THE NP-COMPLETENESS OF FINDING MINIMUM AREA LAYOUTS FOR ARBITRARY VLSI-CIRCUITS

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Abstract. Using Thompson's model for VLSI we prove the following problem to be NP-complete: given a (connected) graph G and an integer A, can G be embedded in a rectangle of area ≤ A.

Keywords and phrases: VLSI, circuit, graph, layout, area, rectangle, NP-completeness.

1. Introduction.

In VLSI-theory circuits are evaluated in terms of their computing time and period, and the area required for an embedding on a chip. For our purposes a circuit will simply be a (finite) graph, with nodes corresponding to gates and edges corresponding to wires that connect gates. (We will continue to refer to edges as wires.) Following Thompson [4] we define a chip to be a simple and connected domain of some shape on the 2-dimensional grid. (We only consider iso-oriented, rectangular chips but sometimes weaker assumptions are made.) Each cell of the grid may contain a node or one or two wire segments. Only one wire may cross a given cell boundary and (hence) two wires can at best cross in a single cell. Also, the number of wires incident to a single node is necessarily bounded by 4. It will be clear what is meant by an embedding of a graph on a chip, or "in a rectangle".

Techniques for embedding graphs in VLSI are largely guided by heuristic. From a theoretical point of view, problems of (approximate) optimality can be studied adequately within the framework of Thompson's model. (I should be noted that the model assumes a unit size for every node and a unit "width" for every wire regardless its length.) Leiserson [3] provides a good survey of some of the results and techniques for obtaining "area efficient" layouts. In this note we shall attempt to assess the principal difficulty of finding absolutely minimum area layouts.

In an earlier study we viewed the embedding problem as consisting of two parts: (a) placement of the nodes and (b) routing of the wires. It was shown that the problem of deciding whether there exists a routing at all for a given placement, in general, is NP-complete (Kramer and van Leeuwen [2]). In view of the general intuition for NP-complete problems this means that optimal wire routing must be computationally hard no matter what criterion of optimality is used. We shall now prove that the general embedding problem, with all freedom of placement, is NP-complete as well*. More precisely, we consider the following problems and prove each one to be NP-complete:

A: given a graph G and a rectangle R, can G be embedded in R,

B: given a graph G and an integer A, can G be embedded in a rectangle of area $\leq A$,

C: given a connected graph G and an integer A, can G be embedded in a rectangle of area $\leq A$.

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*Independently, and apparently before us, Dolev and Trickey ["On linear area embedding of planar graphs", Techn. Rep STAN-CS-81-876, Stanford University, 1981] have shown that finding a minimum area embedding is NP-complete for forests. However, they used a model for VLSI that permits no wire crossings and leave the NP completeness question open for connected graphs. See D.S. Johnson ["The NP-completeness column: an ongoing guide", J. Algor 3 (1982) 89-99, p.95].

We shall discuss in section 2 that problems A and B are essentially (polynomially) equivalent.

We assume familiarity with the theory of NP-completeness (Garey and Johnson [1]). In particular we shall make use of the following problem, which is known to be NP-complete in the strong sense (i.e, "unary NP-complete", see Garey and Johnson [1] sect. 4.2.2):

> 3-PARTITION: given positive integers m, B and a set of $3m$ integer $a_1, \dots, a_{3m}$ such that $\frac{1}{4}B < a_i < \frac{1}{2}B$ and $\sum_i a_i = mB$, does there exist a partition of this set into m disjoint subsets such that each subset sums to B.

(Note that by the size constraint on the a_i each subset of a "3-partition" is forced to contain exactly 3 elements.)

2. Some preliminary facts, and the NP-completeness of problems A and B.

Problem B clearly reduces to $\sqrt{A}$ instances of problem A but, because area is given in binary, this is not a polynomial reduction. For practical purposes, though, it is because of the following result (from which it follows that A can be assumed to be polynomially bounded in the size of the graph):

<u>Theorem</u> 2.1 (Valiant [5]) Every graph of n nodes and degree ≤ 4 can be laid out in $O(n^2)$ area.

For other purposes we need a very special kind of graph called a "frame graph" (see figure 1).

<u>Definition</u> For integers α, γ and b, ℓ with $\alpha > b$ and $\gamma > \ell$ the frame graph $F(\alpha, \gamma, b, \ell)$ is defined to be the α-by-γ grid, with a b-by-ℓ "window" cut out in the middle.

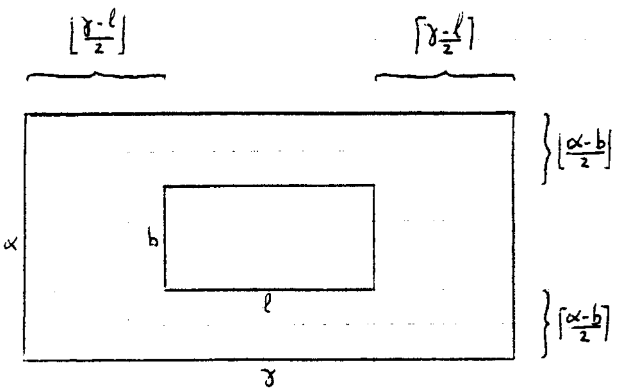


figure 1 (the shaded part is grid connected)

$F(\alpha, \gamma, b, l)$ has $\alpha\gamma - bl$ nodes, and its "natural" embedding (see figure 1) occupies an α-by-γ rectangle.

<u>Theorem</u> 2.2 Let $\alpha \geq b + 36bl$ and $\gamma \geq l + 36bl$. Then the only embedding of $F(\alpha, \gamma, b, l)$ possible in $\alpha\gamma$ area is the natural embedding, in a α-by-γ rectangle.

<u>Proof</u>.

By just counting how many cells are required for nodes alone, it follows that an embedding in a rectangle of $\alpha\gamma$ area can "waste" no more than bl cells on open space and wiring. Consider an embedding of $F(\alpha, \gamma, b, l)$ in an α_1-by-γ_1 rectangle, with $\alpha_1\gamma_1 = \alpha\gamma$. For the argument below we use the following terminology. Nodes in the left α-by-$\lfloor\frac{\gamma-l}{2}\rfloor$ part of $F(\alpha, \gamma, b, l)$ will be called "red", those in the upper $\lfloor\frac{\alpha-b}{2}\rfloor$ by-γ part "green". (Never mind that in this way some nodes are both red and green.) We will show that the red and green nodes, and likewis the remainder of $F(\alpha, \gamma, b, l)$ must occur in natural position, or else mor than bl cells would be needed for additional wiring. We consider essentially two different cases.

<u>Case</u> I : $\alpha_1 \leq \alpha$.

Partition the rectangle in 3-by-3 boxes (we ignore the rounding effect at the borderline). A box is called full when it has red nodes in all its 9 cells. (The nodes in a full box necessarily are in natural position, by just observing the degree of the nodes.) Note that there must be at least

$\frac{1}{9} \cdot \alpha \cdot \lfloor \frac{\gamma - l}{2} \rfloor \geq 2\alpha bl$ boxes that contain at least one red node, hence at least $2\alpha bl / \alpha_1 \geq 2bl$ strips of α_1-by-3 that contain such boxes in the partitioned rectangle. Now suppose that any one of these $2bl$ strips contains (i) a non-full box with a red node, or (ii) only full boxes of red nodes but two of them separated by a "blank" box or some other sort of waste on extra wiring. Then $\geq 2bl$ cells would be wasted on necessary excess wiring, contradicting that at most bl cells were available for this purpose. We conclude that there must be an α_1-by-3 strip that only contains full boxes of red nodes, with no waste due to wiring the red nodes to other (red) nodes. The full boxes must be adjacent and are (necessarily) a natural strip from the grid. There are two possibilities: (a) it is a complete "vertical" strip from the red part of the graph (and $\alpha_1 = \alpha$) or (b) it is a complete "horizontal" strip from the red part. In either case the strip would be at least $18bl$ red nodes long. Consider the immediately adjacent single column or row of (red) nodes connected to the strip. If it is not connected to the strip in natural order but (say) one node is further out, then a simple packing consideration shows that none of the nodes from the column or row can be directly adjacent to the strip either (the problem being the routing of the wires to and within the added column or row). But this gives an excess in cells for wiring $\geq 18bl$, which is more than is permitted. Continuing this one can argue that every part of $F(\alpha, \gamma, b, l)$ must be in its natural place and order.

<u>Case II</u>: $\alpha_1 > \alpha$

This case is the same as requiring that $\gamma_1 < \gamma$. It can be handled exactly like case I, considering the green nodes in stead of the red ones to find a solid strip and force the natural embedding of the graph from there.

The rounding effect in the argument is easily absorbed in the slack of the bounds on α and γ, assuming that b and l are not too small. $\square$

<u>Proposition</u> 2.3 Problems A and B are "polynomial time Turing equivalent", i.e., equivalent in the sense of Cook.

<u>Proof</u>.

Considering an instance of problem A, requesting the embedding of an n-node graph G in some b-by-ℓ rectangle (b and ℓ given in binary) By theorem 2.1 we may assume that b and ℓ are at most $O(n^2)$. Now let $\alpha = b + 36\,b\ell$ and $\gamma = \ell + 36\,b\ell$, and consider the instance of problem B requesting the embedding of $G \cup F(\alpha, \gamma, b, \ell)$ in $\alpha\gamma$ area. Note that the instance of problem B has size bounded by $O(n^8)$, which is polynomial in the size of problem A. By theorem 2.2 the only embedding of $F(\alpha, \gamma, b, \ell)$ in $\alpha\gamma$ area is the natural embedding, and thus the instance of problem B is solvably if and only if G fits in the middle b-by-ℓ window.

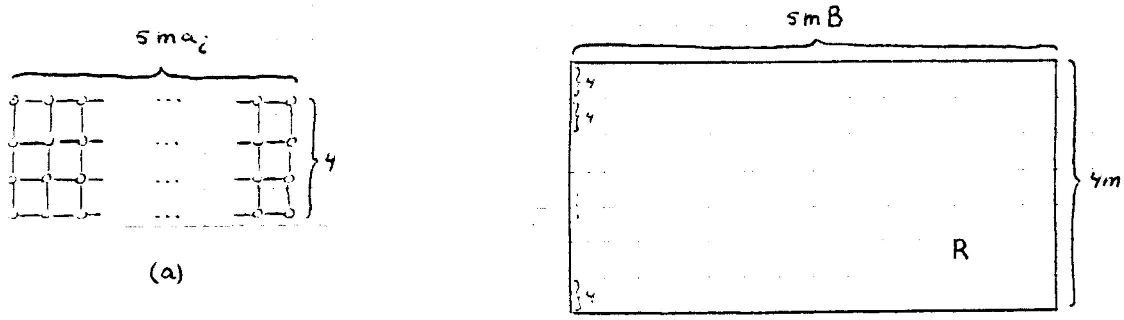
Next consider an instance of problem B. By theorem 2.1 we may assume that the area of the embedding is bounded by $O(n^2)$. Thus problem B can be solved by solving $O(n)$ instances of problem A. □

(From the results below it follows that problems A and B are, in fact, polynomially equivalent in the sense of Karp. See Garey and Johnson [1], sect 5.2, for a discussion of these concepts.)

<u>Theorem</u> 2.4 Problem A is NP-complete (in the strong sense).

<u>Proof</u>.

Problem A obviously is in NP. We shall design a pseudopolynomial transformation from 3-PARTITION to problem A. Let an instance of 3-PARTITION be given. For each integer a_i design a component $C(a_i)$ that is a 4-by-$5ma_i$ grid, as shown in figure 2(a). (It should be obvious that the natural embedding of a component is the only one that is minimum, i.e., that uses no extra cells for wiring.) Let G be the collection of components $C(a_i)$, $1 \le i \le 3m$. Consider the instance of problem A that requests the embedding of G in the rectangle R of size $4m$-by-$5mB$, as



(a) (b)

figure 2

shown in figure 2(b). If the instance of 3-PARTITION has a solution, then so does the instance of problem A. The (3) components corresponding to a subset that sums to B can be packed into a 4-by-$5mB$ strip, and m such strips just fit in R (horizontally). However, the converse is true also For suppose we have a solution to the instance of problem A. G has as many nodes as there are cells in R, and hence every component is forced to be embedded in minimum area. This is the "brick" form shown in figure 2(a) Because $5ma_i > 4m$, no brick can be placed vertically. It follows that R decomposes into m horizontal strips of width 4 that are completely packed. The strips translate back into subsets that are a solution to the instance of 3-PARTITION.

Observe that the reduction can be computed in time polynomial in m and B. The remaining conditions for pseudopolynomial reductions (Garey and Johnson [1], p. 101) are easily verified. By Garey and Johnson [1], lemma 4.1, problem A is NP-complete (in the strong sense). □

<u>Theorem</u> 2.5 Problem B is NP-complete (in the strong sense).

<u>Proof</u>.

Problem B again, obviously, is in NP. We shall design a pseudopolynomial reduction from 3-PARTITION that is very similar as in the proof of theorem 2.4. Let an instance of 3-PARTITION be given. For each a_i we design a component $C(a_i)$ as before. Let G be the graph consisting of all $C(a_i)$ $(1 \le i \le 3m)$ together with one $F(4m+4, 5mB+4, 4m, 5mB)$, and let $A = (4m+4)(5mB+4)$. See figure 3 for an illustration of the last component of G. Consider now the instance of problem B that requests an embedding of G in "A" area.

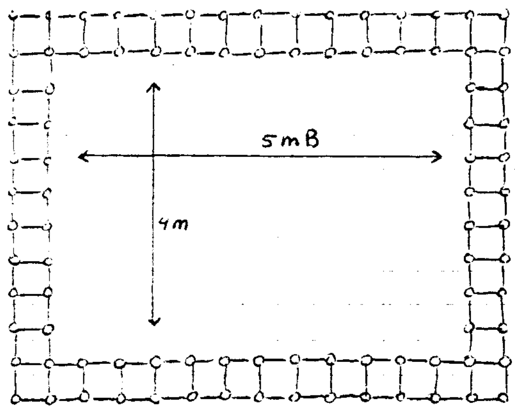


figure 3

The remainder of the proof is very similar as for theorem 2.4. Note that G has as many nodes as there cells in A area, thus forcing each component to be minimally embedded. But this forces the frame graph into its natural form, and the remaining components of G to fill the interior part in completely the same way as before. □

3. <u>The NP-completeness of problem C</u>.

The proofs of theorems 2.4 and 2.5, using basically the same reduction from 3-PARTITION, heavily depend on being able to construct a graph G with many components ("a very disconnected graph"). In this section we will modify the proof of theorem 2.5 so as to end up with a connected graph. For an introduction we first discuss a way to get a "smaller" proof of theorem 2.5. Recall that in the proof we used a wide frame with a single window (see figure 3).

<u>Second proof of theorem 2.5</u>

Recall that the given proof of theorem 2.5 ended up with a graph G that, as only possibility, had an embedding that divided naturally into m disjoint strips. Thus we could have used a "frame" with m windows (of width 4, and separated by chains of width 2), and still get the same effect and a valid reduction. The next step is to place the windows horizontally rather than vertically in sequence, and to shrink them in size by cutting the factor m in width. Thus we use for $C(a_i)$ the 4-by-$5a_i$

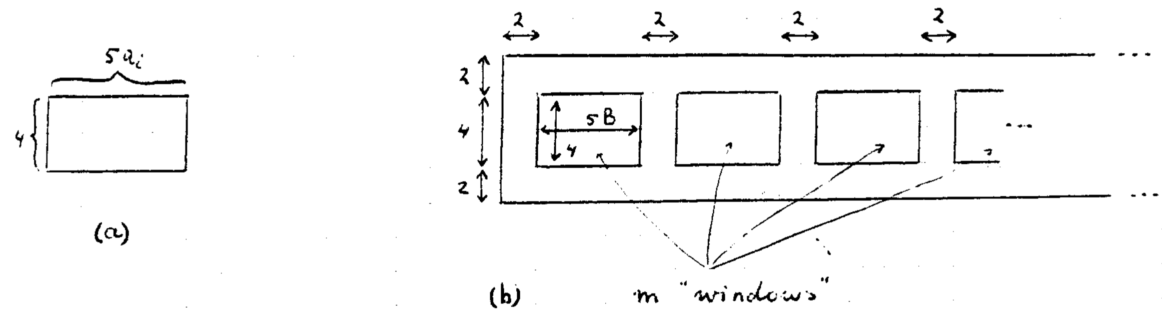


figure 4

"brick" from figure 4 (a) and as "frame" the F(8, 5mB + 2m + 2, 4, 5mB + 2m - 2) with m interior, separated windows as in figure 4 (b). It is not hard to see that the new construction still yields a valid, pseudopolynomial reduction from 3-PARTITION to problem B. Observe tha A went down from about $20m^2B$ to about $40mB$ here. □

To get a connected graph from the frame in figure 4 (b) and the 3m separate components we proceed as follows. First we "stretch" the frame by 3m, by inserting nodes in the supporting columns. (This gives the windows height 3m+4, but their width remains 5B.) Next we span exactly 3m lifelines (just edges) through the middle of the frame all the way acros from one end to the other, and finally we pin each $C(a_i)$ to its own life line. In this way each component is connected to the frame, but their position is not more restricted than before ("components can be freely shifted along their lifeline"). The supporting frame will be made bigger, so as to force a unique embedding.

<u>Theorem</u> 3.1 Problem C is NP-complete (in the strong sense).

<u>Proof</u>.

Problem C obviously is in NP. We shall design a pseudopolynomial reduction from 3-PARTITION to problem C very much like in the previous theorems. Let an instance of 3-PARTITION be given. We shall construct a connected graph G as outlined above. To this end, we start with a

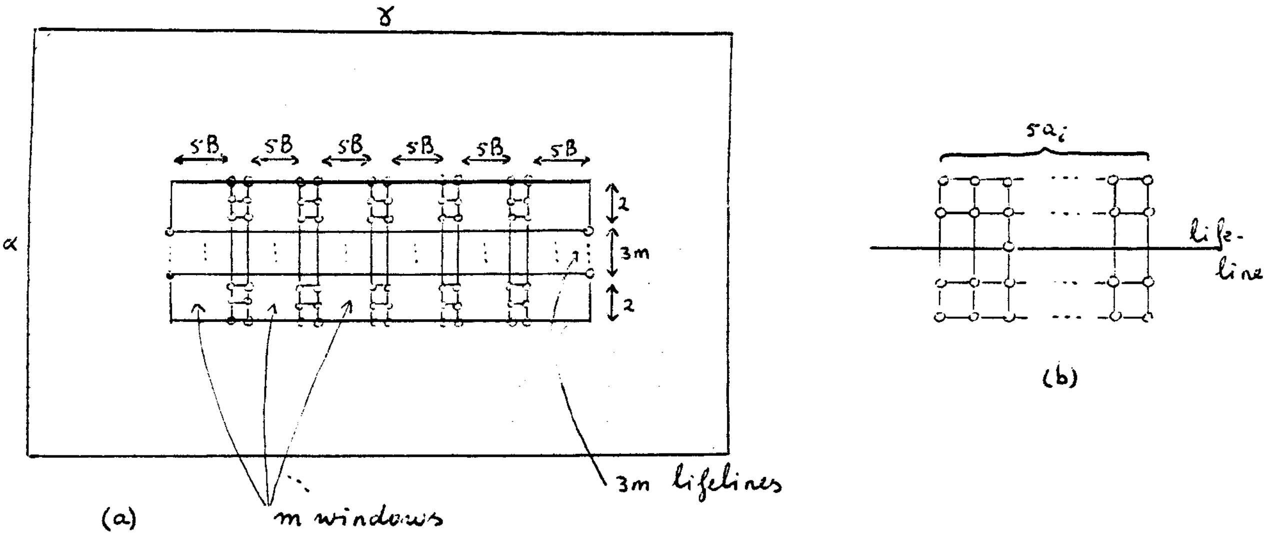


figure 5

frame $F(\alpha, \gamma, 3m+4, 5mB+2m-2)$ and divide its interior into m compartments of width $5B$ as shown in figure 5(a). (α and γ will be chosen later.) Span the $3m$ lifelines across. For each integer a_i ($1 \le i \le 3m$) design a component $C(a_i)$ as shown in figure 5(b), which is essentially the 4-by-$5a_i$ brick but with an extra row in the middle with one node that connects $C(a_i)$ to the i^{th} lifeline. (Note that besides the end points the lifelines all carry precisely one additional node this way, which enables us to position the bricks freely in any one of the compartments.) Consider the instance of problem C that requests the embedding of G in $A = \alpha\gamma$ area, with $\alpha = (3m+4) + 36(3m+4)(5mB+2m-2)$ and the width of the frame chosen as $\gamma = (5mB+2m-2) + 36(3m+4)(5mB+2m-2)$. If the instance of 3-PARTITION has a solution, then it follows by design that G can be laid out in the natural way with the components corresponding to every subset neatly arranged in the m windows. Conversely, suppose we have a layout of G in $\alpha\gamma$ area. By the choice of α and γ it follows from theorem 2.2 that the surrounding frame can be laid out in only one way, namely the natural embedding. The remaining part of G must be laid out in the $(3m+4)$-by-$(5mB+2m-2)$ interior window. Note that all cells beside those occupied by the lifelines in figure 5(a) are needed just to accomodate the nodes from all $C(a_i)$ and the separating columns. This forces the lifelines to occupy no more cells than they do in the natural embedding, or otherwise no embedding could exist for the remaining part of G. It is easily seen that this forces the lifelines to be embedded exactly as in the natural embedding (it is the only possible minimum embedding within the frame) and all other parts to be minimally embedded as well. This puts the $C(a_i)$'s neatly in groups of 3 in the windows of 5B wide, and the embedding translates directly into a solution of the instance of 3-PARTITION. The reduction is easily seen to be pseudopolynomial, and as before we can conclude that problem C is NP-complete. □

4. Final comments.

The main result of this note was the NP-completeness of finding a minimum embedding of a connected graph, as a general problem motivated by the theory of VLSI. It is one (theoretical) indication of the alleged complexity of laying out circuits in as small an amount of area as possible.

Several further observations can be made. Theorem 2.2. remains valid even if we measure "area" just by the number of occupied cells, and not by the size of the (smallest) enclosing rectangle. This means that theorem 3.1 (the embedding problem for connected graphs) remains valid for this, least restrictive notion of area as well. (This is also true for theorem 2.5 by a direct argument.) By adding more lifelines and frame connections to the graph constructed in the proof of theorem 3.1 one can easily show that the embedding problem remains NP-complete for connected graphs of any (fixed) higher degree of connectivity.

Although the proof of theorem 3.1 came "close", it stopped short of proving the NP-completeness of embedding a planar connected graph in minimum area. Also, the NP-completeness question seems open for (connected) graphs of degree ≤ 3.

5. References.

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