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RUU-CS-83-1

Januari 1983

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Abstract. The existence of double diagonal and cross Latin squares for all order (except 2 and 3 in the first case) was shown by Hilton in 1971. Whereas a greatly simplified construction for double diagonal Latin squares was presented immediately afterwards by Gergely in 1972, it has apparently remained open to give simple methods for obtaining larger double diagonal or cross Latin squares from smaller ones. We show that for both types of Latin squares a Kronecker product construction can be devised, using an arbitrary (double diagonal or cross) Latin square of order p to obtain a (double diagonal or cross) Latin square of order pq for any $q \geq 1$. The construction is shown to require only linear time in the size of the constructed object in both cases. We also give a simple direct construction of cross Latin squares of all orders.

1. Introduction.

A Latin square of order n is any $n \times n$ array a over the integers $1, \cdots, n$ such that each integer i $(1 \leq i \leq n)$ occurs exactly once in each row and exactly once in each column of a. Latin squares are of traditional interest to mathematics and statistics, and new uses for them have been found for solving various problems in computing. In this paper we study two special types of Latin squares that received attention in the early 70's*.

A Latin square a is called double diagonal (dd, for short) if each in-

*According to [2] their study was suggested by J. Dénes in 1970.

teger i $(1 \leq i \leq n)$ also occurs exactly once on the main diagonal and exactly once on the off diagonal. A cross Latin square is only slightly harder to define. For even order it is a Latin square in which all entries on the main diagonal are the same integer and all entries on the off diagonal are the same integer. For odd order the definition reads analogous, except that at the intersection of the two diagonals (see figure 1) two entries of the off diagonal are "rotated" to let the main diagonal pass unbroken. The following results are due to Hilton [2] (the first result was also obtained by Lindner [3]):

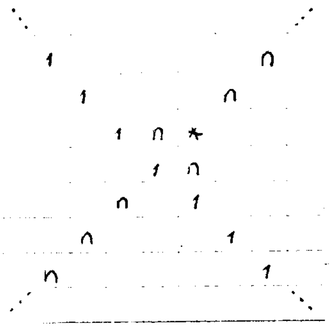


figure 1

Theorem A. There exist double diagonal Latin squares for all orders n with $n \geq 1$, $n \neq 2, 3$.

Theorem B. There exist cross Latin squares for all orders n with $n \geq 1$.

Hilton used sophisticated techniques to prove his results. In 1972 Gergely [1] gave a much simpler proof of theorem A using the elegant method of "projecting transversals". His proof is easily seen to imply a linear time algorithm for constructing a double diagonal Latin square of given order. In an appendix we include a simplified proof of theorem B in the same spirit, and prove that cross Latin squares can be constructed in linear time (i.e. linear in the size of the squares) as well.

The major part of this paper (sections 2 and 3) is devoted to a proof that the familiar direct ("Kronecker") product construction for Latin squares (trivial for ordinary squares) can be modified so as to hold in a rather direct way for both double diagonal and cross Latin squares. It is clear that this will lead to alternative proofs for theorems A and B, together with the conclusion that these Latin squares are not as elusive to traditional construction methods as was hitherto believed.

Theorem C. Given a cross (double diagonal) Latin square of order p, it can be composed with a cross (suitable) Latin square of order q to obtain a cross (double diagonal) Latin square of order pq, for any integer $q \geq 1$. (For double diagonal Latin squares it is assumed that $q=1$ or $q \geq 4$ when $p=1$.

The "closure" of double diagonal Latin squares under direct (Kronecker-) product is straightforward, but only covers the case when $q=1$ or $q \geq 4$ (cf. theorem A).

2. Composition of cross Latin squares.

Let a be a cross Latin square of order p over the (barred) symbol set $\bar{1}, \cdots, \bar{p}$ and let b be a cross Latin square of order q over the symbol set $1, \cdots, q$. Let $p, q > 1$ (or else theorem C follows trivially) and assume that the main and off diagonals of a and b consist of $\bar{1}$'s and $\bar{p}$'s and 1's and q's respectively.

If p is even (no intersecting diagonals), then replace every $\bar{i}$-cell of a $(1 \leq i < p)$ by the instance of b with $(i-1) \cdot q$ added to each entry. Replace every $\bar{p}$-cell by a similar instance of b, but reflected so as to have the unbroken (main) diagonal from the lower left to the upper right corner. (Note that this is essentially [2], lemma 1.) The result is seen to be a cross Latin square of order pq. If p is odd but q even, then one can do a similar composition with the roles of a and b interchanged.

The remaining and slightly harder case is when both p and q are odd integers > 1. Let the diagonals of a cross as in figure 2, with $\bar{r}$ as implied $(1 < r < p)$. Carry out the same composition as suggested above to obtain the result indicated in figure 3. It nearly is a cross Latin square, but for the filling of the off diagonals in the centre $\bar{1}$- and $\bar{r}$-cell. To patch it we should like to make the substitutions $q \rightarrow pq$ in the central $\bar{1}$-cell, $pq \rightarrow q$ in the two bordering $\bar{p}$-cells and $rq \rightarrow pq$ in the $\bar{r}$-cell.

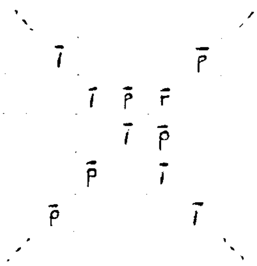


figure 2

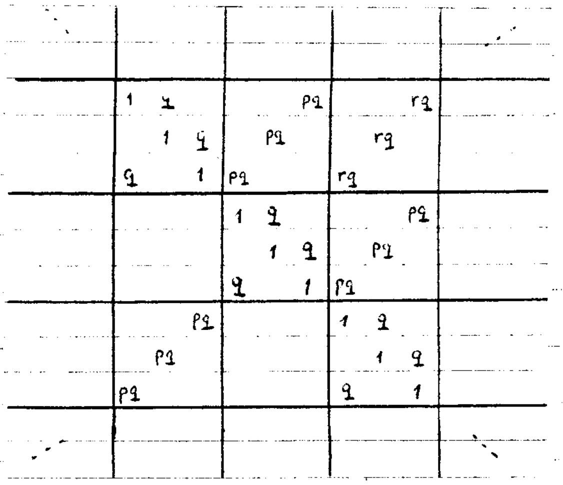


figure 3

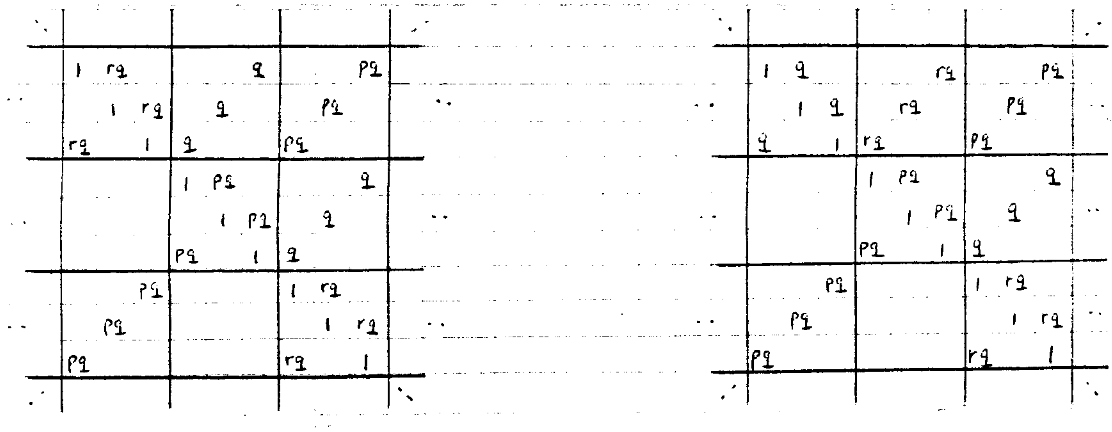
This has to be followed by $q \to rq$ substitutions in the $\bar{1}$-cells in the row and column of the $\bar{r}$-cell, hence by $rq \to q$ substitutions in the $\bar{r}$-cell in their column and row. And this process must be continued. Consider only the $\bar{1}$- and $\bar{r}$-cells of a and assume that with each $\bar{1}$ ($\bar{r}$-) cell the unique links hor and vert are given to the $\bar{r}$ ($\bar{1}$-) cell in the same row and column, respectively. The following is obvious:

Lemma 2.1 Start at an arbitrary $\bar{1}$ ($\bar{r}$-) cell and visit cells by following hor, vert, hor, ... links (alternatingly). The resulting path ends with a vert link into the starting cell and hence is a cycle of even length in which $\bar{1}$- and $\bar{r}$-cells alternate.

Consider the cycle thus obtained when we start with the off-center $\bar{r}$-cell. There are two cases to consider.

Case I: the central $\bar{1}$-cell is not included in the cycle.

Substitute $rq \to q$ in every $\bar{r}$-cell and $q \to rq$ in every $\bar{1}$-cell along the cycle. This leaves a Latin square but gives both the central $\bar{1}$-cell and the $\bar{r}$-cell off-diagonal elements q. Now make the "cyclic shift" $pq \to q$ and $q \to pq$ in the four cells in the upper right corner of the central 3×3 block. The result is a correct cross Latin square of order pq, as can be seen from figure 4a. (This was the situation suggested as an introduc-



(a) (b)

figure 4

tion. In the next case a different renumbering procedure must be followed, as we shall see!)

Case II: the central $\bar{1}$-cell is included in the cycle.

Do not substitute during the "first" part of the cycle, until the central $\bar{1}$-cell is hit. In the subsequent $\bar{r}$-cells and $\bar{1}$-cells on the cycle (but not including the final $\bar{r}$-cell, i.e., the one we started from) substitute $rq \to q$ and $q \to rq$ respectively. Finally substitute $q \to pq$ and $rq \to pq$ in the central $\bar{1}$-cell and $\bar{r}$-cell, and $pq \to rq$ in the $\bar{p}$-cell just above the central $\bar{1}$-cell and $pq \to q$ in the one right of it. The result is a correct cross Latin square of order pq, as can be seen from figure 4b. (Observe that this does require argument.)

Given the hor and vert links in a, the composition is easily carried out in linear time. The hor and vert links in the composed square can likewise be computed within this bound (by the naive algorithm). This proves theorem C for cross Latin squares.

3. Composition of double diagonal (dd) Latin squares

Let a be a dd Latin square of order p over the (barred) symbol set $\bar{1}, \cdots, \bar{p}$ and let $q \geq 1$. Without loss of generality let $p \geq 4$ (cf. theorem

A). If $q \geq 4$ then theorem A implies that there exists a dd Latin squar b of order q over the symbol set $1, \cdots, q$. Replace every $\bar{i}$-cell of a $(1 \leq i \leq p)$ by the instance of b with $(i-1)q$ added to each entry. The result is easily seen to be a dd Latin square of order pq. (Note that this is essentially the observation in [2], p. 683.) Thus theorem C remains to be proved for $q = 2$ and $q = 3$. We will show that in both cases a suitable replacement of every $\bar{i}$-cell of a $(1 \leq i \leq p)$ can be devised.

<u>Theorem</u> 3.1. Given a double diagonal Latin square a of order p, it can be "composed" into a double diagonal Latin square of order $2p$.

<u>Proof</u>.

Begin by replacing every $\bar{i}$-cell of a $(1 \leq i \leq p)$ as indicated by the following substitution

$$1 \longrightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \qquad 2 \longrightarrow \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 5 & 6 \\ \hline\end{array} \quad \cdots \quad i \longrightarrow \begin{array}{|c|c|}\hline 2i-1 & 2i \\ \hline 2i+1 & 2i+2 \\ \hline\end{array} \quad \cdots \quad p \longrightarrow \begin{array}{|c|c|}\hline 2p-1 & 2p \\ \hline 1 & 2 \\ \hline\end{array}$$

Note that this guarantees that every symbol from $1, \cdots, 2p$ occurs exactly once in every row, on the main diagonal and on the off diagonal (the illustration below shows example diagonals in non-permuted order)

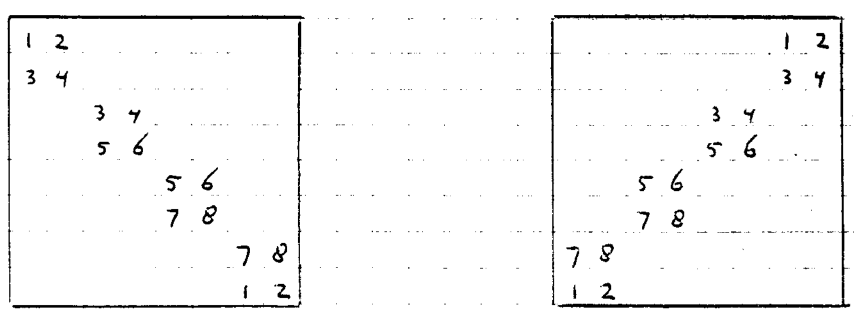


figure 5

However, the columns are not right. To correct this we shall make changes in the cells, while preserving the transversal property for all rows and for the main and off diagonals. The changes consist of "flipping" the upper two entries (the "upper track") and/or the lower two entries (the "lower track") in the appropriate cells. Call an unflipped pair of entries positive a flipped pair negative. Being positive or negative implies a notion of "orientation" in the tracks of the cells:

$\boxed{\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}}$ $\begin{array}{l}\text{(positive)}\\ \text{(positive)}\end{array}$ $\boxed{\begin{array}{cc} 2 & 1 \\ 3 & 4 \end{array}}$ $\begin{array}{l}\text{(negative)}\\ \text{(positive)}\end{array}$ $\boxed{\begin{array}{cc} 1 & 2 \\ 4 & 3 \end{array}}$ $\begin{array}{l}\text{(positive)}\\ \text{(negative)}\end{array}$ $\boxed{\begin{array}{cc} 2 & 1 \\ 4 & 3 \end{array}}$ $\begin{array}{l}\text{(negative)}\\ \text{(negative)}\end{array}$

Observation 1. Flipping the entries in the upper track or in the lower track of a cell does not affect the "Latin square property" for the rows.

An $\bar{r}$-cell and an $\bar{s}$-cell of a are said to be in column-conflict if the cells are in the same column and have symbols in a same position. There can only be two sorts of column-conflict for an $\bar{r}$-cell: (i) its upper track is equal to and has the same orientation as the lower track of an $\bar{s}$-cell in the same column, and (ii) its lower track is equal to and has the same orientation as the upper track of an $\bar{s}$-cell in the same column.

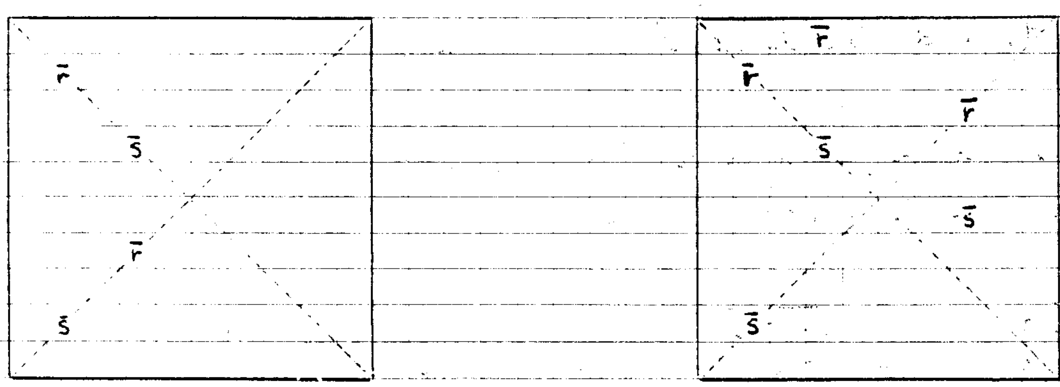
Observation 2. The upper (lower) track of an $\bar{r}$-cell can create a column-conflict with exactly one lower (upper) track of a cell in the same column.

(To be precise, the first type of column-conflict occurs with the $\bar{s}$-cell with $s = r-1$ ($s = p$ if $r = 1$) and the second type occurs with the $\bar{s}$-cell with $s = r+1$ ($s = 1$ if $r = p$)). A column-conflict can be resolved by flipping one of the tracks involved. Clearly one should not flip the track of the cell that lies on the main or off diagonal, if this happens to be the case, because it would destroy the transversal property on the diagonal without there being a reason for it. A problem arises if both cells involved in a column-conflict lie on a diagonal.

Flipping the upper (lower) track of a cell on a diagonal will give rise to a "diagonal-conflict" with the lower (upper) track of exactly one other cell on the same diagonal. (Compare this to observation 2, but note that this time the conflicting tracks have opposite orientation.)

<u>Observation</u> 3. If an $\bar{F}$-cell and an $\bar{S}$-cell are in column-conflict and the $\bar{F}$-cell lies on a diagonal, then flipping the conflicting track in the $\bar{F}$-cell creates a diagonal-conflict precisely with the (unique) $\bar{S}$-cell on the same diagonal. (Recall that n was odd.)

Now consider the case that an $\bar{F}$-cell and an $\bar{S}$-cell are in column-conflict, and both lie on a diagonal (which means that we must flip in a diagonal cell and create a diagonal-conflict). There are essentially



(a) (b)

figure 5

two different cases to distinguish.

<u>Case</u> I. The $\bar{S}$-cell and $\bar{F}$-cell involved in the diagonal-conflicts according to observation 3 are in the same column (see figure 5a).

The conflicts can all be resolved by flipping the appropriate track of both the $\bar{F}$-cell and the $\bar{S}$-cell on one of the diagonals. Note that the diagonal remains a transversal in doing so.

<u>Case</u> II. The $\bar{s}$ - cell and $\bar{r}$ - cell involved are not in the same column (see figure 5b).

The column- and diagonal - conflicts can now be resolved by flipping the appropriate track of the $\bar{r}$ - cell and the $\bar{s}$ - cell on one of the diagonals (i.e., starting with the $\bar{r}$ - cell that was in column-conflict), with the suitable flip in the off-diagonal $\bar{r}$ - cell in the same column as the latter $\bar{s}$ - cell. The resulting changes on the one diagonal leave it a correct transversal.

Thus we have shown that all column-conflicts can be resolved. The resulting Latin square is correct and dd, and of the desired order.

□

<u>Theorem</u> 3.2 Given a double diagonal Latin square a of order p, it can be "composed" into a double diagonal Latin square of order 3p.

<u>Proof</u>

The argument is very similar to that used in theorem 3.1. Begin by replacing every $\bar{i}$ - cell of a $(1 \le i \le p)$ as follows.

$$1 \longrightarrow \boxed{\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}} \qquad 2 \longrightarrow \boxed{\begin{matrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{matrix}} \qquad \cdots \qquad p-1 \longrightarrow \boxed{\begin{matrix} 3p-6 & 3p-5 & 3p-4 \\ 3p-2 & 3p-1 & 3p \\ 1 & 2 & 3 \end{matrix}} \qquad p \longrightarrow \boxed{\begin{matrix} 3p-2 & 3p-1 & 3p \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{matrix}}$$

Note that this guarantees again that all rows are right, and it can easily be verified that the main and off diagonal are right as well. A problem arises because of column-conflicts.

Observe that a track of a cell can be in conflict with precisely two other tracks (of the same "contents", and in separate cells) in the same column. A column-conflict can be resolved by shifting one of the tracks over one position (cyclically) and a second over two. If one of the three tracks occurs in a cell on the diagonal, then this track should obviously be the one that remains unaffected (and thus leaves the diagonal a correct transversal).

If two of the three tracks involved in a column-conflict lie on a diagonal (clearly this is the worst case that can occur), then it cannot be avoided that a track of one cell on the diagonal is shifted and a diagonal-conflict is created. If the column-conflict was created by the upper/lower/middle track of an $\bar{F}$-cell, an $\bar{3}$-cell and a $\bar{A}$-cell with (say) the $\bar{F}$-cell on the main diagonal and the A-cell on the off diagonal, then the diagonal-conflict involves again the $\bar{F}$-cell and the $\bar{3}$- and $\bar{A}$-cells on the particular diagonal. Considering the possible cases as in the proof of theorem 3.1 shows that the column-conflict and the subsequent diagonal-conflict can always be resolved by appropriate cyclic shifts of the tracks.

Removing all column-conflicts leads to a correct dd Latin square of order $3p$.

$\square$

The construction used in theorems 3.1 and 3.2 clearly generalizes and gives a simple and uniform method of building a double diagonal Latin square of order $q \cdot p$ from one of order p ($p \geq 4$) for every integer $q \geq 1$. The construction is easily seen to require only linear time in the size of the resulting square.

Note that we only need theorems 3.1 and 3.2 for a simple proof of the existence of double diagonal Latin squares of all orders $n \neq 2, 3$. Hilton [2] (p. 683) gives a simple "formula" for double diagonal Latin squares of orders $n \equiv 1 \pmod 6$ and $n \equiv 5 \pmod 6$. By theorems 3.1 and 3.2 we can fill in all remaining cases.

4. <u>References</u>.

[1] Gergely, E., A simple method for constructing doubly diagonalised

Latin squares, J. Combin. Theory Ser. A 16 (1974) 266-272.
[2] Hilton, A. J. W., On double diagonal and cross Latin squares, J. London Math. Soc. II 6 (1973) 679-689.
[3] Lindner, C. C., On constructing doubly diagonalized Latin squares, Periodica Math. Hungar. 5 (1974) 249-253.

Appendix. (Simple proof of theorem B.)

The existence of cross Latin squares of all orders (theorem B) follows by only slightly extending Gergely's construction [1]. A transversal of a Latin square is a selection of n entries from distinct rows and columns, which includes every integer from the basic symbol set exactly once. Gergely [1] (p. 269) proves the following result:

Lemma. For every $n \geq 6$, n even, there exists a double diagonal Latin square of order n having a transversal which does not intersect the main and off diagonals.

Theorem B can now be proved as follows. For $n \leq 7$ the theorem follows by constructing suitable examples (see [2] p. 680). For $n \geq 8$, take a La-

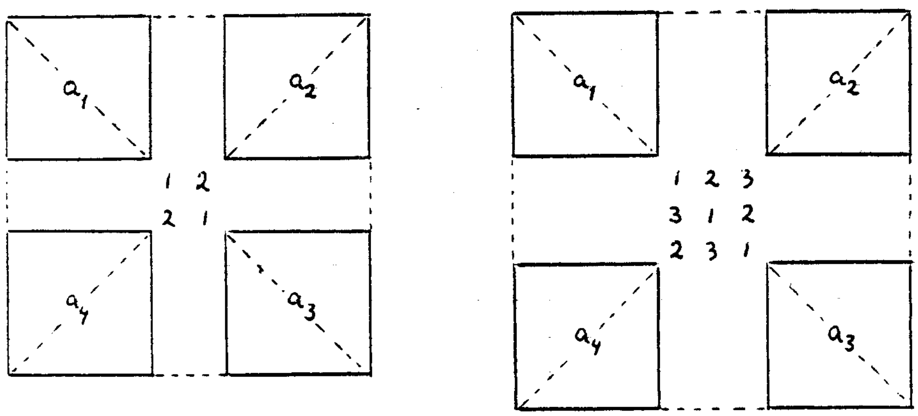


figure 6

tin square a of order n-2 (n even) or of order n-3 (n odd) as implied by the lemma, split it into four quadrants, insert two (n even) or three (n odd) rows and columns and fill in the middle 2×2 (n even) or 3×3 (n odd) square as shown in figure 6. Assuming a was over the symbol set $3 \ldots n$ (n even) or $4 \ldots n$ (n odd) respectively we now proceed as follows. Project the main diagonal, the off diagonal and if n is odd the third transversal onto the first, second and if n is odd the third inserted row/column couple respectively. Replace all entries of the main diagonal by 1, all entries of the off diagonal by 2 and if n is odd all entries of the third transversal by 3. The result is easily seen to be a cross Latin square of order n. As Gergely's construction leads to the required double diagonal Latin squar of order $n-2$ or $n-3$ in linear time (together with a marking of the third, non-intersecting transversal), cross Latin squares of any specified order can be constructed in linear time.