then $< F_1, G_1, \alpha_1, \beta_1 >$ is an initial object in $Res(M)$.*

C Reflexive Objects

A *reflexive* object in a (weak) Cartesian closed category C is an object A satisfying

- $A \Rightarrow A \cong A$

It is well-known that reflexive objects are models of the untyped lambda calculus. In analogy, we define a *linear reflexive* object in a (weak) GC as an object A satisfying

- $A \multimap A \cong A$
- $!A \cong A$

The proof of the following theorem is straightforward.

Theorem 45 *If A is a linear reflexive object in a (weak) GC C, then A is a reflexive object in Kl(C).*

An example of a linear reflexive object is the set of natural numbers ω in the GC Rel. It is easy to see that $\omega \multimap \omega$ and $!\omega$ are denumerable, and hence isomorphic to ω. Notice that ω also satifies the following clauses:

- $\omega \times \omega \cong \omega$
- $\omega \otimes \omega \cong \omega$
- $\neg\omega \cong \omega$

It follows by the theorem that ω is reflexive in the category Kl(Rel). In fact, this is the well-known Graph Model $\mathcal{P}\omega$ [3].

D PRel

A wGC which is in some sense intermediate between the categories Rel and WCohl is the category PRel. The objects A of this category are sets with *predicates*, i.e. tuples $< Dom_A, p_A >$ where $p_A \subseteq Dom_A$. We shall write $p_A(a)$ for $a \in p_A$. The arrows $R : A \to B$ are relations $R \subseteq Dom_A \times Dom_B$ which *preserve truth*, i.e. $p_A(a)$ and aRb implies $p_B(b)$.
On the *Dom*-part of the objects of PRel the linear operators are the same as in Rel. We shall describe their actions on the predicates.

- $p_{A\times B}(< x, i >) \Leftrightarrow (i = 1 \wedge p_A(x)) \vee (i = 2 \wedge p_B(x))$
- $p_I(*) \Leftrightarrow true$

[3] One may wonder how it is possible that $\omega \Rightarrow \omega \cong \omega$, whereas there is only a retraction between $\mathcal{P}\omega$ and its functionspace. Just remember that Kl(Rel) is *weak* Cartesian closed.

- $p_{A\otimes B}(< a, b >) \Leftrightarrow (p_A(a) \wedge p_B(b))$
- $p_{A\multimap B}(< a, b >) \Leftrightarrow (p_A(a) \Rightarrow p_B(b))$
- $p_{\perp}(*) \Leftrightarrow false$
- $p_{A\odot B}(< a, b >) \Leftrightarrow (p_A(a) \vee p_B(b))$
- $p_{!A}(X) \Leftrightarrow \forall a \in X(p_A(a))$
- $p_{?A}(X) \Leftrightarrow \exists a \in X(p_A(a))$
- $p_{\neg A}(a) \Leftrightarrow \neg p_A(a)$

It is easy to see that in PRel the equalities $\times = +$ and $0 = 1$ hold.