From posets to coherence spaces

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Abstract

The aim of this paper is twofold. First we give a brief overview of domain theory by presenting some relevant definitions and theorems. Second we shall give an abstract characterisation of Girard's coherence spaces ([2], [3]).
A *(countable) coherence space* $\mathcal{A}$ is a set of sets which satisfies the following three conditions.

1. **Countability**: $\bigcup \mathcal{A}$ is countable.
2. **Down-closure**: if $a \in \mathcal{A}$ and $a' \subseteq a$, then $a' \in \mathcal{A}$.
3. **Coherence**: if $M \subseteq \mathcal{A}$ and if $\forall a_1, a_2 \in M (a_1 \cup a_2 \in \mathcal{A})$, then $\bigcup M \in \mathcal{A}$.

A coherence space is very concrete: it is a poset of sets ordered by set inclusion. This contrasts with the usual constructions in domain theory, which work with arbitrary posets.
In this paper we characterize coherence spaces abstractly as *coherent dI P-domains*, or as *coherent prime algebraic dcpo's satisfying an extra axiom P*. We shall see that the concrete presentation of a coherence space and the abstract presentation are closely related.
Nothing in this paper is really new, except perhaps the characterisation of coherence spaces as dIP-domains (but see [2]).

1 Posets

Definition 1 *A partially ordered set (poset) P is a pair $(S, \leq)$, where S is a set and $\leq$ is a binary relation on S, such that for all $x, y, z \in S$:*

- $x \leq x$ *(reflexivity)*
- $x \leq y$ *and* $y \leq x$ *imply* $x = y$ *(antisymmetry)*
- $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$ *(transitivity)*

We will often confuse a poset P with its underlying set S.
A special kind of posets, so-called *coherence spaces*, will be the leading examples in this paper ([2], [3]).

Definition 2 *A (countable) coherence space $\mathcal{A}$ is a set of sets which satisfies the following three conditions:*

1. *Countability: $\bigcup \mathcal{A}$ is countable.*
2. *Down-closure: if $a \in \mathcal{A}$ and $a' \subseteq a$, then $a' \in \mathcal{A}$.*
3. *Coherence: if $M \subseteq \mathcal{A}$ and if $\forall a_1, a_2 \in M (a_1 \cup a_2 \in \mathcal{A})$, then $\bigcup M \in \mathcal{A}$.*

Example 3 *A coherence space $\mathcal{A}$ is a poset.*

Definition 4 *Let $P = (S, \leq)$ be a poset, $S' \subseteq S$ and $x \in S$, then*

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1. *x is a* minimal (maximal) *element of S' iff $x \in S'$ and $y \leq x$ ($x \leq y$) implies $x = y$, for each $y \in S'$.*

2. *x is a* lower (upper) bound *of S' iff $x \leq y$ ($y \leq x$) for all $y \in S'$.*

3. *x is a* least (greatest) *element of S' iff $x \in S'$ and x is a lower (upper) bound of S'.*

4. *x is the* greatest lower (least upper) bound *of S' iff x is the greatest (least) element of the set of all lower (upper) bounds of S'.*

The least upper bound (lub) or *join* of a set S is denoted by $\bigvee S$, and the greatest lower bound (glb) or *meet* by $\bigwedge S$. If S consists of two elements x and y then we write $x \vee y$, resp. $x \wedge y$. Special cases: the lub of $\emptyset$ is the least element of the poset, and the glb of $\emptyset$ is the greatest element of the poset.
A suitable kind of mapping for posets preserves the order structure.

Definition 5 *Let $(S, \leq)$ and $(R, \preceq)$ be posets. A function $f : S \rightarrow R$ is* monotone *iff $x \leq y$ implies that $f(x) \preceq f(y)$, for all $x, y \in S$.*

In the sequel we will assume some familiarity with category theory (cf. [7]). We define the category *Pos* as the category with as objects posets, and as arrows monotone functions.

Theorem 6 *Pos is a Cartesian closed category (CCC).*

Proof: Let P_1, P_2 be posets. The product $P_1 \times P_2$ in *Pos* is the usual Cartesian product of sets, ordered pointwise. The exponents $P_1 \Rightarrow P_2$ in *Pos* are the sets of monotone functions ordered pointwise. ■

This means that *Pos* is a model of the typed lambda calculus (we interpret types as posets and terms as monotone functions). Cartesian closure is an important property of a category of domains and we will meet it again and again.
Each poset is a category itself: the objects are the elements of the underlying set and there is exactly one arrow $x \rightarrow y$ iff $x \leq y$. Functors between posets (considered as categories) are just monotone functions.

2 Lattices

Definition 7 *An* upper semilattice *L is a poset in which the lub of each finite subset exists.*

An upper semilattice is sometimes called a *join-semilattice*. Note that L can not be empty, for it must contain a least one element (lub of $\emptyset$). An upper semilattice is sometimes defined as a poset in which the lub of each finite, *non-empty* subset exists.
Dually we define a *lower-* or *meet*-semilattice L as a poset in which the glb of each finite subset exists.

Definition 8 *A* lattice *L is a poset in which the lub and the glb of each finite subset exists.*

Definition 9 *A* complete lattice *L is a poset in which the lub and the glb of each subset exists.*

It makes no sense to define a *complete* upper/lower semilattice, for all lubs exist iff all glbs exist: Let L have all lubs. The glb of an arbitrary subset S of L is given by $\bigvee\{x \in L | \forall x' \in S : x \leq x'\}$.

Example 10

- *The set $\mathcal{F}(S) \cup \{S\}$ (where $\mathcal{F}(S)$ is the set of finite subsets of a set S) ordered by inclusion is a lattice, but not a complete lattice.*

- *The powerset $\mathcal{P}(S)$ of a set S ordered by inclusion is a complete lattice.*

- *The interval* $[0,1]$ *of real numbers ordered as usual is a complete lattice.*

Monotone functions on a complete lattice have the following important property.

Theorem 11 *Let L be a complete lattice, and $f : L \to L$ a monotone function. Then f has a least fixed point, i.e. there is a $x \in L$ such that:*

- $f(x) = x$
- $\forall x' \in L(f(x') = x' \Rightarrow x \leq x')$

Proof: The element $\bigwedge\{x \in L | f(x) \leq x\}$ is the least fixed point. ■

Let *Clat* be the full subcategory of Pos with complete lattices as objects.

Theorem 12 *Clat is a CCC.*

Proof: Product and exponent are just as in *Pos*. ■

Using the fact that every function in *Clat* has a fixed point we see that *Clat* is in fact a model of the typed lambda calculus with a fixpoint combinator, i.e. a combinator Y such that Yt reduces to $t(Yt)$ for a term t. The term Yt is interpreted as the least fixed point of the interpretation of t.

3 Complete posets

The theory of complete lattices is not totally satisfactory. For example, the usual way to consider the set of natural numbers as a lattice is to take $N = \{0, 1, 2, \ldots\} \cup \{\bot, \top\}$ as underlying set, and to order N by $x \leq y \Leftrightarrow (x = y \vee x = \bot \vee y = \top)$. The existence of the least element $\bot$ is intuitively plausible as it stands for "undefined". However the greatest element has not such a direct meaning (sometimes it is considered as standing for the "overdefined element").
In this section a more general type of structures, which nevertheless have the desirable properties of lattices, is defined.

Definition 13 *An ω-*chain *C in a poset $(S, \leq)$ is a countable subset C of S with an enumeration $c_0, c_1, \ldots$ of its elements such that $c_i \leq c_{i+1}$.*

Definition 14 *An* (ω-)complete poset (cpo) *is a poset in which each ω-chain has a lub.*

Example 15

- *Each finite poset is a cpo.*
- *The set $\{0, 1, 2, \ldots\} \cup \{\bot\}$ ordered by $x \leq y \Leftrightarrow (x = y \vee x = \bot)$ is a cpo.*
- *The set of finite and infinite strings over an alphabet Σ ordered by the prefix ordering is a cpo.*
- *A coherence space $\mathcal{A}$ is a cpo.*

Definition 16 *A function $f : P_1 \to P_2$ between the (underlying sets of) two posets is* ω-continuous *iff it preserves lubs of ω-chains, i.e. if C is an ω-chain in P_1 then $f(\bigvee C) = \bigvee\{f(c) | c \in C\}$.*

Note that an ω-continuous function is monotone.

Theorem 17 *Let D be a cpo and $f : D \to D$ an ω-continuous function. If $x \leq f(x)$ for some $x \in D$ then there is a least $y \in D$ such that $x \leq y$ and $y = f(y)$.*

Proof: Take y=$\bigvee\{f^i(x) | i \in N\}$, where $f^0 = id$, and $f^{i+1} = f^i \circ f$. ■

In particular ω-continuous functions on cpo's with a least element have least fixed points. Let *Cpo* be the category with as objects cpo's, and as arrows ω-continuous functions.

Definition 18 *If C is a subcategory of Pos, then $C_\perp$ is the full subcategory of C with as objects those of C with a least element.*

Note that we do not require that functions in $C_\perp$ preserve the least element.

Theorem 19 *Cpo and $Cpo_\perp$ are CCC's.*

Proof: Product in *Cpo* and $Cpo_\perp$ is just as in *Pos*. If P_1, P_2 are cpo's, then the exponent $P_1 \Rightarrow P_2$ is the set of ω-continuous functions from P_1 to P_2. ■

Hence $Cpo_\perp$ is a CCC where each arrow has a least fixed point.

4 Directed complete posets

Directed complete partial orders are often used instead of ω-complete partial orders.

Definition 20 *Let $(S, \leq)$ be a poset, then $S' \subseteq S$ is* directed *iff each finite subset of S' has an upperbound in S'.*

Alternatively we can say that S' is directed iff S is not empty, and $\forall x, y \in S' \exists z \in S'$ such that $x \leq z$ and $y \leq z$ (S' can not be empty for there has to be an upperbound of the empty set in S'). It is easy to see that every ω-chain is a directed set. In fact the concept of directed set is a generalisation of the concept ω-chain.

Definition 21 *A* directed complete poset (dcpo) *is a poset in which each directed subset has a lub.*

Clearly every dcpo is a cpo. In fact all the examples of cpo's in the previous section are dcpo's.

Example 22 *Let P be a poset. A subset S of P is* down-closed *iff $x \leq y \in S$ implies $x \in S$. An* ideal *I of P is a directed, down-closed subset of P. The set $Idl(P)$ of ideals of P ordered by inclusion is a dcpo.*

The following theorem shows that the difference between cpo's and dcpo's is essentially one of cardinality.

Theorem 23 *A poset is a cpo iff it has all lubs of countable directed sets.*

Proof: In every countable directed set S' there is a ω-chain C such that $\bigvee S' = \bigvee C$. ■

The following relation holds between dcpo's and complete lattices.

Theorem 24 *A dcpo D is a complete lattice iff it is a join-semilattice.*

Proof: One side is trivial. For the other let D be a dcpo with all finite lubs. Let S be an arbitrary subset of D. The set consisting of all the lubs of the finite subsets of S is directed, and has the same lub as S. ■

Definition 25 *A function $f : P_1 \to P_2$ between the (underlying sets of) two posets is* continuous *iff it preserves lubs of directed subsets, i.e. if S is a directed subset of P_1 then $f(\bigvee S) = \bigvee\{f(x) | x \in S\}$.*

The theorem of the previous section on fixed points holds also for dcpo's and continuous functions. Define *Dcpo* as the category with as objects dcpo's, and as arrows continuous functions.

Theorem 26 *Dcpo and $Dcpo_\perp$ are CCC's.*

Proof: The product is just as in *Pos*. The exponent $D_1 \Rightarrow D_2$ of two dcpo's D_1, D_2 is the set of continuous functions from D_1 to D_2 ordered pointwise. ■

$Dcpo_\perp$ is a CCC with least fixed points.

5 Domain equations

We can generalize the results of the two previous sections by taking categories instead of posets. While in the previous sections we required special subsets to have lub's, we will now require special diagrams to have colimits.

Definition 27 *An* ω-chain *in a category* C *is a functor* $\omega \to C$, *where* ω *is the poset of natural numbers ordered by* $i \leq i+1$. *A* directed diagram *is a functor* $I \to C$, *where* I *is a directed set.*

Still more general we could consider *filtered* diagrams. However, in the so-called *categories of embeddings*, which are frequently used in domain theory (see definition 32), all the arrows are monomorpisms, and filtered diagrams coincide with directed diagrams.

Definition 28 *An* ω-chain complete *category is a category in which each* ω*-chain has a colimit. A* directed complete *category is a category in which each directed diagram has a colimit.*

Example 29

- *Set (the category of sets and functions) is directed complete.*
- *Dcpo*^{op} *(the dual category of Dcpo) is directed complete.*

We define a *cocontinuous* resp. ω*-cocontinuous* functor as a functor which preserves colimits of directed diagrams resp. ω-chains. There is a fixed point theorem.

Theorem 30 *Let* C *be an* ω*-chain complete category, and* $F : C \to C$ *an* ω*-cocontinuous functor. If* $f : x \to F(x)$ *is an arrow in* C, *then there is an* y *in* C *such that* $F(y) \cong y$.

Proof: Let y be the colimit of the chain $F^i(f)$. ∎

The same holds for directed complete categories.
So we can solve *domain-equations* in *Dcpo*^{op}. For example the equation $D \cong D \times D$ is solved as follows. Take the cocontinuous functor $\Delta : Dcpo^{op} \to Dcpo^{op} : D \mapsto D \times D$. Let $f : \Delta(D_0) \to D_0$ be an arrow in *Dcpo*, so an arrow $f : D_0 \to \Delta(D_0)$ in *Dcpo*^{op}. Then we can find a D which satisfies the equation as in the theorem. By taking a non-trivial f we can ensure that the result will be non-trivial.
It is not possible to solve the equation $D \cong D \Rightarrow D$ in this manner, where $D \Rightarrow D$ is the exponent in *Dcpo*, i.e. the set of all continuous functions from D to D. Because the functor $(.) \Rightarrow (.) : Dcpo^{op} \times Dcpo \to Dcpo$ is contravariant in the first, but covariant in the second component, there is no functor $F : Dcpo^{op} \to Dcpo^{op}$ such that $F(D) = D \Rightarrow D$.
We therefore take a subcategory of *Dcpo* with as arrows certain pairs of arrows of *Dcpo*.

Definition 31 *Let* C *be a subcategory of Pos. An arrow* $f : P \to Q$ *in* C *is a* section *iff there is an arrow* $g : Q \to P$ *in* C *such that* $g \circ f = id_P$. *In this case* g *is called a* retraction. *If* f, g *satisfy the further condition that* $f \circ g \leq id_Q$, *then* f *is called an* embedding, *and* g *a* projection.

Every embedding e determines an unique projection, wich we will denote by e^{-1}.

Definition 32 *Let* C *be a subcategory of Pos.* C^E *is the category having the same objects as* C *and having embeddings as arrows.*

Theorem 33 $Dcpo^E$ *and* $Dcpo_\perp^E$ *are directed complete.*

We can define a functor $F : Dcpo^E \to Dcpo^E$ on objects as $D \mapsto (D \Rightarrow D)$, and on arrows as $(e : D \to E) \mapsto (e \circ (.) \circ e^{-1} : (D \Rightarrow D) \to (E \Rightarrow E))$. Using this cocontinuous functor, and with the help of the theorem on fixed points, domain equations in which $\Rightarrow$ occurs can be solved.

6 Algebraic dcpo's

We define the full subcategory of *Dcpo* consisting of the algebraic dcpo's. A dcpo D is algebraic iff there is a countable subset (a basis) of D such that each element of D is "generated" by this subset.

Definition 34 *Let D be a dcpo. An element $d \in D$ is called* compact *iff for each directed subset S of D we have that $d \leq \bigvee S$ implies $\exists x \in S : d \leq x$.*

A compact element is sometimes called *finite* or *isolated.*

Example 35 *In a coherence space $\mathcal{A}$ the compact elements are precisely the finite sets.*

Definition 36 *D is an* (ω-)algebraic *dcpo iff D is a dcpo and if there is a countable subset C_D of compact elements of D, such that for each $x \in D$ the set $C_D(x) = \{d \in D | d \in C_D, d \leq x\}$ is directed, and $x = \bigvee C_D(x)$.*

In an algebraic dcpo each element is the lub of a directed set of compact elements. The set C_D is called a *basis* for D.

Example 37 *A coherence space $\mathcal{A}$ is an algebraic dcpo.*

Example 38 *If P is a countable poset, then $Idl(P)$ is an algebraic dcpo. As $C_{Idl(P)}$ we take the* principal *ideals $I_x = \{y \in P | y \leq x\}$, for $x \in P$.*
In particular if D is an algebraic dcpo, then $Idl(C_D) \cong D$.

If D, E are algebraic dcpo's, then the function space $D \Rightarrow E$ (the set of continuous functions $D \rightarrow E$) need not be algebraic. For example there may be no *countable* set $C_{D\Rightarrow E}$ of compact elements.

Example 39 *Let N be the set of natural numbers considered as a poset, i.e. $n \leq m \Leftrightarrow n = m$. Let 2 be the set $\{0, 1\}$ considered as a poset. Then each element of the set of functions $N \Rightarrow 2$ is finite. Hence $C_{N\Rightarrow 2}$ is not countable.*

If D, E are algebraic dcpo's and E has a least element $\perp$, we can find a countable subset S of compact elements of $D \Rightarrow E$, such that each continuous function $f : D \rightarrow E$ is the lub of the elements of S under it. The set S consists of the step functions. A *step* function is given by a pair (d, e), with $d \in C_D$ and $e \in C_E$. Application is defined by $(d, e)(x) = e$ iff $d \leq x$ and $(d, e)(x) = \perp$ otherwise. It can easily be shown that each step function is compact in $D \Rightarrow E$. Moreover we have that

$$f = \bigvee\{(d, e) | (d, e) \leq f\}$$

However $D \Rightarrow E$ need not be algebraic, because the set $\{f' \leq f | f' \in S\}$ is not always directed. This is shown by the next example.

Example 40 *Let D be the algebraic dcpo with as underlying set $\{a_0, a_1, \ldots\} \cup \{b_0, b_1, \ldots\} \cup \{\perp, \top\}$ ordered by $x \leq y \Leftrightarrow (x = y \vee x = \perp \vee y = \top \vee (x = c_i \wedge y = c'_j \wedge i < j))$, where $c, c' \in \{a, b\}$.*
The identity function id_D is not compact in $D \Rightarrow D$: It is clear that the set $\{\lambda x.a_i\}$ is directed, and that it has lub $\lambda x.\top$. The identity on D is less than $\lambda x.\top$, but $id_D \not\leq \lambda x.a_i$.
The set $R = \{(\perp, \perp), (a_0, a_0), (b_0, b_0)\}$ contains step functions, and hence compact elements of $D \Rightarrow D$, which are under id_D. Let f be an upperbound of this set, and $f \leq id_D$. Then $f(\perp) = \perp$, and we can prove by induction that $f(a_i) = a_i$ and $f(b_i) = b_i$. So $f = id_D$. Therefore R has no compact upperbound and there is no directed set of compact elements with lub id_D.

Define *Alg* as the full subcategory of *Dcpo* with as objects algebraic dcpo's. By the previous arguments *Alg* is not a CCC. So we are going to look for Cartesian closed subcategories of *Alg*.

7 Saturated posets

In this section we define various sorts of *saturated* posets. The intersection of categories of these posets with *Alg* will give us some Cartesian closed subcategories of *Alg*.

Definition 41 *Let P be a poset, and $S \subseteq P$. S is* κ-consistent *iff for all $S' \subseteq S$ we have that $||S'|| < \kappa$ implies S' has an upperbound in P.*

Let $P \neq \emptyset$, then $\emptyset$ is κ-consistent for each κ. Further every subset of P is 0,1,2-consistent. A 3-consistent subset is a *pairwise consistent* or *coherent* subset, i.e. each pair of elements in the subset has an upperbound in P. In an ω-consistent subset each finite subset has an upperbound in P. Note that each directed set is an ω-consistent set, but not vice versa.
If $\kappa \leq \kappa'$ then each κ'-consistent set is κ-consistent.

Definition 42 *A* κ-saturated *poset is a non-empty poset in which each κ-consistent subset has a lub.*

Remark: In the literature κ-saturated posets are called *κ-complete posets.*

Example 43 *A coherence space $\mathcal{A}$ is a 3-saturated poset.*

Let $\kappa \leq \kappa'$ then each κ-saturated poset is κ'-saturated.

Theorem 44 *If $\kappa \leq \omega$, then each κ-saturated poset is a dcpo with a least element.*

Proof: The empty set is κ-consistent, and the lub of the empty set is the least element.
Each directed set is ω-consistent, hence κ-consistent for $\kappa \leq \omega$, and therefore it has a lub. ∎

It is clear that the κ-saturated posets with $0 \leq \kappa < 3$ are precisely the complete lattices. The cases $\kappa = 3$ and $\kappa = \omega$ also have special names. We call the 3-saturated posets *coherent (complete)* dcpo's, and the ω-saturated posets *bounded complete* dcpo's. The latter are also known as *conditionally* or *consistently complete* dcpo's. The motivation for these last names is given by the following theorem.

Theorem 45 *D is an ω-saturated poset iff D is a dcpo, D has a least element $\bot$, and each bounded subset of D has a lub.*

Proof: ($\Rightarrow$): Let D be an ω-saturated poset. Let S be a bounded subset of D, then each finite subset of S has an upperbound in D, so S is ω-consistent. Therefore S has a lub.
($\Leftarrow$): Let D be a dcpo such that each bounded subset has a lub. Let S be a ω-consistent subset of D. Then each finite subset of S is bounded and has a lub. The set of all these lubs is directed and has a lub in D, which is the lub of S. ∎

In the proof of the theorem we use only lubs of bounded subsets which are finite so we can also state the following.

Theorem 46 *D is an ω-saturated poset iff D is a dcpo, D has a least element $\bot$, and each bounded finite subset of D has a lub.*

Theorem 47 *D is a bounded complete dcpo iff D is a dcpo and each non-empty subset has a glb.*

Proof: ($\Rightarrow$): Let S be a non-empty subset of the bounded complete dcpo D. The set $lwb(S)$ of lowerbounds of S is bounded, and $\bigwedge S = \bigvee lwb(S)$.
($\Leftarrow$): Let D be a dcpo such that each non-empty subset has a glb. Let S be a bounded subset of P. Then the set $upb(S)$ of upperbounds of S is non-empty and $\bigvee S = \bigwedge upb(S)$. ∎

So a bounded complete dcpo is similar to a complete lattice. It may fail to be a complete lattice by not having a greatest element (i.e. the glb of the empty-set).

Theorem 48 *D is a bounded complete dcpo iff D is a dcpo and $D^{\top}$ is a complete lattice.*

Define the category κ-*Sat* of κ-saturated posets and continuous maps.

Theorem 49 *κ-Sat is a CCC.*

8 Saturated algebraic dcpo's

We have the following theorem.

Theorem 50 *If D, E are ω-saturated algebraic dcpo's, then the function space is algebraic.*

Proof: We have seen in the section on algebraic dcpo's that if E has a least element then we can define step functions, and that each step function is compact in the functionspace $D \Rightarrow E$. It is easy to see that in general the lub of a finite set of compact elements, if it exists, is a compact element. If E is an ω-saturated algebraic dcpo, then the compact elements of $D \Rightarrow E$ are exactly the lubs of finite sets of step functions. So we take for $C_{D\Rightarrow E}$ the set of lubs of finite set of step functions, which is countable. ∎

If D, E are κ-saturated algebraic dcpo's ($\kappa \leq \omega$), then they are ω-saturated, so $D \Rightarrow E$ is algebraic.
For an arbitrary κ define κ-*Alg* as the full subcategory of *Dcpo* with as objects the κ-saturated algebraic dcpo's. So 0-*Alg* has as objects algebraic lattices, 3-*Alg* has as objects coherent algebraic dcpo's, and ω-*Alg* has as objects bounded complete algebraic dcpo's. The objects in the last category are the famous *Scott domains.*

Theorem 51 *κ-Alg ($\kappa \leq \omega$) is a CCC.*

Example 52 *A coherence space $\mathcal{A}$ is a coherent algebraic dcpo.*

9 Strongly algebraic dcpo's

There is a larger subcategory of *Alg* than ω-*Alg* that is a CCC.

Definition 53 *A* strongly algebraic *dcpo is a poset which is the colimit of an ω-chain of finite dcpo's in $Dcpo_{\perp}^{E}$.*

The strongly algebraic dcpo's are sometimes called *SFP-objects*, where SFP stands for "Sequences of Finite Partial orders".
It is clear that each strongly algebraic dcpo is a dcpo. There is another characterisation of this class of dcpo's, which shows that they are algebraic dcpo's with some other properties. First we need some definitions.

Definition 54 *Let S be a subset of a partial order P. Define $MUB(S)$ as the set of minimal upperbounds of S. Then S has property M iff*

1. *$MUB(S)$ is finite.*
2. *$MUB(S)$ is complete, i.e. every upperbound of S is above some element of $MUB(S)$.*

Definition 55 *Let S be a subset of a partial order P. Define an operator U^* as follows:*

- $U^0(S) = S$
- $U^{n+1}(S) = \{MUB(S') | S' \subseteq U^n(S)$, *and* S' *finite*$\}$
- $U^*(S) = \bigcup U^n(S)$

Theorem 56 *D is a strongly algebraic dcpo iff D is an algebraic dcpo with a least element, and such that the following holds for every finite subset S of C_D:*

1. *S has property M.*
2. *$U^*(S)$ is finite.*

Define SFP as the full subcategory of *Dcpo* with as objects the strongly algebraic dcpo's. By the previous theorem we have that SFP is a full subcategory of *Alg*.

Theorem 57 *SFP is a CCC.*

It is a consequence of the following theorems that ω-*Alg* is a subcategory of SFP.

Theorem 58 *If D and $D \Rightarrow D$ are algebraic dcpo's, then D is strongly algebraic.*

Theorem 59 *SFP is the largest Cartesian closed full subcategory of Alg.*

10 Continuous dcpo's

We can generalize the notion of algebraic dcpo to that of *continuous* dcpo.

Definition 60 *Let D be a dcpo. Define the* way-below *relation $\ll$ on $D \times D$ as follows:*
$x \ll y$ iff for each directed subset S of D we have that $y \leq \bigvee S$ implies there exists an $y' \in S$ such that $x \leq y'$.

Theorem 61 *The way-below relation $\ll$ has the following properties:*

- $x \ll y \Rightarrow x \leq y$
- $x \ll x \Leftrightarrow x$ *is compact*
- $x' \leq x \ll y \leq y' \Rightarrow x' \ll y'$

Definition 62 *D is a (ω-)*continuous *dcpo iff D is a dcpo and there exists a countable subset B_D of D such that for each $x \in D$ the set $B_D(x) = \{x' \in B_D | x' \ll x\}$ is directed, and $x = \bigvee B_D(x)$.*

The set B_D is called a *basis* for D.
If we define $\Downarrow(x) = \{x' \in D | x' \ll x\}$, then in a continuous dcpo D this set is directed, and $\bigvee \Downarrow(x) = x$.

Example 63 *Every algebraic dcpo D is continuous: Take $B_D = C_D$, then $B_D(x) = C_D(x)$, so $B_D(x)$ is directed and has lub x.*

Theorem 64 *Let D be a continuous dcpo, then for each $x, z \in D$ we have that $x \ll z$ implies that there exists an $y \in D$ such that $x \ll y \ll z$.*

Proof: Take $S = \{d \in D | \exists y \in D : d \ll y \ll z\}$.
S is directed: It is clear that S is non-empty. Suppose $d_1, d_2 \in S$, then there are $y_1, y_2 \in D$ such that $d_i \ll y_i \ll z$. Now $\Downarrow(z)$ is directed, so there exists $y_3 \in D$ such that $y_3 \ll z$ and $y_1, y_2 \leq y_3$. The set $\Downarrow(y_3)$ is directed, so there exists $d_3 \in D$ such that $d_3 \ll y_3$ and $d_1, d_2 \leq d_3$.
Further $\bigvee S$
$= \bigvee\{\bigvee \Downarrow(y) | y \ll z\}$
$= \bigvee B_z$
$= z$.
Now suppose $x \ll z$, then $x \ll z \leq \bigvee S$. So there exists $d \in D$ such that $x \leq d$, so $x \leq d \ll y \ll z$ for certain y. So $x \ll y \ll z$. ∎

Example 65 *Let P be a countable poset, with a relation $R \subseteq P \times P$ such that:*

1. $R(x, y) \Rightarrow x \leq y$
2. $R(x_1, z) \wedge R(x_2, z) \Rightarrow \exists y \in P : x_1, x_2 \leq y \wedge R(y, z)$

3. $x' \leq x \wedge R(x,y) \wedge y \leq y' \Rightarrow R(x',y')$

4. $R(x,z) \Rightarrow \exists y \in P : R(x,y) \wedge R(y,z)$

A rounded *ideal* I *of* P *with respect to* R *is an ideal* I *such that if* $x \in I$, *then there exists an* $y \in I$ *such that* $R(x,y)$. *Define* $RIdl(P)$ *as the set of rounded ideals of* P, *ordered by inclusion. It is easy to see that* $RIdl(P)$ *is a continuous dcpo, with as basis the elements* $I_x = \{x' \in P | R(x',x)\}$, *for* $x \in P$.

We can characterize the continuous dcpo's as follows.

Theorem 66 *D is a continuous dcpo iff there exists a continuous retraction $r : E \to D$, with E an algebraic dcpo.*

Proof: If D is a continuous dcpo, then $\bigvee(-) : Idl(B_D) \to D$ is a retraction, with $\Downarrow(-) : D \to Idl(B_D)$ as section. ∎

Definition 67 *Let C be a full subcategory of Dcpo, then $R(C)$ is the full subcategory of C with as objects the continuous retracts of C, i.e. D is an object of $R(C)$ iff there exists a continuous retraction $r : E \to D$, with E in C.*

If we define *Con* as the full subcategory of *Dcpo* with as objects the continuous dcpo's, then *Con*=*R*(*Alg*).
Con is not a CCC, just like *Alg*. However it is easy to find Cartesian closed subcategories of *Con*.

Theorem 68 *If C is a CCC, then $R(C)$ is a CCC.*

For example $R(\kappa\text{-}Alg)$ ($\kappa \leq \omega$) is a CCC. We can characterize the objects in these categories.

Theorem 69 *D is a κ-saturated continuous dcpo iff there exists a continuous retraction $r : E \to D$, with E a κ-saturated algebraic dcpo.*

11 Prime algebraic dcpo's

Compact elements are defined with respect to lubs of directed sets. *Complete prime* elements are defined in the same manner with respect to arbitrary lubs.

Definition 70 *Let D be a dcpo. An element $d \in D$ is called* complete prime *iff for each subset S of D we have that $d \leq \bigvee S$ implies $\exists x \in S : d \leq x$.*

A *prime* element is defined in the same manner with respect to lubs of finite sets.

Example 71 *In a coherence space $\mathcal{A}$ the complete prime elements are precisely the one element sets.*

It is easy to see that a complete prime element is compact.

Definition 72 *D is an (ω-)prime algebraic dcpo iff D is a dcpo and there is a countable subset P_D of complete prime elements of D such that for each $x \in D$ we have that $\bigvee P_D(x) = x$, where $P_D(x) = \{d \in D | d \in P_D, d \leq x\}$.*

Example 73 *A coherence space $\mathcal{A}$ is a prime algebraic dcpo.*

Let *PAlg* be the full subcategory of *Dcpo* with as objects the prime algebraic dcpo's. Then *Alg* is not a subcategory of *PAlg*. Neither *PAlg* is a subcategory of *Alg*.

Definition 74 *Let D, E be dcpo's, and let E have a least element $\perp$. A* prime step function *is a pair (c,p), where c is compact in D, and p is complete prime in E.*
Application is defined by $(c,p)(x) = p$ iff $c \leq x$, and $(c,p)(x) = \perp$ otherwise.

Theorem 75 *Let D be an algebraic dcpo, and E a prime algebraic dcpo with a least element $\perp$, then every $f \in D \Rightarrow E$ is the lub of the prime step functions under it.*

Theorem 76 *Let D be a dcpo, and E a κ-saturated ($\kappa \leq \omega$) dcpo with a least element $\perp$, then each prime step function is a complete prime element of $D \Rightarrow E$.*

Theorem 77 *Let D be a κ-saturated prime algebraic dcpo ($\kappa \leq \omega$), then D is algebraic.*

Proof: Show that it holds for ω-saturated prime algebraic dcpo's. Take $C_D = \{\bigvee X | X \subseteq P_D, X$ finite and bounded$\}$. ■

Define *κ-Palg* as the full subcategory of *Dcpo* with as objects the κ-saturated prime algebraic dcpo's. By the last theorem we have that *κ-PAlg* is a full subcategory of *κ-Alg*.
We can combine the previous three theorems as follows.

Theorem 78 *κ-Palg ($\kappa \leq \omega$) is a CCC.*

A dcpo D satisfies *axiom d* iff for all $x, y, z \in D$

$$y \uparrow z \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Theorem 79 *Let D be a κ-saturated prime algebraic dcpo, then D satisfies axiom d.*

Proof: We have only to show that if $y \uparrow z$ then $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$ (the reverse inequality is always true). Show now that each complete prime under $x \wedge (y \vee z)$ is also under $(x \wedge y) \vee (x \wedge z)$. ■

In the next section we shall consider a special subclass of the ω-saturated prime algebraic dcpo's.

12 dI-domains

Definition 80 *Let D be a dcpo. An element $d \in D$ is called* very finite *iff there are only finitely many elements below d.*

It is easy to see that in an algebraic dcpo each very finite element is compact. An algebraic dcpo satisfies *axiom I* iff the reverse is also true, i.e. each compact element is very finite.

Definition 81 *A dI-domain D is an ω-saturated algebraic dcpo (i.e. a Scott domain) which satisfies axiom d and I.*

A *coherent* dI-domain is a dI-domain which is coherent, i.e. 3-saturated.

Example 82 *A coherence space $\mathcal{A}$ is a coherent dI-domain*

We will show that the class of dI-domains is a subclass of the class of ω-saturated prime algebraic dcpo's, viz. those which satisfy axiom I. We need two lemma's.

Lemma 83 *Let D be an ω-saturated dcpo, and $d \in D$, then d is a complete prime iff d is compact and prime.*

Lemma 84 *Let D be an algebraic dcpo, then D is prime algebraic iff there is a countable subset P of complete primes of D such that for each $c \in C_D$ we have that $c = \bigvee\{p | p \leq c, p \in P\}$.*

Theorem 85 *D is a dI-domain iff D is an ω-saturated prime algebraic dcpo which satisfies axiom I.*

Proof: The if-part follows by the results of the previous section.
For the only if-part suppose D is a dI-domain. We have to show that D is prime algebraic. By lemma 84 we have to find a countable subset P of complete primes such that each $c \in C_D$ is the lub of elements from P. Take $P = \{p \text{ complete prime} | \exists c \in C_D : p \leq c\}$. Clearly P is countable.
We will now show that a very finite element $d \in D$ is the lub of complete primes by induction to the number n of elements below d.

basis If $n = 0$, then d is the least element $\bot$ of D, and $\bigvee\{p|p \leq \bot\} = \bigvee\emptyset = \bot$ (note that $\bot$ is not a complete prime).

hypothesis Suppose that if there are k elements below d, and $k \in \{0, ..., n\}$, then d is the lub of complete primes.

step Let there be $n+1$ elements below d. If d is itself a complete prime then we are ready. Suppose d is not a complete prime. The element d is very finite, hence it is compact, and by lemma 83 it follows that d is not prime. So there is a finite subset S of D, such that $d \leq \bigvee S$, and for all $x \in S$ $d \not\leq x$. Consider the elements $d \wedge x$ for $x \in S$. We have that $\bigvee\{d \wedge x | x \in S\} = d \wedge \bigvee S$, because S is finite, D satisfies axiom d, and $d \uparrow x$ for all $x \in S$. Further $d \wedge \bigvee S = d$, because $d \leq \bigvee S$. Hence d is the lub of the elements $d \wedge x$.
We have that $d \wedge x < d$, because $d \not\leq x$. So there are less than $n+1$ elements below $d \wedge x$, and we may apply the induction hypothesis to it. Therefore $d = \bigvee\{d \wedge x | x \in S\} = \bigvee\{p$ complete prime$|p \leq d \wedge x, x \in S\}$.

■

The category of dI-domains and continuous functions is not Cartesian closed, because in the function space compact points need not be very finite. However there is a non-full Cartesian closed subcategory of *Dcpo* with as objects the dI-domains.

Definition 86 *Let D, E be dcpo's and $f : D \to E$ a continuous function. f is* stable *iff for every $d \in D$ and $e \leq f(d)$ there exists a $d' \leq d$ such that $e \leq f(d')$ and $\forall d'' \leq d : e \leq f(d'') \Rightarrow d' \leq d''$.*

So a function f is stable iff for every $d \in D$ and $e \leq f(d)$ the set $\{d''|d'' \leq d$ and $e \leq f(d'')\}$ has a least element. This least element will be denoted by $L(f, d, e)$. Intuitively a function is stable iff it uses a definite part of the input for a certain part of the output.

Theorem 87 *Let D, E be ω-saturated dcpo's, and $f : D \to E$ a continuous function, the f is stable iff for every bounded $X \subseteq D$ we have that $f(\bigwedge X) = \bigwedge f(X)$.*

Proof: ($\Leftarrow$): Take $L(f, d, e) = \bigwedge\{d''|d'' \leq d$ and $e \leq f(d'')\}$.
($\Rightarrow$):Let $X \subseteq D$ be bounded by u. Show that $L(f, u, \bigwedge f(X))$ is a lowerbound of X. Then $\bigwedge f(X) \leq f(L(f, u, \bigwedge f(X)) \leq f(\bigwedge X)$, and $\bigwedge f(X) = f(\bigwedge X)$ follows, because $\bigwedge f(X) \geq f(\bigwedge X)$ is trivial.
■

Definition 88 *Let D, E be dcpo's, and $f : D \to E$ a continuous function. f is* conditionally multiplicative (c.m.) *iff $\{d, d'\}$ bounded implies that $f(d \wedge d') = f(d) \wedge f(d')$.*

It is clear that if D, E are ω-saturated, then a stable f is c.m. If we restrict to dI-domains then the reverse is also true.

Theorem 89 *Let D, E be dI-domains, and $f : D \to E$ a continuous function, then f is stable iff f is c.m.*

Let *dI-Dom* be the subcategory of *Dcpo* with as objects dI-domains, and as arrows stable functions.

Theorem 90 *dI-Dom is a CCC.*

13 coherence spaces

Definition 91 *Let D be a dcpo with a least element. An element $d \in D$ is called* primitive *iff there are exactly two elements $\leq d$.*