THE COMPLEXITY OF CUTTING PAPER

(extended abstract)

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1. Introduction.

We consider the following problem:

> Given a polygonal piece of paper with a polygon P drawn on it, cut P out of the piece of paper in the cheapest possible way.

We of course have to define what we mean with cutting and what we mean with the cheapest way.

A cut is a line. It divides the piece of paper in a number of pieces, namely those pieces that lie left of the line and those pieces that lie right of the line. A cut is not allowed to run through the polygon P. After a cut is made we continue with the piece of paper containing P. See figure 1.1. for an example of a cut. A cutting sequence is a sequence of cuts such that, after the last cut we end with a piece that is the polygon P.

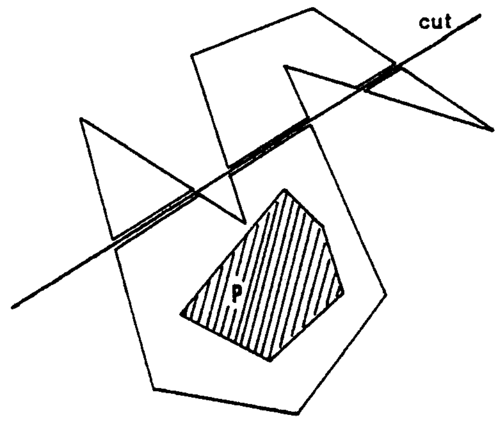

Figure 1.1.

The cost of a cut is the length of the intersection of the cutline with the piece of paper. We ask for the cutting sequence such that the total cost is minimal. This sequence is called the optimal cutting sequence. See figure 1.2. for an example of an optimal cutting sequence. The total cost of the optimal cutting sequence of a

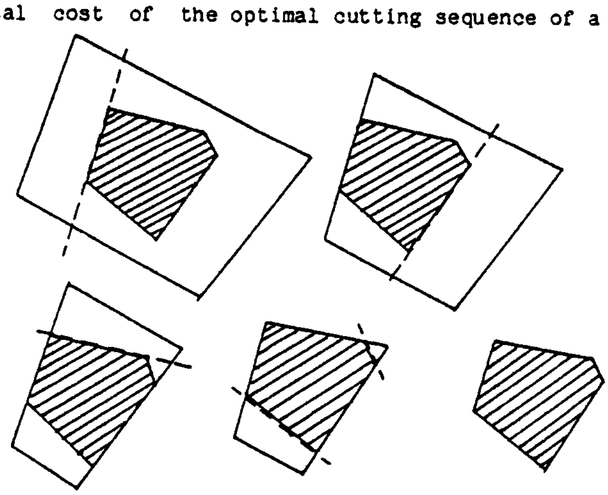

Figure 1.2.

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polygon P on a piece of paper is called the *cut cost*.

The problem is mainly of theoretical interest, although one could think of some practical applications. The problem is interesting because it is much harder than one should expect. A number of things that seem obvious turn out not to be obvious at all or even to be false. We will show some results on the existence of optimal cutting sequences and give efficient algorithms for some restricted cases. A large number of open problems do remain, the most important being whether the general problem of finding an optimal cutting sequence is solvable at all.

2. Existence of optimal cutting sequences.

It is clear that the problem is not solvable when the polygon P we want to cut out is not convex. So from now on we always assume that P is convex. We will show in this section that, assuming that P is convex and the piece of paper is convex there does exist a finite optimal cutting sequence. In the case the piece of paper is not convex the problem seems to be much harder.

> *Theorem 2.1.* When the piece of paper is convex, there are cases in which there is no optimal cutting sequence in which all cuts are along edges of P.

Proof.

See figure 2.1. for an example.

Q.E.D.

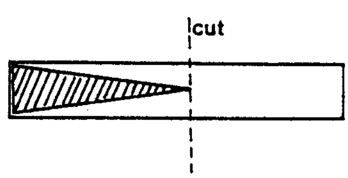

Figure 2.1.

When cuts would run along edges only, it was immediately clear that a finite optimal solution did exist. The above theorem shows that this (unfortunately) is not the case.

> *Theorem 2.2.* Suppose the piece of paper is convex and there exists a finite cutting sequence of total cost c. Then there exists a finite cutting sequence of cost $\leq c$ in which all cuts touch the polygon P.

Proof.

We will not describe the proof in detail. The idea is as follows. Consider the finite cutting sequence of cost c. Out of this sequence we will construct a finite cutting sequence of cost $\leq c$ in which all cuts touch P. This new sequence is exactly the same except that each cut not touching P is either deleted or moved (in parallel) towards P.

Let l be the first cut that does not touch P. Clearly, moving or deleting l does not change the cost of the cuts before l. Let us look how the cost of l plus the total cost of the cuts after l changes when moving l towards or away from the polygon P. It can be shown that this cost behaves as a concave function in the distance of l from P. Hence, the minimum is obtained either when l touches P or when l lies as far from P as possible, i.e., outside the piece of paper. So, either deleting the cut l or moving it such that it touches P gives rise to a cutting sequence of cost $\leq c$.

Applying this to all cuts that do not touch the polygon P yields a cutting sequence of cost $\leq c$ in which all cuts touch P.

Q.E.D.

> *Theorem 2.3.* Suppose the piece of paper is convex and there exists a finite cutting sequence of cost c. Then there exists a cutting sequence of cost $\leq c$ with O(n) cuts (all touching the polygon P), where n is the number of edges of P.

Proof.

We may assume there exists a cutting sequence of cost $\leq c$ in which every cut touches P. Among those we choose the one with the minimal number of cuts. After one cut is made we have the situation in which there is a convex arc with some paper hanging on it. (See figure 2.2. for an example.) It can easily be shown that after a small number of cuts through the endpoints of the arc (never more than 5) a cut has to run through another point of the arc. (An improvement on this number 5 is very likely.) This demonstrates that after a constant number of cuts the problem reduces to two arcs of smaller size. We can continue like this until we get to arcs of only one edge, when we obviously just cut along the edge. Hence, when f(n) denotes the minimal number of cuts necessary to cut out an arc of n edges (with cost below a given bound) then we have $f(1)=1$ and

$$f(n) \leq \max_{1\leq k\leq n-1}(f(k)+f(n-k)+5)$$

and the theorem follows.

Q.E.D.

This still does not show that a finite optimal cutting sequence does exist. We only showed that, assuming a finite optimal cutting sequence does exist, there exists an optimal cutting sequence with O(n) cuts in which each cut touches the polygon P. We will now show that a finite optimal cutting sequence indeed exists.

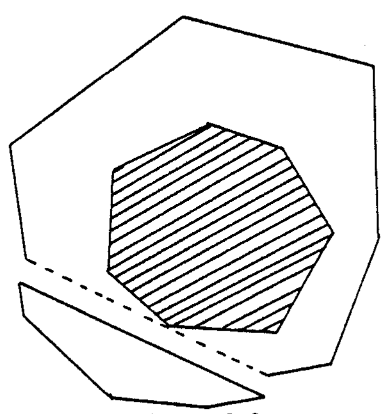

Figure 2.2.

This is not as trivial as one might expect. (See the remark after the proof of the theorem.)

Theorem 2.4. If the piece of paper is convex, then there exists an optimal cutting sequence with O(n) cuts in which each cut touches the polygon P, where n is the number of edges in P.

Proof.

Given a piece of paper and a polygon P, let c be the infinum of the costs of all cutting sequences which cut P out of the paper. (Hence, when a finite optimal solution exists, c is the cut cost.) By the above theorem c is also the infinum of all such sequences with at most c'n cuts (for some fixed constant c') and all cuts touching the polygon. Taking a cutting sequence, look at the sequence of corners and edges in which the cuts touch P. Clearly there are only a finite number of such sequences of corners and edges. Hence, there must be one such sequence of corners and edges such that for each $\varepsilon>0$ there is a cutting sequence with this sequence of corners and edges with cost $\leq c+\varepsilon$. Hence, c is also the infinum of the cutting sequences that touch the polygon P in this fixed order. We will now show that for such a fixed sequence of touching corners and edges an optimal solution does exist.

We will prove this by induction on the number of cuts that still have to be made. When only one cut is left, it is clear that there is only one optimal one namely the one along the last edge. So assume it holds when k cuts are left. Now look at the k+1'st cut (from the back). When this cut must run along an edge (remember it is fixed how the cut touches the polygon P) there is only one possibility. So we only have to consider the case the cut touches P in a corner. Now one can show that turning the cut around the corner, the optimal cost of the remaining cuts changes continuously (because the paper is convex). Moreover, the cost of this cut changes continuously as well. Hence, there is an optimal cut through this corner.

Q.E.D.

For the case the piece of paper is not convex the problem is much harder. First of all there is no analog of theorem 2.2. as the following theorem shows:

Theorem 2.5. When the piece of paper is not convex there are cases in which there is no optimal cutting sequence in which all cuts touch the polygon P.

Proof.
See figure 2.3. for an example.

Q.E.D.

And even worse, if we consider a piece of paper as a closed object (in the topological sense, as opposed to open), there are situations in which no optimal cutting sequence does exist. See figure 2.4. for an example of such a situation. It is clear that moving the cut downwards improves the cutting sequence. But, at the moment the cut touches P the cost increases considerable. Hence, an optimal cutting sequence does not exist.

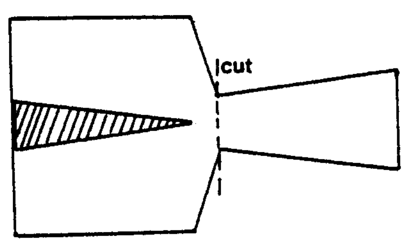

Figure 2.3.

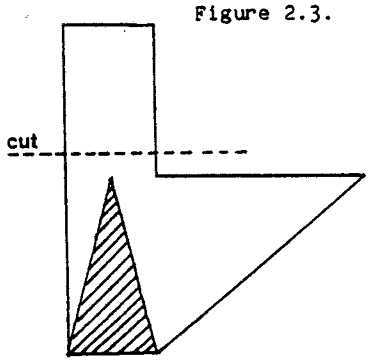

Figure 2.4.

If we consider the piece of paper to be an open object, the argument no longer holds. We are not able to prove or disprove the existence of a finite optimal cutting sequence in this case.

3. Algorithms for restricted cases.

In this section we will give two algorithms for two restricted cases of the problem. In the first case we allow cuts to run along edges of the polygon P only. Hence, we have to cut along each edge of P exactly once. We only have to determine the order in which we have to make the cuts. We assume that the piece of paper is convex. n denotes the number of edges of the polygon P, N the number of edges of the piece of paper. We number the edges of P from 1 to n around the polygon.

The method we apply is dynamic programming. This is based on the following observation: Assume that at some moment edges i and j have been cut and the edges in between i and j not. Then the order in which we have to cut the edges between i and j is completely independent from the order in which we cut the other edges. So we can precompute the order in which we have to make the cuts between i and j.

The algorithm computes optimal cutting sequences for all arcs of edges, in order of length of the arc, i.e., we start with arcs consisting of one edge, next arcs consisting of 2 edges, and so on. An optimal cutting sequence for the edges between i and j consists of a cut along some edge k ($i<k<j$) followed by an optimal cutting sequence for the edges between i and k and for the edges between k and j. Hence, when we have computed the optimal cutting sequences for smaller arcs, we can compute the optimal cutting sequence for this arc by trying all possibilities, i.e., all possible choices for k. Trying a choice consists of computing the cost of the cut along edge k and adding the already computed costs of the two remaining arcs. By doing some $O(n+N)$ preprocessing, such a try can be performed in $O(1)$ time. Hence, finding the optimal cutting sequence for this arc takes time $O(j-i)$. Clearly the total number of arcs is bounded by $O(n^2)$ and, hence, the

total amount of work is bounded by $O(n+N)$ preprocessing and $O(n^3)$ time for computing the optimal cutting sequence.

Theorem 3.1. When we are only allowed to cut along edges of the polygon P, an optimal cutting sequence can be found in $O(N+n^3)$ time.

We will now restrict the problem further to obtain a linear time solution. We assume that the lines extending the edges of the polygon P to infinity do not intersect inside the piece of paper (except at the corners of P). See figure 3.1. for an example. Again we number the edges of the polygon P in clockwise order from 1 to n. Then we divide the cost for a cut along an edge i in three parts: the cost l_i of the part of the cut left of the edge, the cost e_i of the edge itself and the cost r_i of the part of the cut right of the edge. We do not have to consider the costs e_i because we have to pay them in each cutting sequence.

Let us now look at a corner of the polygon (see figure 3.2.). There we encounter a right part r_i of an edge and the left part l_{i+1} of the next edge. Independent of the order of cutting we either have to cut r_i or l_{i+1}. Hence we can reduce the cost of both parts with $a_i=\min(r_i,l_{i+1})$ cause we always have to pay this amount. So let $r'_i=r_i-a_i$ and $l'_{i+1}=l_{i+1}-a_i$. Now either $r'_i=0$ or $l'_{i+1}=0$. We distinguish the following cases:

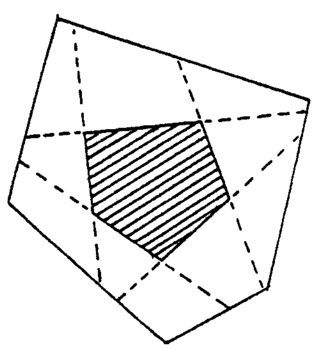

Figure 3.1.

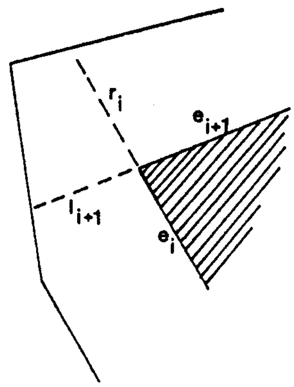

Figure 3.2.

(1) Either all l'_i are zero or all r'_i are zero. Suppose all l'_i are zero. Then we start our cutting sequence with the cut along the edge k with minimal r'_k. As a result of this cut r'_{k-1} "becomes" zero. Hence, next we can cut along the edge k-1 with no real cost (except for the cost we always have to pay). Next we can cut along the edge k-2, etc. It is easy to see that this gives an optimal cutting sequence with total cost

$$\sum_{i=1}^{n} (e_i+a_i) + r'_k.$$

(2) Otherwise there must be an edge k with both l'_k and r'_k zero. Of course, we can start the cutting sequence by cutting along this edge k. In fact, we can continue cutting along edges with both left and right part having zero (extra) cost. This leads to an optimal cutting sequence with total cost

$$\sum_{i=1}^{n} (e_i+a_i).$$

The values for l_i, r_i, r'_i and l'_i can be determined during one simultaneous walk around the piece of paper and the polygon P. The cutting sequence can then be determined in another walk around the polygon P. This lead to the following result:

Theorem 3.2. When we are only allowed to cut along edges of the polygon P and these cuts do not intersect inside the piece of paper (except at the corners of P), an optimal cutting sequence can be determined in O(N+n) time.

4. Concluding remarks and open problems.

In this paper we introduced the problem of cutting a polygon out of a piece of paper, which seems to be interesting to study. We solved some part of it and pointed out some of its difficulties. A large number of open problems do remain.

The main open problem, of course, is to find an algorithm that finds an optimal cutting sequence for each piece of paper and convex polygon P. Some smaller steps in this direction would be: a proof of existence of a finite optimal cutting sequence in the case the piece of paper is not convex (regarding a piece of paper to be an open object), and an algorithm for solving the problem in the case the piece of paper is convex. Both might not exist. Because the problem is not discrete it might be possible that an optimal cutting sequence cannot be computed at all.

Also algorithms that approximate the optimal cutting sequence are interesting. The algorithms described in section 3 can be very bad in some cases, i.e., cutting only along edges is not a good approximation for the general problem.

As noted, the polygon P needs to be convex. On the other hand, it is possible to cut a number of convex polygons out of one piece of paper. An extension to this more general situation might be interesting (after the problems mentioned above have been solved).

Finally there exists the problem of cutting with half-rays. This corresponds more with our usual way of cutting: a cut is a half-ray running from infinity to some point in the paper. In this case not only convex polygons can be cut out but also some other classes. These can easily be identified, but the problem of optimal cutting sequences with half-rays is completely open.

5. References.

No references on this topic seem to exist and no useful results could be found.