Dialogue Games for Inconsistent and Biased Information

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Abstract

A dialogue game is presented that describes coherent conversational sequences with inconsistent and biased information between agents at the speech act level. These types of information are represented with the use of a multi-valued logic based on a bilattice structure. This approach makes it possible to devise a dialogue game in which agents can deliberate their cognitive states with inconsistent and biased information. An example dialogue is analysed by describing: a) the agent’s cognitive state as a set of multi-valued theories, b) dialogue rules that define applicable communicative acts based on the agent’s cognitive state, and c) update rules that change the agent’s cognitive state as a result of communicative acts. We show that an example dialogue with biased and inconsistent information can be described with our dialogue game in a consistent manner.

1 Introduction

Even in simple conversation agents often need to convey information with different modalities, e.g. alethic modalities of necessarily or contingently true and false, or probabilistic modalities of certainly or possibly true and false. A different type of modality is epistemic inconsistency and ‘biasedness’ of statements. For example, an agent may have evidence to believe something and may also have an equal amount of evidence to believe the contrary. In this case, the agent’s belief state is inconsistent. A belief state is called biased when more evidence exists to believe than to disbelieve something or \textit{vice versa}. How to cope with this contradictory and biased information in dialogues?

In Gärdenfors [5], an epistemic description of a belief state is given with epistemic commitment functions in which, given an epistemic input, changes

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of a belief state are described. A similar approach in the context of agents is
given in Beun [2], who analyses locutions between agents and identifies three
structures with accompanying properties forming a dialogue game that agents
need to possess to communicate in a sensible way. Agents need to have a cog-
nitive state to represent the information of e.g. their belief state. In addition,
agents need to have dialogue rules that convey information to other agents
and update rules that process incoming information. In the same vein of the
FIPA work on ACLs, we are trying to discover the semantics of speech acts.
In Labrou [9], a modal logic is used to describe the agent interactions. We
introduce and use a multi-valued logic makes it possible to focus solely on in-
consistent and biased information transfer in conversation. Another approach
to define the semantics of dialogue games is given in McBurney [10]; they use
a geometric semantics to compare dialogue game protocols.

In real life conversations, humans can state inconsistent or biased informa-
tion without any difficulty. Inconsistent locutions are observed when informa-
tion of different locutions or one single locution is contradictory, e.g. when it
is said to be true and false that it rains outside. Information is biased when
an agent has more evidence to believe a statement than to disbelieve it, or
*vice versa*. Such an information state may be inconsistent but biased in the
true or false directions. What is still lacking in an analysis of dialogue games
in multi-agent systems is a description of these epistemic modalities.

Our objective is to show that inconsistent and biased information can be
dealt with consistently in dialogue games. To do this, we introduce truth-
values from bilattice structures [6] [4] in all aspects of the dialogue game.
As a result, the cognitive state of the agent is formulated as a set of multi-
valued (logical) theories based on the bilattice structure. The update and
dialogue rules are refined to cope with the new types of information. In this
approach, agents can deal with monotonic information addition leading to
possible inconsistent information states without belief revision. An example
dialogue is proven to be semantically valid in our dialogue game.

A fictitious dialogue from Sesame Street serves as a running example. Elmo
wants to know (which is not shown) whether he can go play outside. He states
this in a question² to Oscar (line 1); the answer is distributed between Tv,
Grover and Oscar, and, after a number of communicative acts, Oscar gives
two answers (line 5 and 7). Note that the information becomes inconsistent
and that no belief revision occurs.

**Dialogue 1 (Sesame Street)**

1. Elmo to Oscar ‘Hi Oscar, do you know whether I can go play outside?’
2. Oscar to Tv ‘Hey Tv, is it dry outside?’
3. Oscar to Grover ‘Grover, are you wearing a raincoat?’

² Please notice that the questions in the example dialogue are confined to simple ones that
can be answered with a yes or no. Questions with answers from an enumerable domain,
e.g. ‘what type of weather is it outside?’ could be addressed by splitting up the question
in a finite number of yes or no type of questions.
4.  Tv to Oscar  ‘I think it is dry outside, but I'm not sure.’
5.  Oscar to Elmo  ‘Hey Elmo, maybe you can go play outside.’
6.  Grover to Oscar  ‘Yes Oscar, I’m wearing a raincoat.’
7.  Oscar to Elmo  ‘Elmo! You cannot play outside!
   But on the other hand, maybe you can.’

In Sec. 2, inconsistent and biased information is formalised using bilattice structures. In addition, a multi-valued logic based on these structures is described. In Sec. 3, our dialogue game based on bilattice structures and multi-valued logics is presented. The example dialogue is translated into a formal one in which the contents of the conversation is presented with sentences from the multi-valued logic. We conclude with the formal analysis of the dialogue game and the example dialogue in Sec. 4.

2  Representing Information with Bilattice Structures

Stating inconsistent information does seem to be necessary when agents cannot make up their mind or when they are not in a position to resolve the inconsistency. To cope with this type of information in multi-agent systems, a theoretically sound way is needed to represent and reason with inconsistent and biased information.

In classical logic, a valuation of a statement can be true or false depending on a particular world. In multi-valued logics new truth-values are introduced to represent epistemic attitudes towards this world. In this section, truth-values representing unknown and inconsistent information states are described. Furthermore, an infinite number of truth-values are introduced to represent biased information states. A biased information state is epistemically between true and false, the value ‘biased true’ is, for example, closer to true than to false. In addition, a biased information state has a level of information between unknown and inconsistent.

In the example dialogue, classical information is used when Oscar asks Tv whether it is true that it rains outside (line 2). This information state is represented with truth-value true. The answer conveys a biased information state, because, when Tv says that it rains outside and also that he is not sure (line 4), Tv states that his epistemic state is biased to true that it rains outside. An inconsistent information state occurs when Oscar says that Elmo cannot play outside and also that he maybe can (line 7). The information in this utterance is biased to false, because Oscar says that it is false and partially true that Elmo can play outside. Of course, this information is not very helpful to Elmo. Elmo needs a method to handle incoming (inconsistent) information, if he is to decide later what to do with the information.

2.1  Bilattice Structure

Before the bilattice structure is described other general notions are presented. Given the set $P$ ordered according to $\leq$ and $S \subseteq P$, an element $x \in P$ is an
The following equations:

and

the structure orders are defined as follows (given

Given two complete lattices

Definition 2.2 (bilattice) Given two complete lattices \( \langle B, \leq \rangle \) and \( \langle D, \leq \rangle \), the structure \( B(B, D) = \langle B \times D, \leq_k, \leq_i \rangle \) is a complete bilattice if the partial orders are defined as follows (given \( \sim \subseteq B \times D \)): \( b_1 \sim d_1 \leq_k b_2 \sim d_2 \) if \( b_1 \leq_1 b_2 \) and \( d_1 \leq_2 d_2 \), and \( b_1 \sim d_1 \leq_i b_2 \sim d_2 \) if \( b_1 \leq_1 b_2 \) and \( d_2 \leq_2 d_1 \), cf. [4]. To each ordering is associated join (\( \oplus \)) and meet (\( \otimes \)) operations according to the following equations:

Knowledge ordering \( (\leq_k) \)

\[
\begin{align*}
    b_1 \sim d_1 \otimes_k b_2 \sim d_2 &= (b_1 \cap_1 b_2) \sim (d_1 \cap_2 d_2) \\
    b_1 \sim d_1 \oplus_k b_2 \sim d_2 &= (b_1 \cup_1 b_2) \sim (d_1 \cup_2 d_2)
\end{align*}
\]

Truth ordering \( (\leq_i) \)

\[
\begin{align*}
    b_1 \sim d_1 \otimes_i b_2 \sim d_2 &= (b_1 \cap_1 b_2) \sim (d_1 \cap_2 d_2) \\
    b_1 \sim d_1 \oplus_i b_2 \sim d_2 &= (b_1 \cup_1 b_2) \sim (d_1 \cup_2 d_2)
\end{align*}
\]

The intuition is that \( B \) provides the evidence for believing a statement and \( D \) provides evidence to disbelieve a statement. We assume \( B \) and \( D \) both have at least two elements: 0 for lack of evidence or disbelief respectively, and 1 for maximal evidence or disbelief respectively. Consequently, the bilattice has at least four truth-values: 0\sim 0, 1\sim 0, 0\sim 1 and 1\sim 1, abbreviated with \( u, t, f \) and \( i \) respectively. With truth-value \( t \) (1\sim 0) full evidence for believing and no doubt for a statement is represented; this is considered the orthodox ‘true’ from classical logic. With \( f \) (0\sim 1) no evidence for believing but maximal doubt exists, this is the orthodox ‘false’. In \( u \) (0\sim 0) neither evidence for believing nor for doubting exists, i.e. information is lacking completely. In \( i \) both maximal evidence for believing and for doubting exists, i.e. information is inconsistent. These truth-values do not represent fact-related, ontological uncertainty or inconsistency but epistemic attitudes towards the world.

The intuitive space of truth-values is ordered according to \( \leq_i \), describing differences in the ‘measure of truth’, i.e. the evidence about the truth of a particular statement. If \( b_1 \sim d_1 \leq_i b_2 \sim d_2 \), then we have more reason to believe

\(^3\) which means that it is defined like the upper bound with reversed order.
in situation 2 than in 1, because the reasons for believing the statement either increases or the reasons against it are weaker, i.e. $b_2 \sim d_2$ is ‘more true’ than $b_1 \sim d_1$. In the example dialogue, Grover says to Oscar that it is true that he is wearing a raincoat (line 6), a different answer with lower degree of truth is, ‘No, I’m not wearing a raincoat’.

However, truth-values can also be ordered according to $\leq_k$ describing the ‘measure of information’ about a statement. If $b_1 \sim d_1 \leq_k b_2 \sim d_2$, then situation 2 has more information than 1, i.e. knowledge is increased in truth or falsity. In the example dialogue, line 5 has less information than line 7; this is discussed next. The two orderings behave differently under negation, viz. a statement that is true becomes false after being negated, and a statement in which nothing is known remains unknown after being negated, cf. [6]. We do not use negation and leave it aside. The upper bound of $\leq_k$ is denoted $\downarrow_k$, analogous for the other three combinations.

The smallest complete bilattice is presented graphically in Fig. 1(a) and is, in fact, a well-known four-valued logic introduced by Belnap [1]. The meet, join and negation operations associated with the $\leq_t$ ordering confined to $\{f, t\}$ are those of classical logic. When $u$ is added to these truth-values, Kleene’s strong three-valued logic [8] is obtained.

Informally, the greatest lower bound $\theta \otimes_k \vartheta$ can be thought of as the truth-value representing information which is mutual in $\theta$ and $\vartheta$ or the consensus of $\theta$ and $\vartheta$, e.g. $f \otimes_k t = u$. Likewise, the least upper bound $\theta \oplus_k \vartheta$ is thought of as the information that results after combination of both $\theta$ and $\vartheta$, e.g. $f \oplus_k t = i$, see Fig. 2(a). $\theta \otimes_t \vartheta$ is the least amount of truthfulness, e.g. $f \otimes_t t = f$; and likewise is $\theta \oplus_t \vartheta$ the most amount of ‘truthness’, e.g. $f \oplus_t t = t$, see Fig. 2(b).

### 2.2 Biased Information with Bilattices

An information state is biased if there is more evidence to believe a statement than to doubt it or, \textit{vice versa}, if there is more disbelief than evidence supporting a statement. In the former, the information state is biased true, in the latter, is said to be biased false. The biased information corresponding with $1 \sim 0.5$ states that full evidence to believe a statement exists, while the doubt about it is partial, so that in effect true is biased over false. See Fig. 1(b), for a bilattice with biased information states.

When Oscar says to Elmo that he maybe can go play outside (line 5),
or, stated differently, when Oscar makes a statement in which it is partially true that Elmo can go play outside, we denote this information state with 0.5~0. Later Oscar says to Elmo that he cannot go play outside, but, on the other hand, maybe he can (line 7). Oscar has a statement with inconsistent information in which it is false that Elmo can go play outside and partially true that he can; this information state is denoted 0.5~1.

Some statements are more general than others. The information in the less general statements is subsumed under the information of the more general ones. For example, when Oscar says to Elmo that he maybe can go play outside (line 5, 0.5~0), this information state is subsumed under the information of the sentence in which Oscar says to Elmo that he cannot play outside and also that he maybe can (line 7, 0.5~1). Formally, we have 0.5~0 ≤ₖ 0.5~1.

2.3 Multi-valued Logics

Sentences from a multi-valued logic (MVL) are constructed in a fashion that is considered truth-value bearing, capable of being the object of belief or ignorance. In Sec 3.2, these sentences represent the epistemic contents of an agent’s attitude and theories of MVL represent mental constructs.

**Definition 2.3 (language of MVL)** Given complete bilattice $\mathcal{B} = (B \times D, \leq_k, \leq_t)$ and ontology $^4 \mathcal{O}$, the language of MVL $\mathcal{L}^\mathcal{B}$ is the smallest set satisfying:

(i) if $\psi \in \mathcal{O}$ and $\theta \in B \times D$ then $\psi: \theta \in \mathcal{L}^\mathcal{B}$, (atomic sentence)

(ii) if $\nu, \mu \in \mathcal{L}^\mathcal{B}$ and $\theta \in B \times D$ then $(\nu \rightarrow \mu): \theta \in \mathcal{L}^\mathcal{B}$. (conditional sentence)

Atomic sentences consist of a propositional formula from an ontology assigned a truth-value from a bilattice; sentence $\psi: \theta$ is read as: ‘$\psi$ has at least truth-value $\theta$’ (with respect to $\leq_k$). Conditional sentences resemble the conditionals from classical logic. Remark that the truth-values of the antecedent and consequent are embedded in the conditional and that other connectives like the ‘or’ or ‘not’ are not defined. A theory of MVL is defined as a set of sentences from the language of MVL with the following semantic properties.

**Definition 2.4 (Theory of MVL)** Given a language of MVL $\mathcal{L}^\mathcal{B}$, a theory

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$^4$ For our current purpose, an ontology is nothing more than a set of concepts with unique propositional formulas identifying the concepts.
of MVL $T^B \subseteq L^B$ is the smallest set of sentences satisfying:

(i) if $\psi : \theta \in L^B$ then $\psi : u \in T^B$,  
    \hspace{1cm} \text{(founded information)}
(ii) if $\psi : \theta \in T^B$ and $\theta \leq_k \vartheta$ then $\psi : \vartheta \in T^B$,  
    \hspace{1cm} \text{(subsumed information)}
(iii) if $\psi : \theta, (\psi : \theta \rightarrow \varphi : \vartheta) t \in T^B$ then $\varphi : \vartheta \in T^B$.  
    \hspace{1cm} \text{(implied information)}

A dual theory of MVL $T^{B0} \subseteq L^B$ is the smallest set of sentences satisfying:

(i) if $\psi : \theta \in L^B$ then $\psi : i \in T^{B0}$,  
    \hspace{1cm} \text{(dual founded information)}
(ii) if $\psi : \theta \in T^{B0}$ and $\theta \leq_k \vartheta$ then $\psi : \vartheta \in T^{B0}$.  
    \hspace{1cm} \text{(dual subsumed information)}

$T^B$ is complete if $\psi : \theta \in T^B$ and $\psi : \theta \oplus_k \vartheta \in T^B$.

$T^{B0}$ is complete if $\psi : \theta \in T^{B0}$ and $\psi : \theta \otimes_k \vartheta \in T^{B0}$.

Truth-value assignment $\psi : \theta$ states that at least (with respect to $\leq_k$) evidence $\theta$ holds for $\psi$. Consequently, the total lack of information associated with truth-value $u$ always applies to all sentences in a theory. This least possible amount of information provides the founding lower bound of information a theory encodes. The reading of the truth-value assignment enforces that all truth-values below a truth-value from a sentence already part of a theory (with respect to $\leq_k$), are also part of the theory. Information in these sentences is said to be subsumed. Implied information results from a semantics generalisation of the classical implication: if a conditional has a designated truth-value and the antecedent of the conditional is part of theory, then so is the consequent, cf. [11]. Only $t$ is considered designated because other truth-values have no clear intuitive reading for conditionals. Dual founded information and dual subsumed information are dual counterparts of the normal theory: a dual theory has a reversed $\leq_k$ order. Conditionals for a dual theory can be defined in a similar way, but in our use to represent ignorance (Sec. 3.2) there is no need for dual implied information. The upper bound of a theory, $\text{up}_k(T^B)$, is defined as the set of sentences from $T^B$ with a truth-value that is the upper bound with respect to $\leq_k$ for all sentences of the theory. Formally, $\text{up}_k(T^B) = \{ \psi : \theta \in T^B \mid \theta \in \text{up}_k(\{ \psi : \theta \in T^B \}) \}$. The other three combinations are formalised analogously.

In the following section, we need the notion of prerequisite. If a background theory $T_1$ is extended to include $T_2$ by adding $T_3$ set theoretically to background theory $T_1$, then $T_3$ is called a prerequisite for extension $T_2$. $\mathcal{E}_{T_1}(T_2)$ is the set of prerequisites to obtain extension $T_2$ from the background theory $T_1$. Formally:

**Definition 2.5 (Prerequisite set)** Given a closed theory of MVL $T_1 \subseteq L^B$ and a set of sentences $T_2 \subseteq L^B$, a prerequisite set $\mathcal{E}_{T_1}(T_2)$ is the set of complete theories of MVL satisfying: if $T_3 \subseteq L^B$ and $T_2 \subseteq T_1 \cup T_3$ then $T_3 \in \mathcal{E}_{T_1}(T_2)$.
3 Agent Dialogues

The dialogue game by Beun [2] is refined by making it possible for agents to communicate about inconsistent and biased information. In this paper, a dialogue is a sequence of communicative acts governed by a set of dialogue rules. A dialogue game defines a set of different dialogues, prescribed by a set of dialogue rules, a set of update rules and the agents’ initial cognitive states.

For an agent to engage in conversation, it needs to have the following capabilities. In Sec. 3.3, dialogue rules are presented that describe applicable communicative acts (Sec. 3.1) given an agent’s cognitive state (Sec. 3.2). Second, in Sec. 3.4, update rules or epistemic commitment functions are presented that prescribe a cognitive state given a communicative act and a cognitive state.

3.1 Communicative Acts

We take a communicative act to be utterances at the speech act level used by agents to manifest (parts of) their cognitive state, cf. [7]. Communicative acts to grant requests for a belief addition or to set up obligations, etc. are not discussed. In the following, agents use the language of MVL based on a static ontology in three different communicative acts: questions, statements of belief and statements of ignorance, tagged with the following markers: ？, ？+, and ？− respectively.

A question is a request for a belief addition, i.e. an agent x asks an agent y whether sentence ψθ may be added to x’s belief, [x, y, ψθ]？. In a statement of belief, an agent x states to y that ψθ is part of x’s belief and that y may add this to its belief state, [x, y, ψθ]？+. A belief statement can be an approval of a request for a belief addition. On the other hand, this request can also be denied, which is in effect a statement of ignorance, i.e. an agent x states to y that x is ignorant about ψθ, [x, y, ψθ]？−.

Both questions and ignorance statements express the agent’s lack of evidence supporting the content sentence. The difference between these two acts is that in the former, the agent (implicitly) requires an answer; with ignorance statements an agent does not. Only additions of content sentences are considered; consequently, communicative acts only result in expansions of the cognitive state. The three communicative acts have their duals for derogations of content sentences, which would result in contractions of information from the agent’s cognitive state [5], but these have not been analysed yet.

An ontology O with \{p, r, c\} ⊆ O is needed to formalize the Sesame Street example. The propositional formula r denotes the concept ‘it rains outside’, or stated differently, ‘it is not dry outside’, c denotes ‘Grover is wearing a raincoat’, and p denotes ‘Elmo can go play outside’. The assignment r:f denotes ‘it is (at least with respect to ≤k) false that it rains outside’, or equivalently, ‘it is (at least) true that it is dry outside’. Question [elmo, oscar, p:t]？ is a shorthand for the sequential uttering of [elmo, oscar, p:t]？ and [elmo, oscar, p:f]？.
The example dialogue can now be represented by the following sequence of communicative acts.

**Dialogue 2 (Sesame Street)**

1. \([\text{elmo, oscar, p}\sim 0.0\sim 0.1]\) ‘Hi Oscar, do you know whether I can go play outside?’
2. \([\text{oscar, tv, r}\sim 0.0\sim 0.1]\) ‘Hey Tv, is it dry outside?’
3. \([\text{oscar, grover, c}\sim 0.1\sim 0.0]\) ‘Grover, are you wearing a raincoat?’
4a. \([\text{tv, oscar, r}\sim 0.5\sim 0.1]?+\) ‘I think it is dry outside,’
4b. \([\text{tv, oscar, r}\sim 0.0\sim 0.1]?−\) ‘but I’m not sure (that it is dry outside).’
5. \([\text{oscar, elmo, p}\sim 0.5\sim 0.1]?+\) ‘Hey Elmo, maybe you can go play outside.’
6. \([\text{grover, oscar, c}\sim 0.1\sim 0.0]?+\) ‘Yes I’m wearing a raincoat.’
7. \([\text{oscar, elmo, p}\sim 0.5\sim 1.0]?+\) ‘Elmo! You cannot play outside!
   
   But on the other hand, maybe you can.’

3.2 The Agent’s Cognitive State

In our dialogue game, an agent’s cognitive state consists of a set of mental constructs which are theories of MVL expressing the agent’s attitude towards aspects of its world. Three different mental constructs are used, viz. the agent’s belief, desire and ignorance. In addition, two types of mental construct are distinguished: private constructs represent the agent’s own attitudes and manifested constructs represent other agents’ attitudes that have been communicated explicitly. For example, formula \(r\) represents ‘it rains outside’, but whether Tv believes this to be the case depends on the truth-value assigned to \(r\) that is part of Tv’s private mental construct known as its belief. Oscar is a priori not aware that Tv believes this to be the case, therefore this assignment is not part of Oscar’s manifested belief of Tv. Apart from that, Oscar is also a priori not aware that Tv is ignorant about this, and subsequently this assignment is not part of Oscars manifested ignorance of Tv.

**Private belief** \(B_x \subseteq \mathcal{L}^B\) is a complete theory of MVL that represents \(x\)’s belief. \(\psi: \theta\) in \(B_x\) states that \(x\) believes that formula \(\psi\) has at least truth-value \(\theta\). By default, an agent does not believe anything, i.e. for all \(\psi: \theta\) in \(B_x\) holds that \(\theta\) equals \(u\).

**Private desire** (to believe) \(D_x \subseteq \mathcal{L}^B\) is a theory of MVL that represents \(x\)’s desire. \(\psi: \theta\) in \(D_x\) states that \(x\) desires \(\psi: \theta\) to be part of its private belief. By default, an agent does not desire anything, i.e. for all \(\psi: \theta\) in \(D_x\) holds that \(\theta\) equals \(u\).

Private desire is not a complete theory. As a result, the upper bound \(\text{up}_{k}(D_x)\) can have more than one truth-value for a propositional formula. This happens when, for example, Elmo likes to know whether he can go play outside (line 1); he is interested to believe that he can go play outside or that he cannot. Elmo is not interested in the inconsistent state of believing both. We chose to represent this ‘or’ in the desire state with a non-complete theory. Perhaps one would like to represent this disjunction also in the object language by introducing the semantics of the or-operator; this is future research. An agent’s private ignorance is not represented because it equals those sentences not part of the agent’s private belief state. Another cognitive state of interest
which is not discussed is the dual of the private desire (to believe), i.e. the private desire to be ignorant, which would motivate questions to retract a proposition from the speaker’s belief state.

**Manifested belief** $M_x \mathcal{B}_y \subseteq \mathcal{L}^B$ is a complete theory of MVL that represents the manifested beliefs of $y$ which $x$ is aware of. $\psi : \theta$ in $M_x \mathcal{B}_y$ states that $x$ is aware that $y$ has private belief $\psi : \theta$. By default, an agent is not aware of beliefs of other agents, i.e. for all $\psi : \theta$ in $M_x \mathcal{B}_y$ holds $\theta$ equals $u$.

**Manifested ignorance** $M_x \mathcal{I}_y \subseteq \mathcal{L}^B$ is a dual theory of MVL that represents the manifested ignorance of $y$ which $x$ is aware of. $\psi : \theta$ in $M_x \mathcal{I}_y$ states that $x$ is aware that $y$ is ignorant about $\psi : \theta$. By default, an agent is not aware of the ignorance of other agents, i.e. for all $\psi : \theta$ in $M_x \mathcal{I}_y$ holds that $\theta$ equals $i$.

Manifested ignorance is not the set theoretic complement of manifested belief, because the ignorance of other agents needs to be manifested explicitly and cannot be derived from manifested beliefs. For example, if agent $x$ has stated $\psi : \theta$ to agent $y$, $y$ may not assume that $x$ is ignorant about the complement of the sentence because $x$ may have stated only parts of its belief. An agent $x$ can never be completely aware what an agent $y$ believes, because yet another agent $z$ may have stated $\psi : \theta$ to $y$ just after $y$ has stated $\psi : \vartheta$ (with $\vartheta \leq_k \theta$) to $x$. Notice that sentences may be part of both the manifested ignorance and the manifested belief. This is encountered when an agent $x$ first poses a question to $y$ asking for $\psi : \theta$ and later $y$ states $\psi : \vartheta$ (with $\theta \leq_k \vartheta$) to $x$, then $\psi : \theta \in M_x \mathcal{I}_y$ and also $\psi : \vartheta \in M_y \mathcal{B}_x$.

**Manifested desire** $M_x \mathcal{D}_y \subseteq \mathcal{L}^B$ is a theory of MVL that represents the manifested desires of $y$ which $x$ is aware of. $\psi : \theta$ in $M_x \mathcal{D}_y$ states that $x$ is aware that $y$ has desire $\psi : \theta$. By default, an agent does not believe anything about the desires of another agent, i.e. for all $\psi : \theta$ in $M_x \mathcal{D}_y$ holds that $\theta$ equals $u$.

**Manifested-manifested ignorance** $M_x \mathcal{M}_y \mathcal{I}_x \subseteq \mathcal{L}^B$ is a dual theory of MVL that represents the manifested ignorance of $x$ which $x$ is aware of that $y$ is aware of. $\psi : \theta$ in $M_x \mathcal{M}_y \mathcal{I}_x$ states that $x$ is aware that $y$ is aware that $x$ is ignorant about $\psi : \theta$. By default, an agent is not aware about manifested ignorance of other agents, i.e. for all $\psi : \theta$ in $M_x \mathcal{M}_y \mathcal{I}_x$ holds that $\theta$ equals $i$.

### 3.3 Dialogue Rules

Dialogue rules define which communicative acts are applicable (denoted $\sim$) in a dialogue game. Applicability of a communicative act is a semantic relation
between the agent’s cognitive state and a speech act, and not an analytical relation that enforces question-answer pairs. If different acts are simultaneously applicable, agents need to make a choice.

Different motivations to communicate can be identified. First, the agent’s incentive to reduce the imbalance between a desire and a belief state, in the sense that an agent desires (to believe) a sentence which is not part of its belief state [2]. We denote this agent to be interested to add such a sentence. Other motivations are based on the pragmatics of the Gricean maxims of cooperation [7]. In a nutshell, these maxims state that, 1) agents are neither allowed to utter information that is not part of their belief state, 2) nor are they allowed to ask for information that they already believe or are aware of that the other is ignorant about, and in addition, 3) questions always need to be answered. In this sense, applicability is a normative relation that all agents adhere to. Yet another motivation can be to reduce inconsistency in a belief state as described in the theory of belief revision [5]. The following dialogue rules respect the Gricean maxims and balance the belief/desire state.

3.3.1 Questions

Two roles of questions are distinguished: first, to reduce the imbalance between a desire and a belief state, i.e. when an agent is interested in a sentence, and second, when an agent considers another agent interested. This is described in Section 3.3.3. An agent is said to be interested in $\psi : \theta$ when $\psi : \theta \in \text{up}_k(D_x)$ and $\psi : \theta \not\in B_x$.

Questions are restricted to be sensible by the Gricean maxims of cooperation, i.e., an agent is not allowed to pose questions if it is aware that the addressed agent cannot answer them. A question from $x$ to $y$ is sensible if the sentence of the question is not part of $y$’s ignorance as $x$ is aware of, formally, $\psi : \theta \not\in M_x I_y$. In addition, a question needs to be fresh, i.e. $x$ is not allowed to pose a question with the same information more than once because $x$ may assume the other agent $y$ is already aware that $x$ is interested in the sentence, formally, $\psi : \theta \not\in M_x M_y I_x$. A question from $x$ to $y$ is applicable when $x$ is interested in a sentence and this sentence is sensible and fresh for $y$.

**Definition 3.1 (question)** If $\psi : \theta \in \text{up}_k(D_x)$, $\psi : \theta \not\in B_x$, $\psi : \theta \not\in M_x I_y$ and $\psi : \theta \not\in M_x M_y I_x$ then a question $[x, y, \psi : \theta]$ is applicable, denoted $\Rightarrow$.

3.3.2 Belief Statements

Three dialogue rules that define applicability of belief statements are identified. An agent $x$ assumes another agent $y$ to be interested in a sentence $\psi : \theta$ when $x$ is aware that, $y$ desires the sentence, and second, that $y$ is ignorant about this sentence. Formally, $\psi : \theta \in \text{up}_k(M_y D_y)$ and $\psi : \theta \not\in M_y B_y$. All agents are sincere and can, consequently, only state sentences that are part of their private belief. Formally, $\psi : \theta \in B_x$.

Depending on the cognitive state of the agents, belief statements can have different roles in a dialogue. That is, a belief statement can be a partial, over-informative or minimal answer to a question. A belief statement from $x$ to $y$
is applicable as a partial answer when \( x \) considers \( y \) interested in a sentence, but this sentence does not (necessarily) balance the belief and desire state of \( y \). The answering agent is sincere.

**Definition 3.2 (partial answer)** If \( \psi: \theta \in M_x D_y, \psi: \theta \not\in M_x B_y \) and \( \psi: \theta \in B_x \) then belief statement \([x, y, \psi: \theta]^2\) is applicable with the role of a partial answer, denoted \( \sim \Rightarrow_{+1} \).

Partial answers do not necessarily balance the belief and desire state because there may exist sentences that are part of agent’s desire state that are not subsumed under the sentence from the (partial) answer. On the other hand, agents can also state sentences in which all sentences of interest are subsumed, but these sentences may have more information than asked for, resulting in an over-informative answer.

**Definition 3.3 (over-informative answer)** If \( \psi: \theta \in M_x D_y, \psi: \theta \not\in M_x B_y \) and \( \psi: \theta \not\in B_x \) then belief statement \([x, y, \psi: \theta]^2\) is applicable with the role of an over-informative answer, denoted \( \sim \Rightarrow_{+2} \).

If an agent states a sentence that is part of the upper bound of a manifested desire state, then this statement is less informative than that of an over-informative answer. Nevertheless, it balances the desire and belief state without stating more information than asked for. This belief statement has the role of a minimal answer.

**Definition 3.4 (minimal answer)** If \( \psi: \theta \in up_k (M_x D_y), \psi: \theta \not\in M_x B_y \) and \( \psi: \theta \not\in B_x \) then belief statement \([x, y, \psi: \theta]^2\) is applicable with the role of a minimal answer, denoted \( \sim \Rightarrow_{+3} \).

### 3.3.3 Counter-questions

If an agent is aware that another agent is interested in a particular sentence that is not part of its belief but it does know a prerequisite for this sentence, then it may pose a question asking for this prerequisite. Answers to this question may result in an extension of the agent’s belief in which the particular sentence is included. A question from \( x \) to \( y \) is applicable as a counter-question when \( x \) considers \( y \) interested in sentence \( \psi: \theta \), this sentence is not part of \( x \)’s belief, but, in the prerequisite set for \( \psi: \theta \) a sentence exists that is fresh and sensible.

**Definition 3.5 (counter-question)** If \( \psi: \theta \in up_k (M_x D_y), \psi: \theta \not\in M_x B_y \) and \( \psi: \theta \not\in B_x \), \( \varphi: \vartheta \in T \in E_{B_x} (\{ \psi: \theta \}) \), \( \varphi: \vartheta \not\in M_x I_z \) and \( \varphi: \vartheta \not\in M_x M_z I_x \) then question \([x, z, \varphi: \vartheta]^1\) is applicable with the role of counter-question, denoted \( \sim \Rightarrow \).

A special case of a counter-question is when the agent considers itself interested in a sentence, this is not elaborated.

### 3.3.4 Ignorance statement

If an agent cannot balance a desire and belief state of another agent, i.e. it can neither utter a belief statement nor a counter-question, than it states its ignorance. An ignorance statement from \( x \) to \( y \) is applicable when \( x \) is aware
that \( y \) is interested in \( \psi : \theta \) which is not part of \( x \)'s belief, and in addition, all sentences (possibly none) that are part of the set of prerequisites of \( \psi : \theta \) are neither fresh nor sensible to \( y \), in this case \( x \) states its ignorance to \( y \) regarding \( \psi : \theta \).

**Definition 3.6 (ignorance statement)** If \( \psi : \theta \in \text{up}_k(M_xD_y) \), \( \psi : \theta \not\in M_xB_y \), \( \psi : \theta \not\in B_x \), \( (\forall T \in \mathcal{E}_B, \{(\psi : \theta)\})(T \subseteq M_yM_zI_x \cup M_zI_x) \) and \( \psi : \theta \not\in M_xM_yI_x \) then ignorance statement \( [x, z, \psi : \theta]^? \) is applicable, denoted \( \sim \sim \).

### 3.4 Update Rules

In the previous section, dialogue rules were identified that describe applicable communicative acts given an agent’s cognitive state. Update rules or epistemic commitment functions describe the opposite: they prescribe the cognitive state of both sending and receiving agent after the information in a communicative act is accepted by both agents. Different attitudes that agents may have towards incoming information are represented with different types of update rules. Contrary to belief revision [5] in which agents have an aversion to inconsistent information, no restriction on incoming information is assumed, i.e. agents accept all communicative acts. The following rules are confined to \( k \)-monotonous updates.

If \( x \) has uttered a belief statement to \( y \), \( y \) believes sentence \( \psi : \theta \), i.e. \( \psi : \theta \in B_y \); \( y \) is aware that \( x \) believes the sentence, i.e. \( \psi : \theta \in M_yB_x \); and \( x \) is aware that \( y \) believes the sentence, i.e. \( \psi : \theta \in M_xB_y \).

**Definition 3.7** If \( [x, y, \psi : \theta]^? \) is uttered, then \( \psi : \theta \in B_y \), \( \psi : \theta \in M_yB_x \) and \( \psi : \theta \in M_xB_y \) hold; this update after a belief statement is denoted \( \triangleright \sim \).

If \( x \) has uttered a question to \( y \), \( y \) is aware that \( x \) has the desire to believe sentence \( \psi : \theta \), i.e. \( \psi : \theta \in M_yD_x \); and \( y \) is aware that \( x \) is aware that \( y \) is ignorant of the sentence, i.e. \( \psi : \theta \in M_xM_yI_x \).

**Definition 3.8** If \( [x, y, \psi : \theta]^? \) is uttered, then \( \psi : \theta \in M_yD_x \) and \( \psi : \theta \in M_xM_yI_x \) hold; this update after a question is denoted \( \triangleright \).

If \( x \) has uttered an ignorance statement to \( y \), \( y \) is aware that \( x \) is ignorant of sentence \( \psi : \theta \), i.e. \( \psi : \theta \in M_yI_z \); and \( x \) is aware that \( y \) is aware that \( x \) is ignorant of the sentence, i.e. \( \psi : \theta \in M_xM_yI_x \).

**Definition 3.9** If \( [x, y, \psi : \theta]^? \) is uttered, then \( \psi : \theta \in M_yI_z \) and \( \psi : \theta \in M_xM_yI_x \) hold; this update after an ignorance statement is denoted \( \triangleright \).

### 4 Example Dialogue

An overview of the proof that the example dialogue follows from the dialogue game is presented next (the full proof is presented in the appendix). First the initial states of the agents’ cognitive states are specified. Such a state is the set of all mental constructs of the cognitive states of the agents participating in the dialogue. The domain description in Sec. 3.1 is used. Only the following
sentences are part of the agents’ initial cognitive states (state 0). Elmo desires to believe that he can go play outside or that he cannot play outside, Tv believes that it possibly rains outside, and Grover believes that he is wearing a raincoat. Oscar believes a lot: 1) if it rains outside, then Elmo cannot play outside; 2) if it does not rain outside, then Elmo can play outside; 3) if it possibly rains outside, then it is possible that Elmo cannot play outside; 4) if it possibly does not rain outside, then Elmo can possibly play outside; and 5) if Grover wears a raincoat, then it rains outside. Formally:

\[ p_t, p_f \in D_{elmo}, r: 0.0 \sim 0.5 \in B_{tv}, c_t \in B_{grover}, (r:t \rightarrow p:f); t, (r:f \rightarrow p:t); t, (r:0.5 \sim 0 \rightarrow p:0 \sim 0.5); t, (r: 0 \sim 0.5 \rightarrow p:0.5 \sim 0); t, (c:t \rightarrow r:t) ; t \in B_{oscar} . \]

**Proof (overview)** The following nine lemmas show that Dialogue 2 follows from the initial state and the rules of the dialogue game.

1a. state 0 \[ \sim \? [elmo, oscar, p:t] \? \? \rightarrow \? \] state 1,
1b. state 0 \[ \sim \? [elmo, oscar, p:f] \? \? \rightarrow \? \] state 2,
2. state 1 \[ \sim \? [oscar, tv, r:f] \? \? \rightarrow \? \] state 3,
3. state 2 \[ \sim \? [oscar, grover, c:t] \? \? \rightarrow \? \] state 7,
4a. state 3 \[ \sim \? [tv, oscar, r:f] \sim \? \rightarrow \? \] state 4,
4b. state 3 \[ \sim \? [tv, oscar, r:0 \sim 0.5] \sim \? \rightarrow \? \] state 5,
5. state 5 \[ \sim \? [oscar, elmo, p:0.5 \sim 0] \sim \? \rightarrow \? \] state 6,
6. state 7 \[ \sim \? [grover, oscar, c:t] \sim \? \rightarrow \? \] state 8,
7. state 6 + 8 \[ \sim \? [oscar, elmo, p:0.5 \sim 1] \sim \? \rightarrow \? \] state 9.

**5 Conclusions**

The aim of this article was to show that with the introduction of multi-valued logics to dialogue games, dialogues with inconsistent and biased information are possible. The dialogue game by Beun [2] was refined to cope with these epistemic information states, by (1) representing the agents’ cognitive states as sets of multi-valued theories, (2) by identifying dialogue rules for stating partial, minimal and over-informative answers to questions and (3) by adapting update rules to process incoming information. Two different types of dialogue rules for questions were distinguished, viz. (normal) questions and counter-questions. One dialogue rule was given for ignorance statements and three different rules for belief statements.

In the fictitious dialogue from Sesame Street it was shown that a complex dialogue between four participants is possible with truth-values other than the orthodox true and false. In addition, the formal counterpart of the example dialogue was proven to follow from an initial state and the presented dialogue game. What we have shown in essence is that the semantic relations between utterances in a dialogue do not have to suffer from inconsistent epistemic states the agents have of their environment. The semantics of a domain as described in an ontology is decoupled from the epistemic attitude agents can have to this domain. Concepts from the ontology are combined with truth-values from the bilattice structure to form mental constructs, which in their turn form
cognitive states of agents. The update rules and especially the dialogue rules arrange the possible communicative acts independent of the domain theory (ontology) and the epistemic theory (bilattice), making the dialogue game a multi-agent notion by delegating use of domain and epistemic theory to agents.

Future research includes: (1) Definition of communicative acts for derogation of sentences which result in contractions of the agent’s cognitive state, and accompanying dialogue and update rules that can cope with $k$-non-monotonous behaviour. Introduction of derogations will make it possible to define belief revision at the multi-agent level of description. (2) Research to use these dialogue games for the semantics of FIPA ACLs restricted to information communication performatives. (3) Definition of specialised dialogue rules for ontological relations such as classifications, taxonomic and mereologic relations.

References

A Proofs

In the following proofs usage of the intermediate states of the cognitive states of the agents in conversation is used. When we use state 0 e.g. in lemma 2 to prove that state 3 follows from state 2, we refer to the properties of state 0 that still hold in state 2 owing to monotonicity.

**lemma 1a: state 0 to state 1** Given state 0 (initial state), we have:

(i) \( pt \in u_p(D_{elmo}) \) because \( pt \) and \( pf \) are part of \( D_{elmo} \) and for all \( p\theta \in D_{elmo} \) holds \( \theta \leq_k t \) or \( \theta \leq_k f \) (state 0),

(ii) \( pt \notin B_{elmo} \) because for all \( p\theta \in B_{elmo} \) holds \( \theta = u \) (state 0),

(iii) \( pt \notin M_{elmo}I_{oscar} \) because for all \( p\theta \in M_{elmo}I_{oscar} \) holds \( \theta = u \) (state 0),

(iv) \( p:t \notin M_{elmo}M_{oscar}I_{elmo} \) because for all \( p: \theta \in M_{elmo}M_{oscar}I_{elmo} \) holds \( \theta = u \) (state 0).

With i to iv and \( \leadsto ? \) follows \([\text{elmo}, \text{oscar}, p:t] \) and with \( \triangleright ? \) follows state 1:

\( pt \in M_{oscar}D_{elmo} \) and \( pt \in M_{elmo}M_{oscar}I_{elmo} \).

**lemma 1b: state 0 to state 2** Given state 0 (initial state), we have:

(i) \( pf \in u_p(D_{elmo}) \) because \( pt \) and \( pf \) are part of \( D_{elmo} \) and for all \( p\theta \in D_{elmo} \) holds \( \theta \leq_k t \) or \( \theta \leq_k f \) (state 0),

(ii) \( pf \notin B_{elmo} \) because for all \( p\theta \in B_{elmo} \) holds \( \theta = u \) (state 0),

(iii) \( pf \notin M_{elmo}I_{oscar} \) because for all \( p\theta \in M_{elmo}I_{oscar} \) holds \( \theta = u \) (state 0),

(iv) \( p:f \notin M_{elmo}M_{oscar}I_{elmo} \) because for all \( p: \theta \in M_{elmo}M_{oscar}I_{elmo} \) holds \( \theta = u \) (state 0).

With i to iv and \( \leadsto ? \) follows \([\text{elmo}, \text{oscar}, p:f] \) and with \( \triangleright ? \) follows state 2:

\( pf \in M_{oscar}D_{elmo} \) and \( pf \in M_{elmo}M_{oscar}I_{elmo} \).

**lemma 2: state 1 to state 3** Given state 1, we have:

(i) \( p:t \in u_p(M_{oscar}D_{elmo}) \) because \( p:t \in M_{oscar}D_{elmo} \) and for all \( p: \theta \in M_{oscar}D_{elmo} \) holds \( \theta \leq_k t \) (state 1 + state 0),

(ii) \( pt \notin M_{oscar}B_{elmo} \) because for all \( p\theta \in M_{oscar}B_{elmo} \) holds \( \theta = u \) (state 0),

(iii) \( pt \notin B_{oscar} \) because for all \( p\theta \in B_{oscar} \) holds \( \psi = u \) (state 0),

(iv) \( r:f \in T \in E_{oscar}(\{p:t\}) \) because \( r:f \in B_{oscar} \cup \{p:t\} \) because \( r:f \) and \( (r,f \rightarrow pt)t \) result in \( pt \),

(v) \( r:f \notin M_{oscar}I_{tv} \) because for all \( r\theta \in M_{oscar}I_{tv} \) holds \( \theta = i \) (state 0),

(vi) \( r:f \notin M_{oscar}M_{tv}I_{oscar} \) because for all \( r: \theta \in M_{oscar}M_{tv}I_{oscar} \) holds \( \theta = i \) (state 0).

With i to vi and \( \leadsto ? \) follows \([\text{oscar}, \text{tv}, r:f] \) and with \( \triangleright ? \) follows state 3:

\( rf \in M_{tv}D_{oscar} \) and \( rf \in M_{oscar}M_{tv}I_{oscar} \).

**lemma 3: state 2 to state 7** Given state 2, we have:
Lemma 4a: state 3 to state 4
Given state 3, we have:

(i) \(p: f \in up_k(M_{oscar}D_{elmo})\) because \(p: f \in M_{oscar}D_{elmo}\) and for all \(p: \theta \in M_{oscar}D_{elmo}\) holds \(\theta \leq_k f\) (state 2 + state 0),

(ii) \(pf \not\in M_{oscar}B_{elmo}\) because for all \(p\theta \in M_{oscar}B_{elmo}\) holds \(\theta = u\) (state 0),

(iii) \(pf \not\in B_{oscar}\) because for all \(p\theta \in B_{oscar}\) holds \(\theta = u\) (state 0),

(iv) \(c t \in T \in E_{B_{oscar}}(\{p: f\})\) because \(c t \in B_{oscar} \cup \{p: f\}\), because \(c t \in B_{oscar}\) and \((c t \rightarrow r: t): t \in B_{oscar}\) result in \(r: t \in B_{oscar}\), combined with \(r: t \in B_{oscar}\) and \((r: t \rightarrow pt): t \in B_{oscar}\)

(v) \(ct \not\in M_{oscar}I_{grover}\) because for all \(c\theta \in M_{oscar}I_{grover}\) holds \(\theta = i\) (state 0),

(vi) \(ct \not\in M_{oscar}M_{grover}I_{oscar}\) because for all \(c\theta \in M_{oscar}M_{grover}I_{oscar}\) holds \(\theta = i\) (state 0).

With i to iv and \(\sim \gamma\) follows \([oscar, grover, ct]^{2}\) and with \(\triangleright \gamma\) follows state 7: \(ct \in M_{grover}I_{oscar}\) and \(ct \in M_{oscar}M_{grover}I_{oscar}\).

Lemma 4b: state 3 to state 5
Given state 3, we have:

(i) \(r: f \in up_k(M_{tv}D_{oscar})\) because \(r: f \in M_{tv}D_{oscar}\) and for all \(r: \theta \in M_{tv}D_{oscar}\) holds \(\theta \leq_k f\) (state 3 + state 0),

(ii) \(rf \not\in M_{tv}B_{oscar}\) because for all \(r\theta \in M_{tv}B_{oscar}\) holds \(\theta = u\) (state 0),

(iii) \(rf \not\in B_{tv}\) because for all \(r\theta \in B_{tv}\) holds \(\theta = u\) (state 0),

(iv) \(E_{B_{oscar}}(\{r: f\}) = \{\}\) because no conditional sentence exists in which \(r: f\) is a conclusion (state 0).

(v) \(rf \not\in M_{tv}M_{oscar}I_{oscar}\) because for all \(rf \in M_{tv}M_{oscar}I_{oscar}\) holds \(\theta = i\).

With i to v and \(\sim \gamma\) follows \([tv, oscar, r: f]^{2}\) and with \(\triangleright \gamma\) follows state 4: \(rf \in M_{oscar}I_{tv}\) and \(rf \in M_{tv}M_{oscar}I_{tv}\).

Lemma 5: state 5 to state 6
Given state 5, we have:

(i) \(p0.5\sim0 \in M_{oscar}D_{elmo}\) because \(pt \in M_{oscar}D_{elmo}\) (state 1),

(ii) \(p0.5\sim0 \not\in M_{oscar}B_{elmo}\) because for all \(p\theta \in M_{oscar}B_{elmo}\) holds \(\theta = u\) (state 0),

(iii) \(p0.5\sim0 \in B_{oscar}\) because \(r0\sim0.5 \in B_{oscar}\) (state 5) and \((r0\sim0.5 \rightarrow p0.5\sim0): t \in B_{oscar}\) (state 0) then also \(p0.5\sim0 \in B_{oscar}\).

With i to ii and \(\sim \gamma+1\) follows \([oscar, elmo, p0.5\sim0]'\) and with \(\triangleright \gamma+1\) follows state 6: \(p0.5\sim0 \in B_{elmo}\), \(r0\sim0 \in M_{elmo}B_{oscar}\), and \(r0\sim0 \in M_{oscar}B_{elmo}\).
Lemma 7: state 6 and 8 to state 9
Given state 6 and 8, we have:

(i) \( c:t \in \text{up}_k(M_{grover} \text{Ioscar}) \) because \( c:t \in M_{grover} \text{Ioscar} \) and for all \( c: \theta \in M_{grover} \text{Ioscar} \) holds \( \theta \leq_k t \) (state 7 + state 0),

(ii) \( ct \not\in M_{grover} \text{Boscar} \) because for all \( ct \theta \in M_{grover} \text{Boscar} \), holds \( \theta = u \) (state 0)

(iii) \( ct \in B_{grover} \) (state 0).

With i to iii and \( \sim \gamma_+ \) follows \([grover, oscar, ct]^{?+}\) and with \( \triangleright \gamma_+ \) follows state 8: \( ct \in B_{oscar}, ct \in M_{oscar}B_{grover} \), and \( ct \in M_{grover}B_{oscar} \).

Lemma 7: state 6 and 8 to state 9
Given state 6 and 8, we have:

(i) \( p:f \in \text{up}_k(M_{oscar} \text{Delmno}) \) because \( p:f \in M_{oscar} \text{Delmno} \) and for all \( p: \theta \in M_{oscar} \text{Delmno} \) holds \( \theta \leq_k f \) or \( \theta \leq_k t \) (state 1 + state 2),

(ii) \( pf \not\in M_{oscar} \text{Belmo} \) because for all \( pf \theta \in M_{oscar} \text{Belmo} \), holds \( \theta = u \) (state 0),

(iii) \( p0.5\sim1 \in B_{oscar} \) because
* \( p0.5\sim0 \in B_{oscar} \) because \( r:0\sim0.5 \in B_{oscar} \) (state 5) and \( (r:0\sim0.5 \rightarrow p:0.5\sim0); t \in B_{oscar} \) (state 0) then also \( p0.5\sim0 \in B_{oscar} \).
* \( pf \in B_{oscar} \) because \( ct \in B_{oscar} \) (state 8) and \( (ct \rightarrow pf); t \in B_{oscar} \) (state 0) then also \( pf \in B_{oscar} \).

With \( p0.5\sim0 \in B_{oscar} \) and \( pf \in B_{oscar} \) then also \( p0.5\sim1 \in B_{oscar} \) because \( B_{oscar} \) is a complete theory.

(iv) \( f \leq_k 0.5\sim1 \).

With i to iv and \( \sim \gamma_+ \) follows \([oscar, elmo, p:0.5\sim1]^{?+}\) and with \( \triangleright \gamma_+ \) follows state 9: \( p0.5\sim1 \in B_{elmo}, p0.5\sim1 \in M_{elmo}B_{oscar} \), and \( c:0.5\sim1 \in M_{oscar}B_{elmo} \).