THE EQUIVALENCE OF RECTANGLE CONTAINMENT, RECTANGLE ENCLOSURE AND ECDF SEARCHING

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Abstract. We consider three searching problems: d-dimensional rectangle containment searching, d-dimensional rectangle enclosure searching and 2d-dimensional ECDF searching. We show that these three problems are equivalent, and thus have the same complexity, by giving transformations that map the problems into each other.

Keywords and phrases. Rectangle containment problem, rectangle enclosure problem, ECDF searching.

1. Introduction.

Searching problems are problems in which we ask a question (query) with an object with respect to a (static or dynamic) set of other objects. In this paper we consider three familiar searching problems:

I the rectangle containment searching problem: Given a set V of rectilinearly oriented hyper-rectangles in d-dimensional space and another such hyper-rectangle R, determine all elements of V that are contained in R. (See figure 1a.)

II the rectangle enclosure searching problem: Given a set V of rectilinearly oriented hyper-rectangles in d-dimensional space and another such hyper-rectangle R, determine all elements of V that enclose R. (See figure 1b.)

III the ECDF searching problem: Given a set V of points in d-dimensional space and another point p, determine all points p' in V such that $p' \le p$. ($(x_1', x_2', \ldots, x_d') \le (x_1, x_2, \ldots, x_d)$ iff $x_1' \le x_1$, $x_2' \le x_2$ ... and $x_d' \le x_d$). (See figure 1c.)

Note that in all three problems we want the elements of V themselves. Often,

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especially in ECDF searching, we are only interested in the number of answers to the query. In this case we talk about a counting problem instead of a searching problem. From now on we will write "rectangle" when we mean "rectilinearly oriented hyper-rectangle".

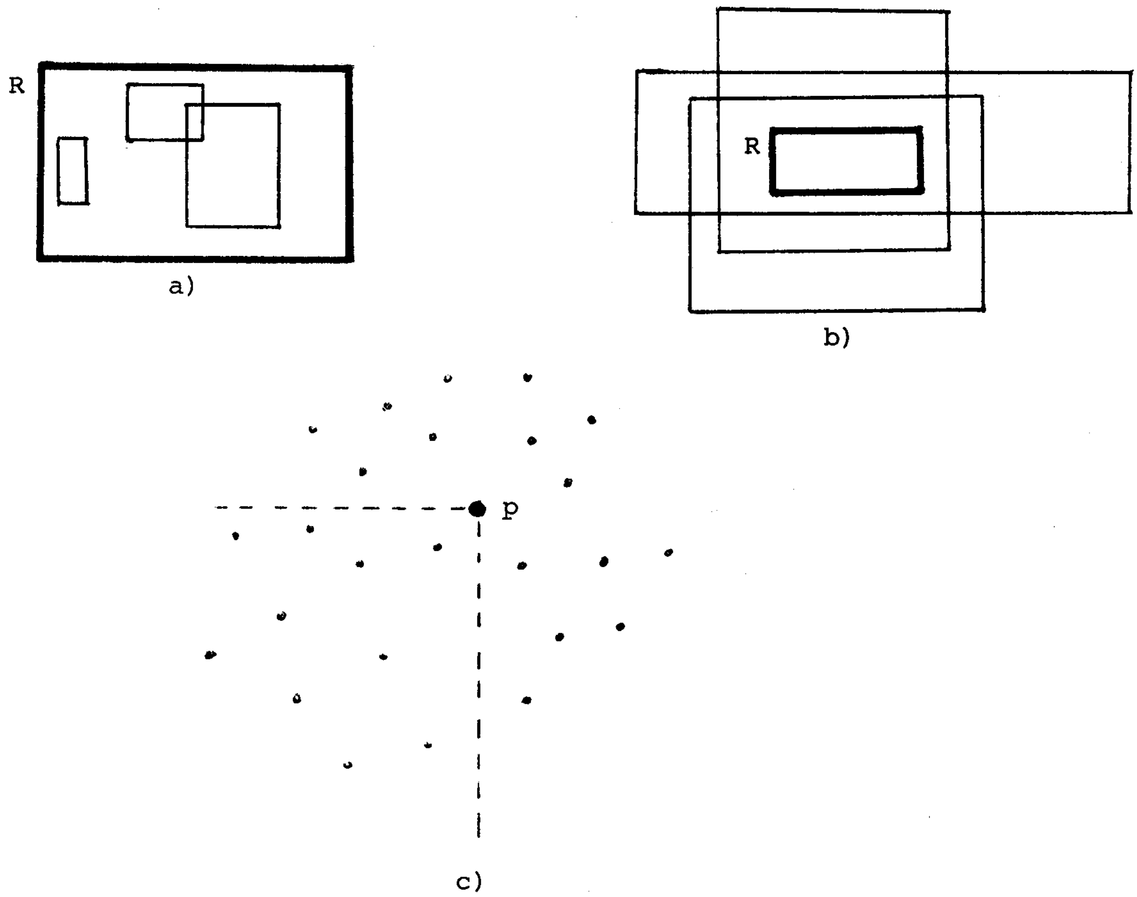

figure 1.

The rectangle containment searching problem and the rectangle enclosure searching problem were recently solved by Lee and Wong [3] by transforming them to the 2d-dimensional range query problem, that asks for all points of a set V that lie within a given rectilinearly oriented range. By using a slightly modified version of a structure of Willard [5] they obtained a dynamic solution to the problem (i.e. such that one can insert and delete rectangles in the set V) with a query time of $O(\log^{2d-1} n + k)$, and an insertion and deletion time of $O(\log^{2d} n)$, where n is the number of rectangles currently in the set and k is the number of reported rectangles. Using a technique of Overmars and van Leeuwen [4], the deletion time can even be lowered to $O(\log^{2d-1} n)$. In fact, Lee and Wong [3] transform the rectangle problems to 2d-dimensional ECDF searching (instead of range searching), and show it can be solved in the bounds stated. When we want to solve the d-dimensional containment and enclosure counting problem the modified structure of Willard [5] can no longer be used. In this case we can transform the two problems to the 2d-dimensional ECDF counting problem and, using results of e.g. Bentley [1] and Bentley and Shamos [2], we can obtain a dynamic solution to the three problems, achieving a query time of $O(\log^{2d} n)$, an insertion of $O(\log^{2d} n)$ and a deletion time of $O(\log^{2d-1} n)$ (using a result of Overmars and van Leeuwen [4]).

In this paper we show that the d-dimensional rectangle containment problem, the d-dimensional rectangle enclosure problem and the 2d-dimensional ECDF problem are equivalent (both in the searching variant and the counting variant), by giving transformations that map the problems into each other. It follows that all three problems have the same complexity.

2. Rectangle containment and rectangle enclosure.

In this section we will show that rectangle containment and rectangle enclosure are equivalent. We do this by describing a tranformation f that maps rectangles into rectangles such that a rectangle R_1 encloses a rectangle R_2 if and only if $f(R_1)$ is contained in $f(R_2)$, and a rectangle R_1 is contained in a rectangle R_2 if and only if $f(R_1)$ encloses $f(R_2)$. It follows that transforming the set of rectangles V using f and transforming the query object R into f(R), we can perform a containment query with f(R) on f(V) to solve an enclosure query with R on V and we can solve a containment query with R on V by performing an enclosure query with f(R) on f(V) (in which f(V) means the set of transformed rectangles).

Let us first consider the one dimensional case. Hence we have a set V of segments [x:y] and we would like to have a tranformation $f = (f_1, f_2)$ such that

i) f maps a segment [x:y] into a segment $[f_1(x):f_2(y)]$

ii) S_1 encloses $S_2 \Leftrightarrow f(S_1)$ is contained in $f(S_2)$

To let f map segments into segments the following condition is needed

$$x \le y \;\Rightarrow\; f_1(x) \le f_2(y) \qquad (2.1.)$$

Condition ii) can easily be translated into

$$x' \le x \;\Leftrightarrow\; f_1(x') \ge f_1(x) \qquad (2.2.)$$

$$y \le y' \;\Leftrightarrow\; f_2(y) \ge f_2(y') \qquad (2.3.)$$

From (2.2.) and (2.3.) it follows that both f_1 and f_2 must be strictly decreasing. From (2.1.) and (2.3.) it follows that

$$\min_y f_2(y) \geq \max_x f_1(x) \qquad (2.4.)$$

A first choice for f would be (see figure 2)

$$f_1(x) = -2^x$$
$$f_2(y) = 2^{-y} \qquad (2.5.)$$

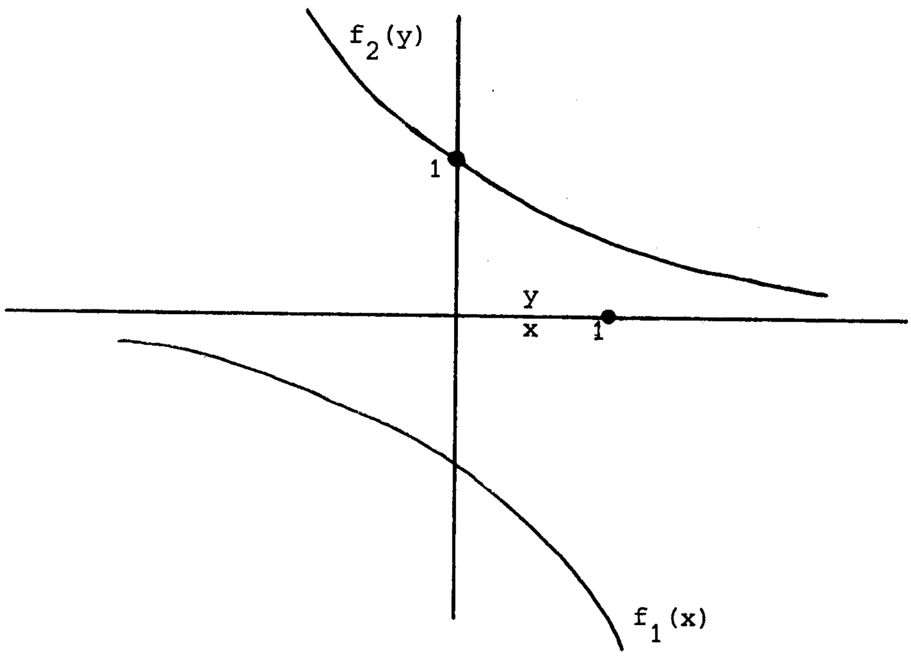

Figure 2.

This function performs the transformation correctly but it is exponential. Having a non-exponential function would be better because it can be computed more easily. Therefore we had better used a transformation f like the following (see figure 3)

$$f_1(x) = \begin{cases} \frac{1}{x} & \text{if } x \leq -1 \\ -x-2 & \text{if } x \geq -1 \end{cases}$$

$$(2.6.)$$

$$f_2(y) = \begin{cases} -y+2 & \text{if } y \leq 1 \\ \frac{1}{y} & \text{if } y \geq 1 \end{cases}$$

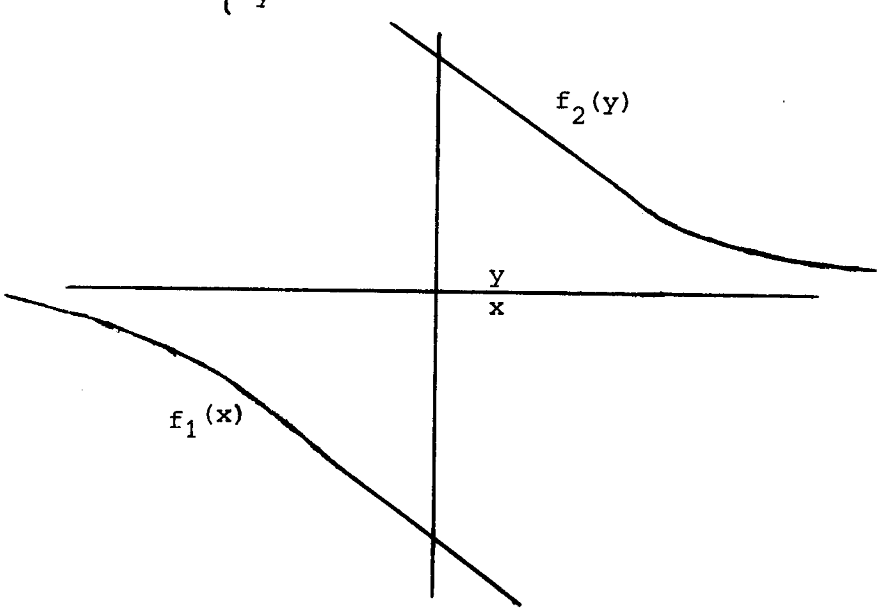

Figure 3.

This function satisfies (2.1.), (2.2.) and (2.3.) and can be computed more easily. Hence, we have shown that 1-dimensional rectangle containment searching and 1-dimensional rectangle enclosure searching are equivalent.

The generalization to d-dimensional space is completely straightforward. A d-dimensional rectangle can be represented by its projection on all coordinate axes. These are segments. Hence a d-dimensional rectangle can be represented by d segments $([x_1:y_1], [x_2:y_2], \ldots, [x_d:y_d])$. A rectangle R_1 contains a rectangle R_2 if and only if for each coordinate axis the projection of R_1 on it contains the projection of R_2 (and similar for enclosure). Performing on each projection the transformation f, mapping $([x_1:y_1], [x_2:y_2], \ldots, [x_d:y_d])$ to $([f_1(x_1):f_2(y_1)], [f_1(x_2):f_2(y_2)], \ldots, [f_1(x_d):f_2(y_d)])$, transforms containment into enclosure in each dimension and hence maps d-dimensional rectangle containment to rectangle enclosure (similar for enclosure to containment). We have shown.

Theorem A. d-dimensional rectangle containment searching and d-dimensional rectangle enclosure searching are equivalent.

3. Rectangle containment and ECDF searching.

In this section we will show that the d-dimensional rectangle containment problem and the 2d-dimensional ECDF problem are equivalent. Although Lee and Wong [3] already show how to transform d-dimensional rectangle containment into 2d-dimensional ECDF we will give a more explicit proof here. Let us first consider the case in which d = 1. We would like to have a transform $f = (f_1, f_2)$ that maps a segment [x:y] into the point $(f_1(x), f_2(y))$ such that containment is transformed into domination. Therefore, the following conditions are needed

$$x \le x' \Leftrightarrow f_1(x) \ge f_1(x') \quad (3.1.)$$
$$y' \le y \Leftrightarrow f_2(y') \le f_2(y) \quad (3.2.)$$

From (3.1.) it follows that f_1 must be strictly decreasing, and from (3.2.) it follows that f_2 must be strictly increasing. Hence, we can choose the following transformation

$$f_1(x) = -x \quad (3.3.)$$
$$f_2(y) = y$$

The generalization to d-dimensional rectangle containment and 2d-dimensional ECDF is straightforward.

Next, we have to develop a transformation that maps 2d-dimensional ECDF into d-dimensional rectangle containment. Again, let us first consider the case in which d = 1. Hence we have a 2-dimensional pointset V and another point p in the plane and we would like to have a transformation f that maps points p into segments on the line such that $p' \leq p$ if and only if $f(p')$ is contained in $f(p)$. Let f map a point $p = (x, y)$ into $[f_1(x):f_2(y)]$. To let $[f_1(x):f_2(y)]$ be a segment we need

$$\forall\, x, y \qquad f_1(x) \leq f_2(y) \tag{3.4.}$$

To take care that p dominates p' if and only if f(p) contains f(p') we need

$$x \geq x' \Leftrightarrow f_1(x) \leq f_1(x') \tag{3.5.}$$

$$y \geq y' \Leftrightarrow f_2(y) \geq f_2(y') \,. \tag{3.6.}$$

From (3.5.) it follows that f_1 must be strictly decreasing and from (3.6.) it follows that f_2 must be strictly increasing. (3.4.) can be rewritten as

$$\min_y f_2(y) \geq \max_x f_1(x) \tag{3.7.}$$

like in section 2. A first choice for f would be

$$\begin{aligned} f_1(x) &= -2^x \\ f_2(y) &= 2^y \end{aligned} \tag{3.8.}$$

If we prefer non-exponential functions we can choose (see figure 4)

$$f_1(x) = \begin{cases} \frac{1}{x} & \text{if } x \leq -1 \\ -x-2 & \text{if } x \geq -1 \end{cases}$$

$$f_2(y) = \begin{cases} -\frac{1}{y} & \text{if } y \leq -1 \\ y+2 & \text{if } y \geq -1 \end{cases} \tag{3.9.}$$

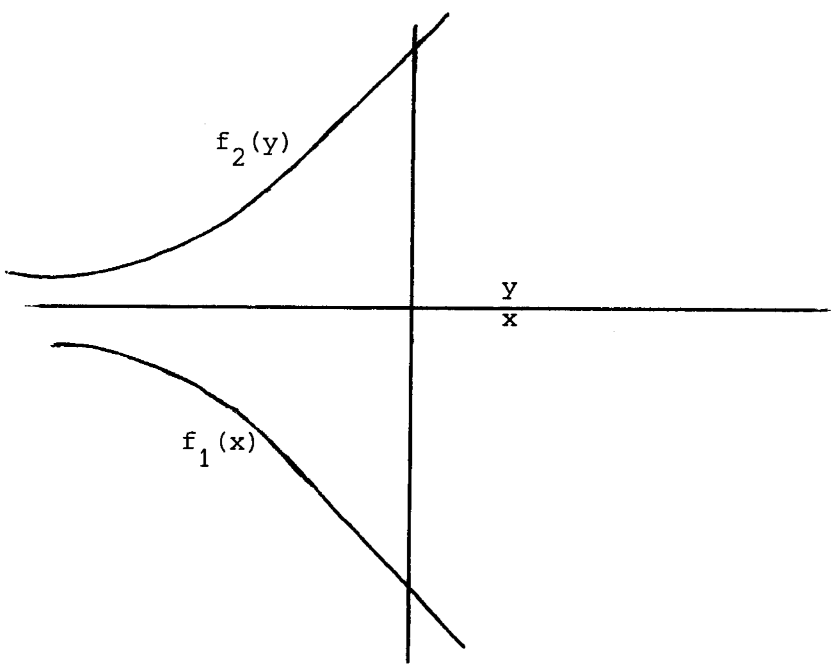

Figure 4.

Generalizing this result to d-dimensional rectangle containment and 2d-dimensional ECDF is straightforward. A point $p = (x_1, y_1, x_2, y_2, \ldots, x_d, y_d)$ is transformed into the hyperrectangle $([f_1(x_1):f_2(y_1)], [f_1(x_2):f_2(y_2)], \ldots, [f_1(x_d):f_2(y_d)])$. It follows that in each dimension domination is transformed into containment and hence that ECDF is transformed into rectangle containment.

Theorem B. The d-dimensional rectangle containment problem and the 2d-dimensional ECDF problem are equivalent.

4. Conclusion.

In section 2 we have shown that the d-dimensional rectangle containment problem and the d-dimensional rectangle enclosure problem are equivalent. In section 3 we have shown that the first one is equivalent to the 2d-dimensional ECDF problem. It follows that all three problems are equivalent to each other. This result holds both for the searching variants as for the counting variants. It follows that the three problem have the same complexity and an improvement of the known bounds for one of them would immediate yield an improvement for the other two problems.

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