FINDING GRID EMBEDDINGS
WITH BOUNDED MAXIMUM EDGE LENGTH
IS NP-COMPLETE

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IS NP-COMPLETE*

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Abstract. The following problem is proven to be NP-complete for every fixed $k\geq 2$: Given a graph $G=(V,E)$, is there an injective mapping f of V to a two-dimensional grid, such that for every edge $(x,y)\in E$, the distance between f(x) and f(y) in the grid is at most k. The same problem is also NP-complete for mappings to d-dimensional grids, with $d\geq 3$, for all fixed $k\geq 1$.

1. Introduction. An embedding of a graph $G=(V_G,E_G)$ in a connected graph $H=(V_H,E_H)$ is an injective mapping of V_G to V_H. The dilation cost (abbreviated dcost) of an embedding f of G in H is the maximum distance between the images of any pair of adjacent nodes in G. Problems concerning graph embeddings arise in a natural way in several problem areas, for instance in the theory of VLSI-layouts, or in the organization of distributed computations on a network of processors. For an extensive list of references on embeddings and their applications, see for instance [4].

In this note we study embeddings in grids. Let $Z^d = (V^d,E^d)$ be the d-dimensional integer grid : $V^d = \{(x_1,\ldots,x_d) | \forall i,\ 1\leq i\leq d:\ x_i \in Z\}$, $E^d=\{((x_1,\ldots,x_d),\ (y_1,\ldots,y_d)) | \sum_{i=1}^{d} | x_i-y_i|=1\}$. Note that if one wants to find an embedding of a graph into Z^d with minimal dcost, then one can restrict oneself to embeddings in a finite subgraph of Z^d.

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We consider the problem to determine efficiently, for any specified integer k, whether a given graph G can be embedded in Z^d, with dilation cost at most k. We assume the reader to be familiar with the theory of NP-completeness (see Garey and Johnson [2]). For d=1 the problem is the well-known BANDWIDTH problem.

[BANDWIDTH]

Instance : Graph G, integer k

Question : Can G be embedded in a line (i.e. Z^1 or a subgraph of Z^1), with dcost at most k?

Papadimitriou showed in 1976 that BANDWIDTH is NP-complete [8,2]. Garey, Grahor, Johnson and Knuth [1,2] showed that BANDWIDTH stays NP-complete, even if G is rectricted to be a tree with all node of degree 3 or less. However, if we fix k and let it no longer be a part of the instance of the problem, then there exist algorithms that solve the problem in polynomial time (Saxe [9], Gurari and Sudborough [3]). For d=2, the problem becomes the following:

[EDGE LENGTH]

Instance : Graph G, integer k

Question : Can G be embedded in Z^2 with d-cost at most k?

Miller and Orlin [7] proved, with a reduction from BANDWIDTH, that the EDGE LENGTH problem is NP-complete. However, unlike the BANDWIDTH problem, we cannot obtain polynomial time algorithms for EDGE LENGTH, when we fix k (unless P=NP). Consider the following problem:

[k-EDGE LENGTH]

Instance : Graph G

Question : Can G be embedded in Z^2 with dcost at most k?

In this note we will prove that k-EDGE LENGTH is NP-complete, for every $k \geq 2$. In section 4 we consider a d-dimensional variant of k-EDGE LENGTH. We argue that d-DIMENSIONAL k-EDGE LENGTH is NP-complete, for every fixed $k \geq 1$ and $d \geq 3$. All known results on the complexity of d-DIMENSIONAL (k-)EDGE LENGTH are summarized in fig. 1.1.

dimension	Problem name	k=1	k fixed,≥2	k variable
d=1	BANDWIDTH	P	P	NPC
d=2	EDGE LENGTH	?	NPC	NPC
d≥3	d-DIM. EDGE LENGTH	NPC	NPC	NPC

fig.1.1. The complexity of d-DIMENSIONAL k-EDGE LENGTH
P = problem is solvable in polynomial time
NPC = problem is NP-complete
? = unknown whether problem is in P and whether it is NP-complete
(Results can be found in [1,3,7,8,9] and this note.)

2. <u>Notations and definitions</u>. For a node $x = (x_1,x_2) \in V^2$ we write $kx=k(x_1,x_2) = (kx_1,kx_2)$. For a set of nodes $V \subseteq V^2$ we write $kV= \{kx|x \in V\}$. Notice $|V|=|kV|$.

Let $x_1,x_2,y_1,y_2 \in Z$. We write $[x_1,y_1] * [x_2,y_2] = \{(z_1,z_2) \in V^2| \; x_1 \le z_1 \le y_1 \;\wedge\; x_2 \le z_2 \le y_2\}$. For $d \ge 1$ we write $[x_1,y_1]^d = \{(z_1,..,z_d) \in V^d \mid \forall i, 1 \le i \le d: \; x_1 \le z_i \le y_1\}$.

For $x,y \in V^2$ we denote the distance from x to y in the graph Z^2 by $d(x,y)$, i.e. $d(x,y) = |x_1-y_1| + |x_2-y_2|$. For a set of nodes $V \subseteq V^2$ we let $[V]_k$ denote the graph with nodes V and edges between nodes that have a distance of at most k.

<u>Definition</u>. Let $V \subseteq V^2$, $k \in N^+$. $[V]_k = (V,E_{V,k}]$ is the subgraph of Z^2 with $E_{V,k} = \{(x,y)|x,y \in V, \; x \neq y, \; d(x,y) \le k\}$.

For V V^2 one denote $[V] = [V]_1$. [V] is called the subgraph of Z^2, induced by V.

Two embeddings f, f' of a graph G in Z^2 will be called equivalent if one can be obtained from the other by applying one or more of the following elementary isometries :

(i) translation in Z^2 $(g((x_1,x_2)) = (x_1+a,x_2+b)$, for all $(x_1,x_2) \in V^2$, and some given $a,b \in Z)$.

(ii) reflexion along the x_1-axis $(g((x_1,x_2)) = (x_1,-x_2)$, for all $(x_1,x_2) \in V^2$.

(iii) reflexion along the x_2-axis $(g((x_1,x_2)) = (-x_1,x_2)$, for all $(x_1,x_2) \in V^2)$.

(iv) reflexion along the line $x_1=x_2$ $(g((x_1,x_2))=(x_2,x_1)$, for all $(x_1,x_2) \in V^2)$.

So f and f' are equivalent, if one can write $f' = g \circ f$, with g a composition of the functions, given in (i) - (iv). We say there is a unique embedding of G in Z^2 with dcost $\leq k$, iff there exists an embedding of G in Z^2 with dcost $\leq k$, and any two embeddings of G in Z^2 with dcost $\leq k$ are equivalent. (These definitions are taken from [7].)

We will make use of following problem:

[HAMILTONIAN CIRCUIT IN A GRID GRAPH]

<u>Instance</u> : Set of nodes $V \subseteq V^2$

<u>Question</u> : Does [V] contain a Hamiltonian circuit?

HAMILTONIAN CIRCUIT IN A GRID GRAPH was proven to be NP-complete by Itai, Papadimitriou and Szwarzfiter [5].

3. <u>NP-completeness of k-EDGE LENGTH</u>.

<u>Theorem 1</u>. For every $k \geq 2$: k-EDGE LENGTH is NP-complete.

<u>Proof</u>.

Let $k \geq 2$ be given. It is obvious that k-EDGE LENGTH is in NP. We can limit ourself to embeddings of graphs G=(V,E) on the subgraph of Z^2, induced by the set of nodes $[0,|V|-1]^2$. One can guess such an embedding and then check in polynomial time whether the dcost of this embedding is at most k. To prove NP-completeness of k-EDGE LENGTH we will reduce the HAMILTONIAN CIRCUIT IN A GRID GRAPH problem to it.

Let a set of node $V \subseteq V^2$ be given. We assume, without loss of generality that $\min \{x_1 | \exists x_2\ (x_1,x_2) \in V\} = \min \{x_2 | \exists x_1\ (x_1,x_2) \in V\} = 0$. Let $m = \max \{x_1 | \exists x_2\ (x_1,x_2) \in V\}$ and $n = \max \{x_2 | \exists x_1\ (x_1,x_2) \in V\}$.

Define $W = [-4k,mk+4k] * [-4k,nk+4k] \setminus kV$. Note that every node of kV is a member of $[0,mk] * [0,nk]$.

<u>Lemma 1.1.</u> $[W]_k$ has a unique embedding with dilation cost $\leq k$.

<u>Proof.</u>

The "trivial" embedding g of $[W]_k$ in Z^2 with $g((x_1,x_2)) = (x_1,x_2)$ has dilation cost k. Now suppose another embedding f of $[W]_k$ in Z^2 with dcost $\leq k$ is given.

Every node $x \in V^2$ has exactly $2k^2+2k$ nodes with distance at most k to it in Z^2. For an example see fig. 3.1.

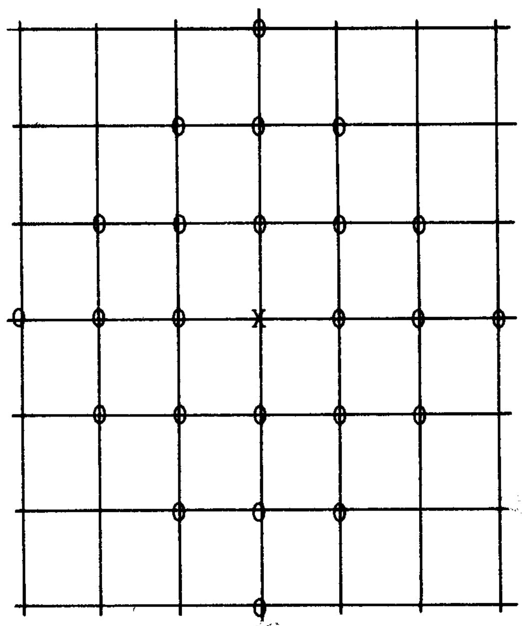


Figure 3.1. A node (X) and the nodes (O) with distance at most 3 to it.

This means that every node $x \in W$ has at most $2k^2 + 2k$ adjacent nodes in $[W]_k$. We call $x \in W$ a <u>diamond</u> <u>centre</u> (as in [7]), iff x has exactly $2k^2 + 2k$ adjacent nodes in $[W]_k$, i.e. for all nodes $y \in V^2$: $d(x,y) \leq k \Rightarrow y \in W$. Let x, $y \in W$ be diamond centres, $x \neq y$. The number of nodes that have distance $\leq k$ to x and to y in Z^2, is equal to the number of nodes that are adjacent to x and to y in $[W]_k$. This number is $2k^2-2$, if x and y are adjacent, or if x and y are one diagonal step away, (i.e. the euclidian distance between x and y is √2) in Z^2. See fig. 3.2.a and 3.2.b.

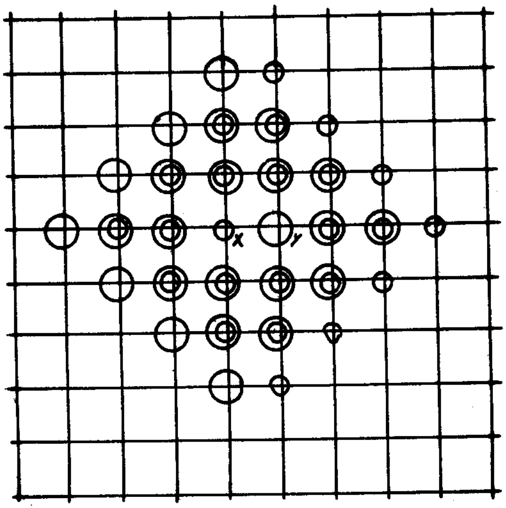


fig. 3.2.a

◯ = node with distance ≤3 to x

o = node with distance ≤3 to y

⊙ = node with distance ≤3 to x and y.

x and y are neighbours, and there are $2.3^2-2=16$ nodes that have distance ≤3 to x and y.

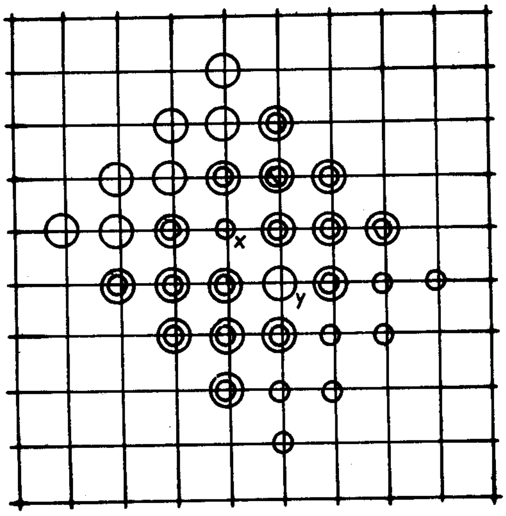


fig. 3.2.b

◯ = node with distance ≤3 to x

o = node with distance ≤3 to y

⊙ = node with distance ≤3 to x and y.

x and y are one diagonal step away and there are $2.3^2-2=16$ nodes that have distance ≤3 to x and y.

We say a node x is near to y, if x and y are neighbours or x and y are one diagonal step away in Z^2. See fig. 3.3.

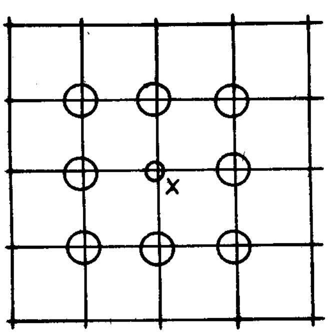
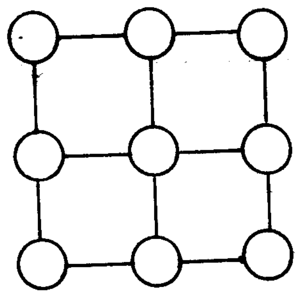


fig. 3.3. A node(x) and the 8 nodes that are near to it.

If x and y are not near, then the number of nodes with distance $\leq k$ to x and to y in Z^2 is less than $2k^2-1$. This means for diamond centres x, y, that if x is near to y, then f(x) must be near to f(y). Now observe that, if a node is a diamond centre, and the 8 nodes that are near to it are also diamond centres, then, when we apply f to the subgraph of Z^2, induced by these 9 nodes (see fig. 3.4.) we must obtain a subgraph of Z^2 isomorphic to it. This subgraph we call G_9.



G_9

fig. 3.4.

Write $W^* = \{(x_1,x_2) \in W \mid -3k \leq x_1 \leq -k-1$ or $-3k \leq x_2 \leq -k-1$ or $mk+k+1 \leq x_1 \leq mk+3k$ or $nk+k+1 \leq x_2 \leq nk+3k\}$. Every node in W^* is a diamond centre. (Recall that $W=[-4k,mk+4k] * [-4k,nk+4k] \setminus kV$, and $kV \subseteq [o,mk] * [o,nk]$.) When we apply f to a subgraph of $[W^*]$ isomorphic to G_9 then we obtain a subgraph of Z^2, again isomorphic to G_9. This means that $[f(W^*)]$ is isomorphic to W^*. One can apply one or more of the 4 elementary operations, given in section 2, such that one obtains an f', with $\forall (x_1,x_2) \in W^*$ $f'((x_1,x_2)) = (x_1,x_2)$. With induction one now can prove that $f'((x_1,x_2)) = (x_1,x_2)$ for all $(x_1,x_2) \in W$. Hence $f'=g$. □

Now choose an arbitrary node $z=(z_1,z_2) \in V$. Let the graph G consist of:

- the graph $[W]_k$
- a cycle with $|V|$ nodes $v^o, v^1, \ldots, v^{|V|-1}$, (so there are edges $(v^i, v^{(i+1) \bmod |V|})$, for $o \leq i \leq |V|-1$.)
- one edge between (kz_1-1,kz_2) and v^o. The resulting graph G is connected and fulfils the following property.

Lemma 1.2. G can be embedded in Z^2 with dcost $\leq k$, if and only if [V] has a Hamiltonian circuit.

Proof.

Suppose [V] has a Hamiltonian circuit. We can obtain an embedding f with dcost $\leq$ k as follows: Number the successive nodes on the Hamiltonian circuit $y^o, y^1, y^2, \ldots, y^{|V|-1}$, with $y^o = z$. Now let $f((x_1,x_2)) = (x_1,x_2)$ and $f(v^i) = ky^i$. It is easy to check that f is an embedding of G with dcost k.

Now suppose f is an embedding of G with dcost $\leq$ k. f restricted to W is an embedding of $[W]_k$ with dcost $\leq$ k. Lemma 1.1. shows that we can assume, without loss of generality, that $f((x_1,x_2)) = (x_1,x_2)$ for all $(x_1,x_2) \in W$. We now prove, with induction that for every v^i in the cycle $f(v^i) \in kV$. $f(v^o)$ must have distance $\leq$ k to (kz_1-1, kz_2), hence $f(v^o) \in [-k-1, mk+k] * [-k, nk+k] \Rightarrow f(v^o) \in kV$. For all $i < |V|-1$, $f(v^i) \in kV$: $[o,mk] * [o,nk] \Rightarrow f(v^{i+1}) \in [-k,mk+k] * [-k,nk+k] \Rightarrow f(v^{i+1}) \in kV$, which completes the inductive argument. Number the nodes in V y^o, $y^1, \ldots, y^{|V|-1}$, such that $f(v^i) = ky^i$. One has $d(f(v^i), f(v^{(i+1) \bmod |V|})) \leq k$, so y^i and $y^{(i+1) \bmod |V|}$ are adjacent. Hence the nodes $y^o, y^1, \ldots, y^{|V|-1}$, in this order, form a Hamiltonian circuit in [V]. □

From lemma 1.2., the NP-completeness of HAMILTONIAN CIRCUIT IN A GRID GRAPH and the fact that G can be constructed in time, polynomial in the size of V, it follows that k-EDGE LENGTH is NP-complete. □

4. Final remarks. The result of section 3 can be generalized to higher dimensions. One can obtain without much difficulty the following result, similar to theorem 1:

Theorem 2. Let $k \geq 1$, $d \geq 3$. The following problem is NP-complete:

[d-DIMENSIONAL k-EDGE LENGTH]

Instance: Graph G

Question: Can G be embedded in Z^d with dcost at most k?

Proof.

We will only give a very brief sketch of the proof. Let $V \subseteq V^2$ be an instance of HAMILTONIAN CIRCUIT IN A GRID GRAPH, and suppose

$V \subseteq [o,m]^2$. Choose $W=[-4k,mk+4k]^d \setminus \{(kx_1,kx_2,o,\dots,o) \in V^d | (x_1,x_2) \in V\}$. Similar to the proof of theorem 1, one can show that every embedding of $[W]_k$ in Z^d is isomorphic to the embedding with $f(x)=x$. Again add a cycle with $|V|$ nodes to $[W]_k$, with one node of the cycle adjacent to $(kx_1-1,kx_2,o,\dots o)$ for a certain $(x_1,x_2) \in V$. The graph G so obtained again can be embedded in Z^d with dilation cost $\leq k$ if and only if [V] contains a Hamiltonian circuit. □

Note that we can also take k=1, if the dimension is 3 or higher. For the 2-dimensional case, the complexity of 1-EDGE LENGTH is an interesting open problem. If we restrict the grid to some given size n,m the problem is NP-complete.

Definition. Let the nxm grid $GR_{nxm} = (V_{nxm}, E_{nxm})$ be the subgraph of Z^2, induced by the set of nodes $V_{nxm} = \{(x_1,x_2) | o \leq x_1 \leq n-1,\ o \leq x_2 \leq m-1\}$.

Theorem 3.1. The following problem is NP-complete:

Instance: Graph G, $n,m \in N^+$

Question: Can G be embedded in GR_{nxm} with dcost at most 1, i.e. is G isomorphic to a subgraph of GR_{nxm}?

The proof is similar to the proof of theorem 3 in [6] and uses a reduction from 3-PARTITION. It is essential in this proof that G does not need to be connected. The problem stays NP-complete if one requires that n=m, or if m is fixed to some constant ≥ 3.

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