Approximation Algorithms for Computing the Earth Mover's Distance Under Transformations

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Abstract. The Earth Mover's Distance (EMD) on weighted point sets is a distance measure with many applications. Since there are no known exact algorithms to compute the minimum EMD under transformations, it is useful to estimate the minimum EMD under various classes of transformations. For weighted point sets in the plane, we will show a 2-approximation algorithm for translations, a 4-approximation algorithm for rigid motions and an 8-approximation algorithm for similarity transformations. The runtime for translations is $O(T^{EMD}(n,m))$, the runtime of the latter two algorithms is $O(nmT^{EMD}(n,m))$, where $T^{EMD}(n,m)$ is the time to compute the EMD between two fixed weighted point sets with n and m points, respectively. All these algorithms are based on a more general structure, namely on reference points. This leads to elegant generalizations to higher dimensions. We give a comprehensive discussion of reference points for weighted point sets with respect to the EMD.

1 Introduction

The Earth Mover's Distance on weighted point sets is a very useful distance measure for e.g. shape matching, colour-based image retrieval and music score matching, see [5], [6], [7] and [11] for more information. For these applications it is useful to have a quick estimation on the minimum distance between two weighted point sets which can be achieved under a considered class of transformations $\mathcal{T}$. Thus we want to find algorithms to compute an approximation where $EMD^{apx}(A,B) \leq \alpha \cdot \min\{EMD(A,\Phi(B)) : \Phi \in \mathcal{T}\}$. This problem was first regarded by Cohen ([5]). He constructed an iterative Flow-Transformation algorithm, which he proved to converge, but not necessarily to the global minimum. In this paper we will take a different approach and use reference points to get an approximation on the problem. These points have already been introduced in [1] and [2] to construct approximation algorithms for matching compact subsets of $\mathbb{R}^d$ under translations, rigid motions and similarity transformations with respect to the Hausdorff-distance. Also approximation algorithms using reference points for matching with respect to the Area of Symmetric Difference have been given,

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see [3] and [12]. A general discussion of reference point methods for matching according to the Hausdorff-distance has been given in [1]. Here we will extend the definition of reference points to weighted point sets and get fast constant factor approximation algorithms for matching weighted point sets under translations, rigid motions and similarity transformations with respect to the EMD. Quite recently, Cabello et al. ([4]) have been working on similar problems. The advantage of our approach is that the results given can be applied to arbitrary dimension and distance measure on the ground set, even on more than the in this abstract mentioned L_p-distances. Therefore the results are widely applicable.

2 Basic Definitions

Definition 1 (Weighted Point Set). *([6]) Let* $A = \{a_1, a_2, ..., a_n\}$ *be a weighted point set such that* $a_i = (p_i, \alpha_i)$ *for* $i = 1, ..., n$*, where* p_i *is a point in* $\mathbb{R}^d$ *and* $\alpha_i \in \mathbb{R}_0^+$ *its corresponding weight. Let* $W^A = \sum_{i=1}^n \alpha_i$ *be the total weight of* A*. Let* $\mathbb{W}^d$ *be the set of all weighted point sets in* $\mathbb{R}^d$ *and* $\mathbb{W}^{d,G}$ *be the set of all weighted point sets in* $\mathbb{R}^d$ *with total weight* $G \in \mathbb{R}^+$*.*

In the following we will use a considered class of transformations on both weighted point sets and discrete subsets of $\mathbb{R}^d$. By a transformation on a weighted point set we mean to transform the coordinates of the weighted points and leave their weights unchanged.

We now introduce the center of mass, which plays an important role in our approximation algorithms. The computation time of this point is linear, so it does not affect the runtime of the presented algorithms.

Definition 2 (Center of Mass). *Let* $A = \{(p_i, \alpha_i)_{i=1,...,n}\} \in \mathbb{W}^{d,G}$ *be a weighted point set for some* $G \in \mathbb{R}^+$*. The center of mass of* A *is defined as* $C(A) = \frac{1}{W^A} \sum_{i=1}^n \alpha_i p_i$*.*

As we will see, the center of mass is an instance of a more general class of mappings, namely reference points. Later we will prove the correctness of abstract algorithms based on this class of mappings. By plugging in the center of mass we will get concrete and implementable algorithms.

Definition 3 (Reference Point). *([1]) Let* $\mathcal{K}$ *be a subset of* $\mathbb{W}^d$ *and* $\delta : \mathcal{K} \times \mathcal{K} \to \mathbb{R}_0^+$ *be a distance measure on* $\mathcal{K}$*. Let* $|| \cdot || : \mathbb{R}^d \to \mathbb{R}_0^+$ *be any norm on* $\mathbb{R}^d$*. A mapping* $r : \mathcal{K} \to \mathbb{R}^d$ *is called a* δ*-reference point for* $\mathcal{K}$ *with respect to a set of transformations* $\mathcal{T}$ *on* $\mathcal{K}$*, if the following two conditions hold:*

1. *Equivariance with respect to* $\mathcal{T}$*: For all* $A \in \mathcal{K}$ *and* $\Phi \in \mathcal{T}$ *we have*
$$r(\Phi(A)) = \Phi(r(A)).$$
2. *Lipschitz-continuity: There is a constant* $c \geq 0$*, such that for all* $A, B \in \mathcal{K}$*,*
$$||r(A) - r(B)|| \leq c \cdot \delta(A, B).$$

We call c *the quality of the* δ*-reference point* r*.*

In section 4.3 we will construct approximation algorithms for similarities. For this reason we will have to rescale one of the weighted point sets. Unfortunately, rescaling in a way that the diameters of the underlying point sets in $\mathbb{R}^d$ are equal, does not work. The key to a working algorithm is to rescale the set in a way that the normalized first moments with respect to their reference points coincide. Here we give the well known definition of the normalized first moment of a weighted point set with respect to an arbitrary point $p \in \mathbb{R}^d$.

Definition 4 (Normalized First Moment). *Let $A = \{(p_i, \alpha_i)_{i=1,\dots,n}\} \in \mathbb{W}^{d,G}$ be a weighted point set for some $G \in \mathbb{R}^+$ and let $p \in \mathbb{R}^d$ be an arbitrary point. We call $m_p(A) = \frac{1}{W^A}\sum_{i=1}^{n} \alpha_i ||p_i - p||$ the normalized first moment of A with respect to p.*

Note that the normalized first moment of a weighted point set with respect to an arbitrary point can be calculated efficiently in linear time.

Next we will introduce the EMD, a distance measure on weighted point sets.

Definition 5 (Earth Mover's Distance). *([5]) Let $A = \{(p_i, \alpha_i)_{i=1,\dots,n}\}$, $B = \{(q_j, \beta_j)_{j=1,\dots,m}\} \in \mathbb{W}^d$ be weighted point sets with total weights W^A, $W^B \in \mathbb{R}^+$. Let $D : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_0^+$ be a distance measure on the ground set $\mathbb{R}^d$. The Earth Mover's Distance between A and B is defined as*

$$EMD(A, B) = \frac{\min_{F \in \mathcal{F}} \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij} D(p_i, q_j)}{\min\{W^A, W^B\}}$$

where $F = \{f_{ij}\}$ is a feasible flow, i.e.

1. $f_{ij} \geq 0, i = 1, ..., n, j = 1, ..., m$
2. $\sum_{j=1}^{m} f_{ij} \leq \alpha_i, i = 1, ..., n$
3. $\sum_{i=1}^{n} f_{ij} \leq \beta_j, j = 1, ..., m$
4. $\sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij} = \min\{W^A, W^B\}$

For the rest of the paper the distance measure D used in the definition of the EMD will be the metric induced by the norm used in the definition of the EMD-reference point. When working with weighted point sets in $\mathbb{R}^d$ we will call $\mathbb{R}^d$ the ground set and $D : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_0^+$ the ground distance. If D is the Euclidean Distance, we will also use EEMD as a notation for the Euclidean Earth Mover's Distance. If D is any L_p-distance for $1 \leq p \leq \infty$ we will write EMD_p to denote the Earth Mover's Distance based on this distance measure.

3 EMD-Reference Points

In this section we discuss the existence of EMD-reference points. We start with a negative result.

3.1 Non-Existence of EMD-Reference Points for Non-Equal Total Weights

Let r be any reference point with respect to translations. Regarding the weighted point sets $A := \{(p, 1)\}$, $B := \{(q, 1)\}$ and $C := A \cup B$, $p \neq q \in \mathbb{R}^d$, we can easily see that $EMD(A, C) = EMD(B, C) = 0$ and therefore $r(A) = r(C) = r(B)$. On the other hand, B is a translation of A by $q - p \neq 0$ and therefore, by equivariance, $r(B)$ has to be a translation of $r(A)$ by $q - p$, which leads to a contradiction. Thereby we have proven the following theorem:

Theorem 1. *There is no EMD-reference point for weighted point sets with unequal total weights with respect to all transformation sets that include translations.*

Unfortunately, Theorem 1 has a deep impact on the usability of the reference point approach for shape matching since it makes it impossible to use this approach for partial matching applications. For a more detailed discussion on partial matching using Mass Transportation Distances, see [6].

3.2 The Center of Mass as a Reference Point

In the next section we will present approximation algorithms for the EMD under transformations using EMD-reference points. Since this would be useless if there was no EMD-reference point, we will restrain the consideration to weighted point sets with equal total weight. In this case, the equivariance of the center of mass under affine transformations is well known and the proof of the Lipschitz-continuity appeared already in [5]. Therefore we can formulate the following theorem:

Theorem 2. *The center of mass is an EMD-reference point for weighted point sets with equal total weight with respect to affine transformations. Its quality is* 1. *This holds for any dimension* d *and any distance measure on the ground set.*

3.3 Lower Bound on the Quality of an EMD-Reference Point

Using the center of mass to construct implementable algorithms raises the question if there is a better reference point. We can prove that the center of mass is optimal in the sence that there is no reference point inducing a better quality. But we do not know so far if there is a reference point inducing approximation algorithms with a better approximation factor.

Theorem 3. *Let* $r : \mathbb{W}^{d,G} \to \mathbb{R}^d$ *be an EMD-reference point with respect to any transformation set including the set of translations for some* $G \in \mathbb{R}^+$ *and some dimension* d*, and let* c *be its quality. Then* $c \geq 1$. *This holds for any distance measure on the ground set.*

4 Approximation Algorithms Using EMD-Reference Points

The following three sections are organized as follows: In each section we consider a class of transformations, construct an approximation algorithm for matching under these transformations for general EMD-reference points and finally use the center of mass to get a concrete algorithm.

For the rest of the paper let $A = \{(p_i, \alpha_i)_{i=1,\ldots,n}\}$, $B = \{(q_j, \beta_j)_{j=1,\ldots,m}\} \in \mathbb{W}^{d,G}$ be two weighted point sets in dimension d with positive equal total weight $G \in \mathbb{R}^+$. Please be reminded that the following results do not hold for weighted point sets with unequal total weight. Further, let $r : \mathbb{W}^{d,G} \to \mathbb{R}^d$ be an EMD-reference point for weighted point sets with respect to the considered class of transformations with quality c. Let $T^{ref}(n)$ be the time to compute the EMD-reference point of A, $T^{EMD}(n,m)$ and $T^{EEMD}(n,m)$ be the time to compute the EMD and $EEMD$ between A and B and $T^{rot}(n,m)$ be the time needed to find a rotation R around a fixed point minimizing $EMD(A, R(B))$.

An upper bound on $T^{EMD}(n,m)$ and $T^{EEMD}(n,m)$ is $O((nm)^2 \log(n+m))$ using a strongly polynomial minimum cost flow algorithm by Orlin ([9]). In practice, an algorithm using the simplex method to solve the linear program will be faster. Since we are developing approximation algorithms anyway, one can consider using an $(1+\varepsilon)$-approximation algorithm for the Earth Mover's Distance by Cabello et al. ([4]) with runtime $O(\frac{n^2}{\varepsilon^2} \log^2(\frac{n}{\varepsilon}))$.

4.1 Translations

The first algorithm will find an approximation for the EMD under translations:

Algorithm *TranslationApx*:
1. Compute $r(A)$ and $r(B)$ and translate B by $r(A) - r(B)$. Let B' be the image of B.
2. Output B' together with the approximate distance $EMD(A, B')$.

Theorem 4. *Algorithm TranslationApx finds an approximately optimal matching for translations with approximation factor* $c+1$ *in time* $O(T^{ref}(\max\{n,m\}) + T^{EMD}(n,m))$. *This holds for arbitrary dimension* d *and distance measure on the ground set.*

Applying the center of mass leads to the following corollary. The approximation factor of 2 is tight, a proof for this can also be found in [8].

Corollary 1. *Algorithm TranslationApx using the center of mass as an EMD-reference point induces an approximation algorithm with approximation factor* 2. *Its runtime is* $O(T^{EMD}(n,m))$.

4.2 Rigid Motions

The following algorithm is a first approach to get an approximation on the EMD under rigid motions, i.e. combinations of translations and rotations:

Algorithm *RigidMotionApx*:
1. Compute $r(A)$ and $r(B)$ and translate B by $r(A) - r(B)$. Let B' be the image of B.
2. Find an optimal matching of A and B' under rotations of B' around $r(A)$. Let B'' be the image of B' under this rotation.
3. Output B'' and the approximate distance $EMD(A, B'')$.

Theorem 5. *Algorithm RigidMotionApx finds an approximately optimal matching for rigid motions with approximation factor* $c+1$ *in time* $O(T^{ref}(\max\{n,m\}) + T^{EMD}(n,m) + T^{rot}(n,m))$. *This holds for arbitrary dimension* d *and distance measure on the ground set.*

Corollary 2. *Algorithm RigidMotionApx using the center of mass as EMD-reference point induces an approximation algorithm with approximation factor* 2 *in time* $O(T^{rot}(n,m) + T^{EMD}(n,m))$. *This holds for arbitrary dimension* d *and distance measure on the ground set .*

Since the position of the EMD-reference point as rotation center is fixed, several degrees of freedom have been eliminated and the problem to find the optimal rotation should be easier than the one finding the optimal rigid motion itself. Unfortunately, even for this problem no efficient algorithm is known so far. Therefore it would be nice to have at least an approximation algorithm for this problem. In the next lemma we will give an approximation for the Euclidean Distance as the ground distance. This result was already published in [4]. Next we will use this lemma to extend the result to all L_p-distances, $1 \leq p \leq \infty$. Unfortunately, the approximation factor will be worse than 2 for $p \neq 2$.

Lemma 1. *Let* $A, B \in \mathbb{W}^d$ *be two weighted point sets and* p^* *be any point. Let* $Rot(p^*)$ *be the set of all rotations around* p^*. *Then there is a rotation* $R' \in Rot(p^*)$ *such that* R' *aligns* p^* *and any two points of* A *and* B, *and*

$$EEMD(A, R'(B)) \leq 2 \cdot \min_{R \in Rot(p^*)} EEMD(A, R(B)).$$

As mentioned above, we will now use the last lemma to extend the result to all L_p-distances, $1 \leq p \leq \infty$. The proof is based on the fact that for $1 \leq p, q \leq \infty$ and any vector $v \in \mathbb{R}^d$ it holds that $||v||_p \leq \sqrt{d}||v||_q$.

Lemma 2. *Let* $A, B \in \mathbb{W}^d$ *be two weighted point sets and* p^* *be any point. Let* $Rot(p^*)$ *be the set of all rotations around* p^*. *Then there is a rotation* $R' \in Rot(p^*)$ *such that* R' *aligns* p^* *and any two points of* A *and* B, *and*

$$EMD_p(A, R'(B)) \leq 2\sqrt{d} \cdot \min_{R \in Rot(p^*)} EMD_p(A, R(B)).$$

An Applicable Algorithm in the Plane Based on the last two lemmata we are able to construct an approximation algorithm for the problem of finding an optimal rotation of a weighted point set around their coinciding reference points. In this section we will discuss the case of weighted point sets in the plane.

Algorithm *RotationApx*
1. Compute the minimum EMD over all possible alignments of the coinciding reference points and any two points of A and B.

Since there are $O(nm)$ possibilities to align the reference point and any two points of A and B, the runtime of this algorithm is $O(nmT^{EMD}(n,m))$. Using this algorithm, we now get an easy to implement and fast approximation algorithm for rigid motions. Unfortunately, the fact that we now constructed an implementable algorithm must be paid by an increased approximation factor. Figure 1 shows an illustration of this algorithm.

Algorithm *RigidMotionApxUsingRotationApx*
1. Compute $r(A)$ and $r(B)$ and translate B by $r(A) - r(B)$. Let B' be the image of B.
2. Find a best matching of A and B' under rotations of B' around $r(A) = r(B')$ where $r(A)$ and any two points in A and B' are aligned. Let B'' be the image of B' under this rotation.
3. Output B'' and the approximate distance $EMD(A, B'')$.

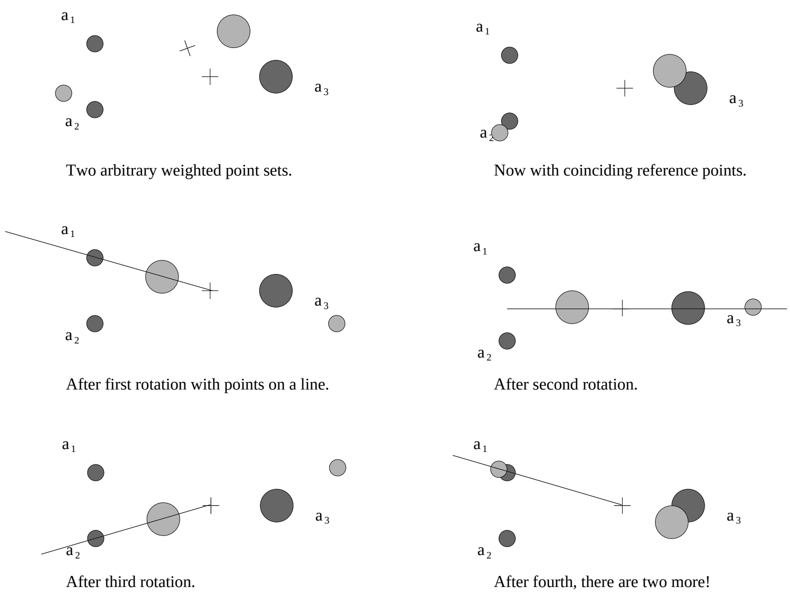

Fig. 1. Illustration of algorithm *RigidMotionApxUsingRotationApx*

Theorem 6. *Regarding EEMD in the plane, Algorithm RigidMotionApxUsingRotationApx finds an approximately optimal matching for rigid motions with approximation factor* $2(c+1)$ *in time* $O(T^{ref}(\max\{n,m\}) + nmT^{EEMD}(n,m))$.

Using Lemma 2 we can extend the result to all L_p-distances:

Theorem 7. *Regarding* EMD_p *in the plane,* $1 \leq p \leq \infty$*, Algorithm RigidMotionApxUsingRotationApx finds an approximately optimal matching for rigid motions with approximation factor* $2\sqrt{2}(c+1)$ *in time* $O(T^{ref}(\max\{n,m\}) + nmT^{EMD_p}(n,m))$.

Application of the center of mass as an EMD-reference point leads to the following corollary:

Corollary 3. *Algorithm RigidMotionApxUsingRotationApx using the center of mass as EMD-reference point induces an approximation algorithm with approximation factor* 4 *in case of the Euclidean Distance in the plane and* $4\sqrt{2}$ *for any other* L_p *distance,* $1 \leq p \leq \infty$*. Its runtime is* $O(nmT^{EMD_p}(n,m))$.

In this section we have constructed approximation algorithms to minimize the EMD of weighted point sets under rigid motions in the plane. These algorithms have elegant generalizations to higher dimensions. In case of dimension $d \geq 3$, the approximation factor of the implementable algorithm using the center of mass as a reference point is $2^{d-1}\sqrt{d}(c+1)$ and the runtime is $O(n^d T^{EMD})$ if $m = O(n)$, see [8].

4.3 Similarities

In the following we present approximation algorithms for matching weighted point sets under similarity transformations, i.e. combinations of translations, rotations and scalings. More precisely, we want to compute $\min_S EMD(A, S(B))$, where the minimum is taken over all similarity transformations S. Note that in this case exchanging A and B makes a difference.

Algorithm *SimilarityApx*:
1. Compute $r(A)$ and $r(B)$ and translate B by $r(A) - r(B)$. Let B' be the image of B.
2. Scale B' by $\frac{m_{r(A)}(A)}{m_{r(B')}(B')}$ around $r(A)$ and let B'' be the image of B' under this scaling.
3. Find an optimal matching of A and B'' under rotations of B'' around $r(A)$. Let B''' be the image of B'' under this rotation.
4. Output B''' and the approximate distance $EMD(A, B''')$.

To show the correctness of this algorithm we use the following two lemmata:

Lemma 3. *Let* $A \in \mathbb{W}^{d,G}$ *for some* $G \in \mathbb{R}^+$ *and let* $m_p(A)$ *be its normalized first moment with respect to some point* $p \in \mathbb{R}^d$*. Let* τ_1, τ_2 *be scalings around the same center* p *and ratios* γ_1 *and* γ_2*, respectively. Then*

$$EMD(\tau_1(A), \tau_2(A)) \leq |(\gamma_1 - \gamma_2)| m_p(A).$$

This lemma gives a new lower bound for the EMD of weighted point sets:

Lemma 4. *Let $A, B \in \mathbb{W}^{d,G}$ for some $G \in \mathbb{R}^+$. Then*

$$|m_{r(A)}(A) - m_{r(B)}(B)| \leq (1+c)EMD(A,B).$$

Using Lemmata 3 and 4 we can prove the following:

Theorem 8. *Algorithm SimilarityApx finds an approximately optimal matching for similarities with approximation factor $2(c+1)$ in time $O(T^{ref}(\max\{n,m\}) + T^{EMD}(n,m) + T^{rot}(n,m))$. This holds for arbitrary dimension d and distance measure on the ground set.*

Corollary 4. *Algorithm SimilarityApx using the center of mass as EMD-reference point induces an approximation algorithm with approximation factor 4. Its runtime is $O(T^{EMD}(n,m) + T^{rot}(n,m))$. This holds for any dimension of the ground set and every distance measure defined on it.*

As for *RigidMotionApx*, *SimilarityApx* depends on finding the optimal rotation, which is impractical. Again, we make this algorithm practical and efficient by using *RotationApx* and again we have to pay by a worse approximation factor.

Algorithm *SimilarityApxUsingRotationApx*

1. Compute $r(A)$ and $r(B)$ and translate B by $r(A) - r(B)$. Let B' be the image of B.
2. Scale B' by $\frac{m_{r(A)}(A)}{m_{r(B')}(B')}$ around $r(A) = r(B')$ and let B'' be the image of B' under this scaling.
3. Find a best matching of A and B'' under rotations of B'' around $r(A) = r(B'')$ where $r(A)$ and any two points in A and B'' are aligned. Let B''' be the image of B'' under this rotation.
4. Output B''' and the approximate distance $EEMD(A, B''')$.

Theorem 9. *Regarding EEMD in the plane, Algorithm SimilarityApxUsingRotationApx finds an approximately optimal matching for similarities with approximation factor $4(c+1)$ in time $O(T^{ref}(\max\{n,m\}) + nmT^{EEMD}(n,m))$.*

Using Lemma 2 we can easily see:

Theorem 10. *Regarding EMD_p in the plane, $1 \leq p \leq \infty$, Algorithm SimilarityApxUsingRotationApx finds an approximately optimal matching for similarities with approximation factor $4\sqrt{2}(c+1)$ in time $O(T^{ref}(\max\{n,m\}) + nmT^{EMD_p}(n,m))$.*

Application of the center of mass as an EMD-reference point leads to the following corollary:

Corollary 5. *Algorithm SimilarityApxUsingRotationApx using the center of mass as an EMD-reference point induces an approximation algorithm with approximation factor 8 in case of the Euclidean Distance in the plane and $8\sqrt{2}$ for any other L_p distance, $1 \leq p \leq \infty$. Its runtime is $O(nmT^{EMD_p}(n,m))$.*

5 Conclusion

In this paper we introduced EMD-reference points for weighted point sets and constructed efficient approximation algorithms for matching under various classes of transformations. In contrast to previous work, this approach allows elegant extension to higher dimensions and more general ground distances. Additionally, we presented the center of mass as an EMD-reference point for weighted point sets with equal total weight. This reference point, in fact, turns out to be an optimal reference point in the sence that there is none with a Lipschitz-constant smaller than 1. Unfortunately, the center of mass is no EMD-reference point if you consider the set of all weighted point sets, including those with different total weights. Even worse, there is no EMD-reference point for all weighted point sets. A variation of the EMD is the Proportional Transportation Distance (PTD). We can show that the center of mass is a PTD-reference point even for weighted point sets with different total weights and all theorems and corollaries mentioned in this paper carry over.

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