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RUU-CS-81-15

September 1981

Rijksuniversiteit Utrecht

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Abstract. We consider three different types of searching problems with d-dimensional rectilinearly oriented hyper-rectangles as objects: the rectangle enclosure, containment and intersection problem. The former two are shown to be equivalent to 2d-dimensional dominance searching; the latter (in its counting version) is shown to be equivalent to the counting version of the d-dimensional dominance problem.

Keywords and phrases. Rectangle intersection searching, rectangle enclosure searching, rectangle containment searching, dominance searching, equivalence, computational geometry.

1. Introduction.

A *rectangle* in d-dimensional space is the Cartesian product of one interval on each of the d coordinate-axes. Hence, a rectangle is assumed to have its sides parallel to the coordinate-axes. A rectangle R *encloses* a rectangle R' if every point of R' is also a point of R, R is *contained* in R' if R' encloses R and R *intersects* R' if R and R' have at least one point in common. See figure 1 for an example. The rectanlge R is enclosed by R_1, contains R_2 and R_4 and intersects R_1, R_2, R_3, R_4 and R_5.

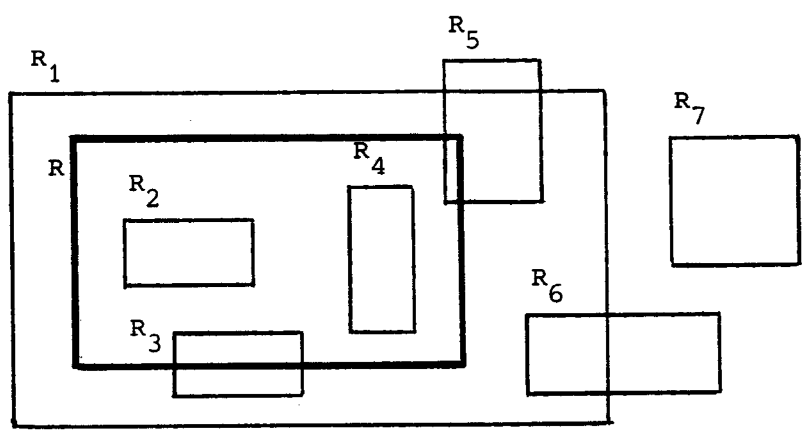

Figure 1.

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* The work of the second author was supported by the Netherlands Organization for the Advancement of Pure Research (ZWO).

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Given a set V of rectangles in d-dimensional space and another such rectangle R, the rectangle enclosure searching problem asks for all rectangles in V that enclose R, the rectangle containment searching problem asks for all rectangles in V that are contained in R, and the rectangle intersection searching problem asks for all rectangles in V that intersect R. Often, we are merely interested in the number of rectangles that enclose, are contained in or intersect the given query rectangle R. In this case, we call the problem the rectangle enclosure, containment respectively intersection counting problem.

In particular the rectangle intersection searching (counting) problem received considerable attention during the past few years (see e.g. Edelsbrunner [2], Edelsbrunner and Maurer [3], Lee and Wong [6], McCreight [7] and Vaishnavi and Wood [10]). The rectangle enclosure and containment searching (counting) problems were treated only recently by Lee and Wong [6] and McCreight [8]. Overmars and van Leeuwen [9] presented a general result by which some of the known solutions could be improved.

In this paper we will show that all three problems are in some sense equivalent to dominance searching: Given a set V of points in d-dimensional space and another point $x = (x_1,\dots,x_d)$, the dominance searching problem asks for all points $p = (p_1,\dots,p_d)$ in V such that $p \leq x$, i.e., $p_1 \leq x_1 \wedge p_2 \leq x_2 \wedge \dots \wedge p_d \leq x_d$. The counting variant of the problem that asks for the number of points in V that are dominated by x is the well-known ECDF-searching problem (see e.g. Bentley [1]).

In section 2 we will show that the d-dimensional rectangle enclosure and containment searching (counting) problems are equivalent to the 2d-dimensional dominance searching (counting) problem. In section 3 we show that the d-dimensional rectangle intersection counting problem is equivalent to the d-dimensional dominance counting problem. It follows that an improvement in the bounds for one of the problems immediately results in an improvement for the other problems. In section 4 we briefly mention how the results can be generalized to other rectangle searching problems. This establishes an additional step towards a unified view of problems involving rectangles that were dealt with separately in the past.

2. Rectangle enclosure/containment searching.

A rectangle $R = ([x_1:y_1],\dots,[x_d:y_d])$ encloses a rectangle $R' = ([x_1':y_1'],\dots,[x_d':y_d'])$ if and only if $x_1 \leq x_1' \wedge \dots \wedge x_d \leq x_d'$ and $y_1 \geq y_1' \wedge \dots \wedge y_d \geq y_d'$. Hence when we transform a rectangle R into the 2d-dimensional point $p(R) = (x_1,\dots,x_d, -y_1,\dots,-y_d)$ then R encloses R' if and only if $p(R) \leq p(R')$. Hence, the 2d-dimensional dominance searching problem can be used to solve the d-dimensional rectangle enclosure problem. Similarly, transforming a rectangle R into the point $p(R) = (-x_1,\ldots,-x_d, y_1,\ldots,y_d)$, R is contained in R' if and only if $p(R) \leq p(R')$. Hence, also the d-dimensional rectangle containment searching problem can be solved using the 2d-dimensional dominance searching problem.

It remains to be shown that the 2d-dimensional dominance searching problem can be solved by both the d-dimensional rectangle enclosure and containment problem. Let us first use the enclosure problem to solve the dominance problem. We want to map a given point $p = (x_1,\ldots,x_d, y_1,\ldots,y_d)$ into a rectangle $R(p) = ([f_1(x_1):f_2(y_1)],[f_1(x_2),f_2(y_2)], \ldots,[f_1(x_d):f_2(y_d)])$ such that

(i) $f_1(x) \leq f_2(y)$ for all x, y

(ii) $x \leq x' \Leftrightarrow f_1(x) \leq f_1(x')$, and
$y \leq y' \Leftrightarrow f_2(y) \geq f_2(y')$

(i) guarantees that points are mapped into rectangles and (ii) guarantees that p' dominates p if and only if R(p) encloses R(p'). A possible solution to (i) and (ii) is

$$f_1(x) = \begin{cases} x - 2 & \text{if } x \leq 1 \\ -\frac{1}{x} & \text{if } x \geq 1 \end{cases}$$

$$f_2(y) = \begin{cases} -y + 2 & \text{if } y \leq 1 \\ \frac{1}{y} & \text{if } y \geq 1 \end{cases}$$

To tranform 2d-dimensional dominance searching into d-dimensional containment searching, we like to map a point $p = (x_1,\ldots,x_d,y_1,\ldots,y_d)$ into a rectangle $R(p) = ([f_1(x_1),f_2(y_1)],\ldots,[f_1(x_d),f_2(y_d)])$ under the following conditions:

(i) $f_1(x) \leq f_2(y)$ for all x, y

(ii) $x \leq x' \Leftrightarrow f_1(x) \geq f_1(x')$, and
$y \leq y' \Leftrightarrow f_2(y) \leq f_2(y')$

One can choose, for instance:

$$f_1(x) = \begin{cases} \frac{1}{x} & \text{if } x \leq -1 \\ -x - 2 & \text{if } x \geq -1 \end{cases}$$

$$f_2(y) = \begin{cases} -\frac{1}{y} & \text{if } y \leq -1 \\ y + 2 & \text{if } y \geq -1 \end{cases}$$

The transformations clearly hold for the counting variants also.

Theorem 2.1. The d-dimensional rectangle enclosure and containment searching/counting problems are both equivalent to the 2d-dimensional dominance searching/counting problem.

3. Rectangle intersection counting.

The d-dimensional dominance searching problem is a special instance of the d-dimensional rectangle intersection searching problem in which the rectangles in the set are degenerated to points and the query rectangle is $([-\infty,x_1],\ldots,[-\infty,x_d])$. Hence it only needs to be shown that d-dimensional rectangle intersection can be solved using the d-dimensional dominance problem. We are only able to show this for the counting variants.

Lemma 3.1. The d-dimensional rectangle intersection counting problem can be solved using the d-dimensional dominance counting problem.

Proof

Let us first consider the 1-dimensional case. So we are given a set V of intervals. An interval [x:y] does not intersect an interval [a:b] if and only if either $x > b$ or $y < a$. Hence, we make two set of points: V_1 containing $-x$ for each begin point x of a segment in V and V_2 containing all end points of segments in V. To perform an intersection (counting) query with [a:b] we perform a dominance (counting) query with $-b$ on V_1 and with a on V_2. The answer to the intersection query is n (the total number of segments in V) minus the answers to the two dominance queries.

In the 2-dimensional case we perform 4 dominance queries to locate the number of rectangles that lie left, right, above and below the query rectangle. In this way we count a number of rectangles twice, namely those that lie e.g. left and above the query rectangle. The number of these double counted rectangles can again be located by 4 dominance queries. Hence the problem can be solved by means of 8 dominance queries.

The generalizations to the d-dimensional case are straightforward and left as an easy exercise to the reader. It follows that the d-dimensional rectangle intersection counting problem can be solved using a number of instances of the d-dimensional dominance counting problem.

□

We have shown:

Theorem 3.2. The d-dimensional rectangle intersection counting problem and the d-dimensional dominance counting problem are equivalent.

4. Extensions.

Beside the three rectangle searching problems considered in the previous sections, numerous other rectangle searching problems can be defined. For example, one may ask for all rectangles that lie completely to the right of a given rectangle x or for those rectangles whose boundaries do not intersect the boundary of x (i.e., those rectangles that enclose x, are contained in x or do not intersect x.) In the 1-dimensional case, i.e., when rectangles are intervals on a line, one can define 64 different types of rectangle searching problems. By an exhaustive case study it can be shown that the counting variant of each of these 64 problems is equivalent to the 0-, 1- or 2-dimensional dominance counting problem (see Edelsbrunner and Overmars [4]). Defining a d-dimensi nal rectangle searching problem as being the Cartesian product of 1-dimensional rectangle problems, there are 64^d different d-dimensional rectangle searching problems. Edelsbrunner and Overmars [4] show:

Theorem 4.1. ([4]) The counting variant of each d-dimensional rectangle searching problem is equivalent to the d'-dimensional dominance counting problem for some d' with $0 \leq d' \leq 2d$.

For some of the problems equivalence can even be shown for the searching versions of the problems but in most cases it remains an open question whether or not the searching versions are equivalent.

Finally, some remarks on lowerbounds. Fredman [5] proved lowerbounds for the range searching problem (a generalization of the dominance problem in which we ask for those elements in a d-dimensional pointset that lie within a given rectangle) but these do not apply to the dominance searching problem. Lowerbound for the dominance searching/counting problem would immediately lead to lowerbounds for all rectangle searching/ counting problem, due to the equivalences described above.

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