# Game Theory for auctions

Uncertainty

Bayesian Nash equilibrium

Expected utility

# Uncertainty

- Exogenous uncertainty in static games
- Exogenous uncertainty in dynamic games
- Endogenous uncertainty in static games

# Exogenous uncertainty, static

- The world is uncertain:

60% chance for oil

| | ¬drill | narrow | wide |
|---|---|---|---|
| ¬drill | 0,0 | 0,44 | 0,31 |
| narrow | 44,0 | 14,14 | -1,16 |
| wide | 31,0 | 16,-1 | 1,1 |

40% chance of no oil

| | ¬drill | narrow | wide |
|---|---|---|---|
| ¬drill | 0,0 | 0,-16 | 0,-29 |
| narrow | -16,0 | -16,-16 | -16,-2 |
| wide | -29,0 | -29,-16 | -29,-2 |

# Exogenous uncertainty, static

Combine the two possible worlds in an expected payoff matrix

$$V_{ij} = 0.6xV^1_{ij} + 0.4xV^2_{ij}$$

| | ¬drill | narrow | wide |
|---|---|---|---|
| ¬drill | 0,0 | 0,20 | 0,7 |
| narrow | 20,0 | 2,2 | -7,-2 |
| wide | 7,0 | -2,-7 | -11,-11 |

Nash equilibrium

# Exogenous uncertainty, dynamic

- Example
- Macrosoft introduces new product
- Choose slick or simple campaign
- The demand can be either high or low
- There might be competition from Microcorp after one year

# Profits Macrosoft, no comp.

| | slick | | simple | |
|---|---|---|---|---|
| | High demand | Low demand | High demand | Low demand |
| Gross profit year 1 | $900 | $600 | $200 | $200 |
| Gross profit year 2 | $700 | $200 | $1200 | $400 |
| Gross profit | $1600 | $800 | $1400 | $600 |
| Ad costs | -$600 | -$600 | -$200 | -$200 |
| **Net profit** | **$1000** | **$200** | **$1200** | **$400** |

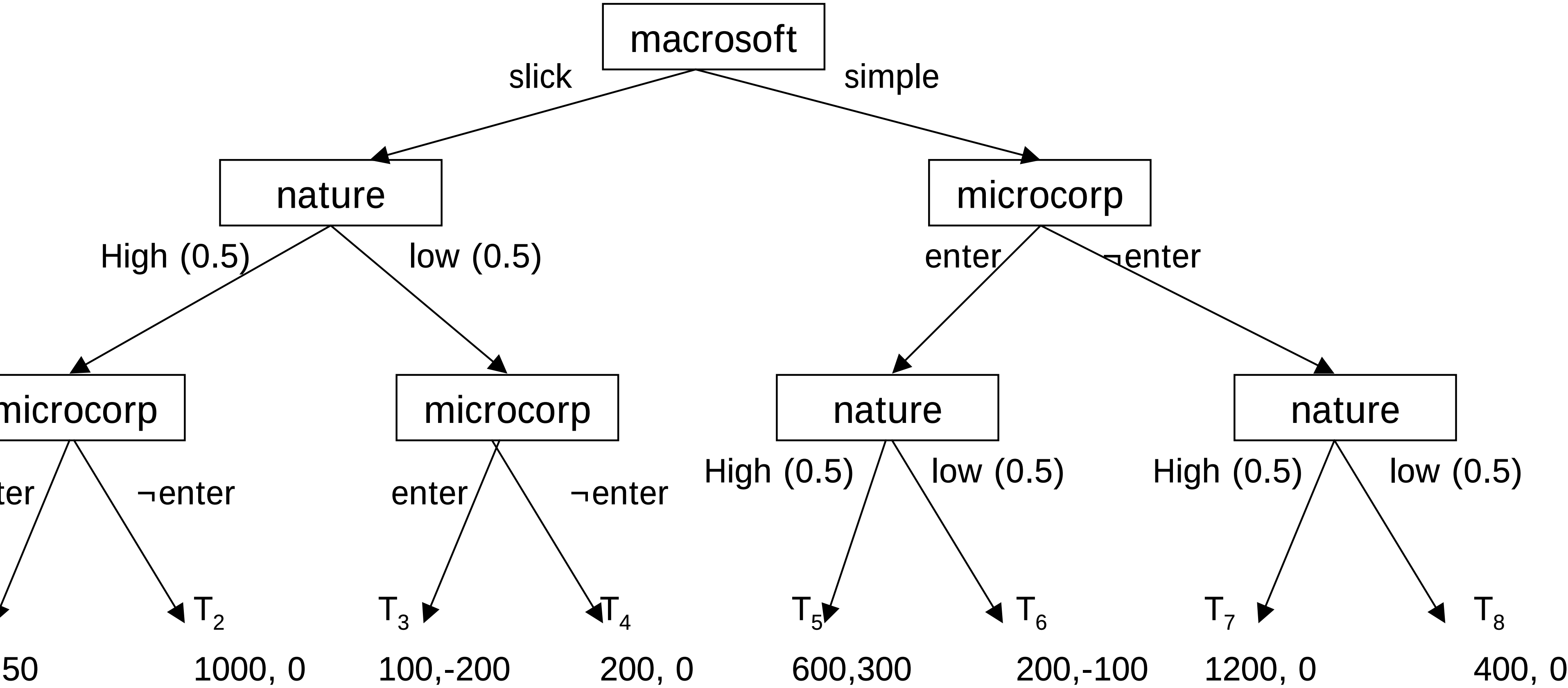
# Profits Macrosoft, if Microcorp enters after 1 year

| | slick | | simple | |
|---|---|---|---|---|
| | High demand | Low demand | High demand | Low demand |
| Gross profit year 1 | $900 | $600 | $200 | $200 |
| Gross profit year 2 | $350 | $100 | $600 | $200 |
| Gross profit | $1250 | $700 | $800 | $400 |
| Ad costs | -$600 | -$600 | -$200 | -$200 |
| **Net profit** | **$650** | **$100** | **$600** | **$200** |

# Profits Microcorp if it enters after 1 year

| | slick | | simple | |
|---|---|---|---|---|
| | High demand | Low demand | High demand | Low demand |
| Gross profit year 1 | $0 | $0 | $0 | $0 |
| Gross profit year 2 | $350 | $100 | $600 | $200 |
| Gross profit | $350 | $100 | $600 | $200 |
| entry costs | -$300 | -$300 | -$300 | -$300 |
| **Net profit** | **$50** | **-$200** | **$300** | **-$100** |

# Model as extensive game

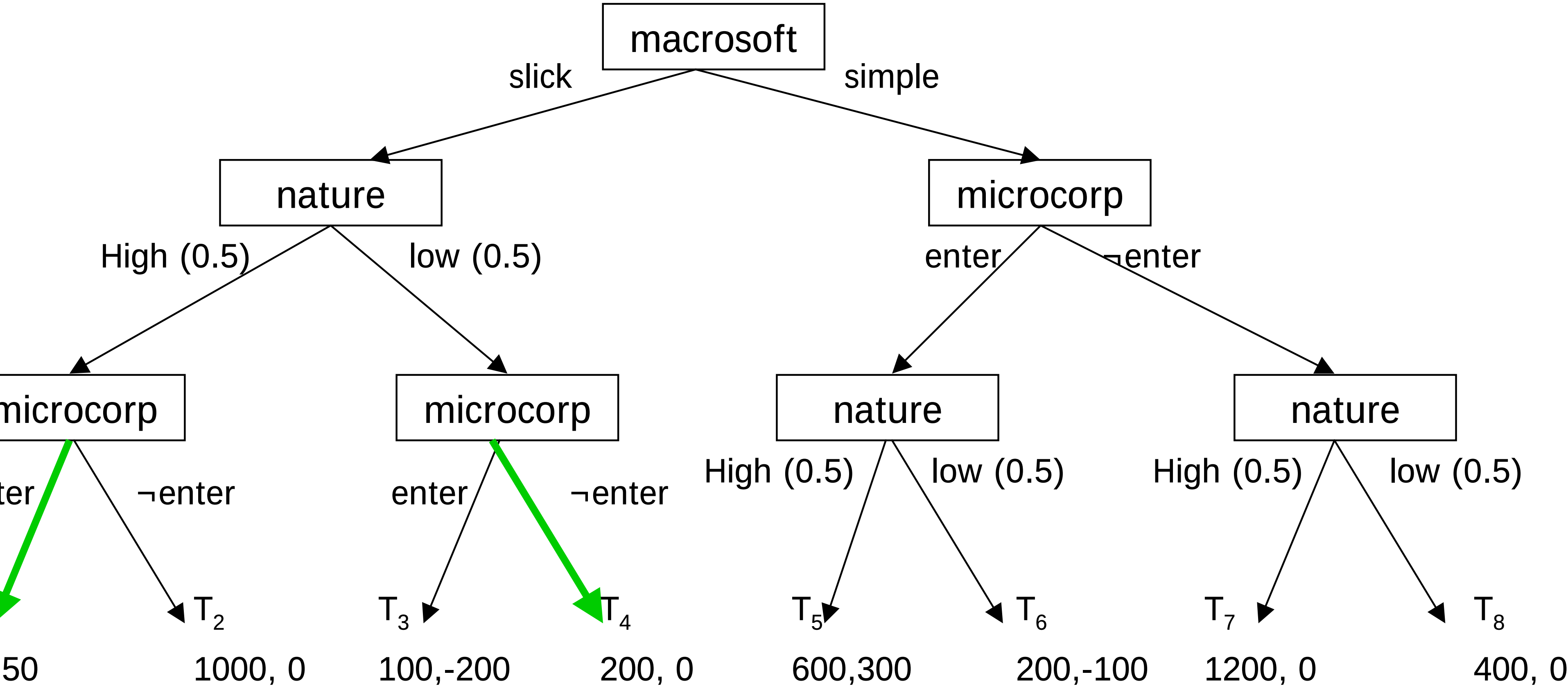


# Strategic form

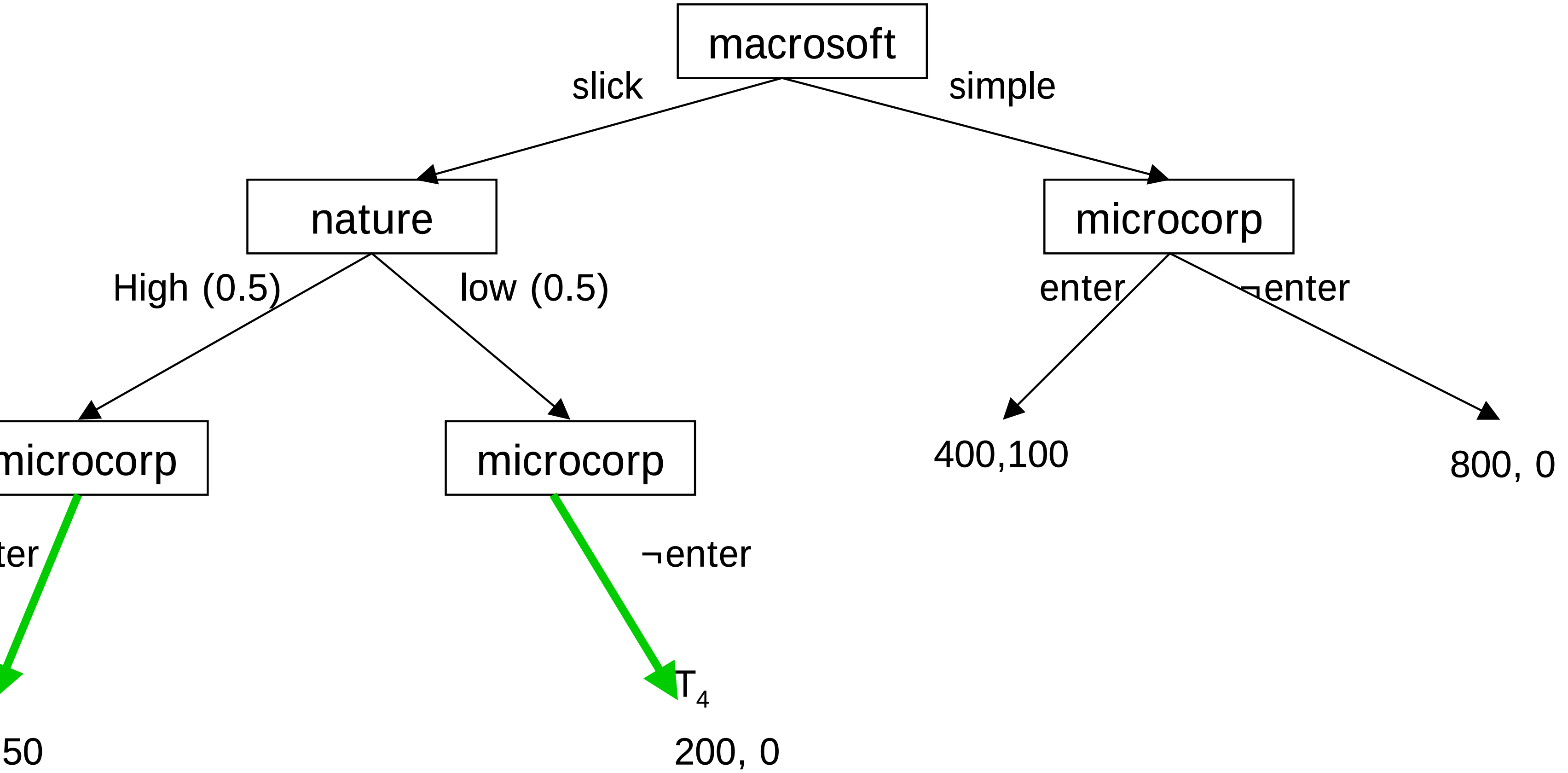
| | Macrosoft | |
|---|---|---|
| | slick | simple |
| ¬enter, ¬enter, ¬enter) | \$600,\$0 | \$800,\$0 |
| ¬enter, ¬enter, enter) | \$600,\$0 | \$400,\$100 |
| ¬enter, enter, ¬enter) | \$550,-\$100 | \$800,\$0 |
| ¬enter, enter, enter) | \$550,-\$100 | \$400,\$100 |
| enter, ¬enter, ¬enter) | \$425,\$25 | \$800,\$0 |
| enter, ¬enter, enter) | \$425,\$25 | \$400,\$100 |
| enter, enter, ¬enter) | \$375,-\$75 | \$800,\$0 |
| enter, enter, enter) | \$375,-\$75 | \$400,\$100 |

Nash equilibria

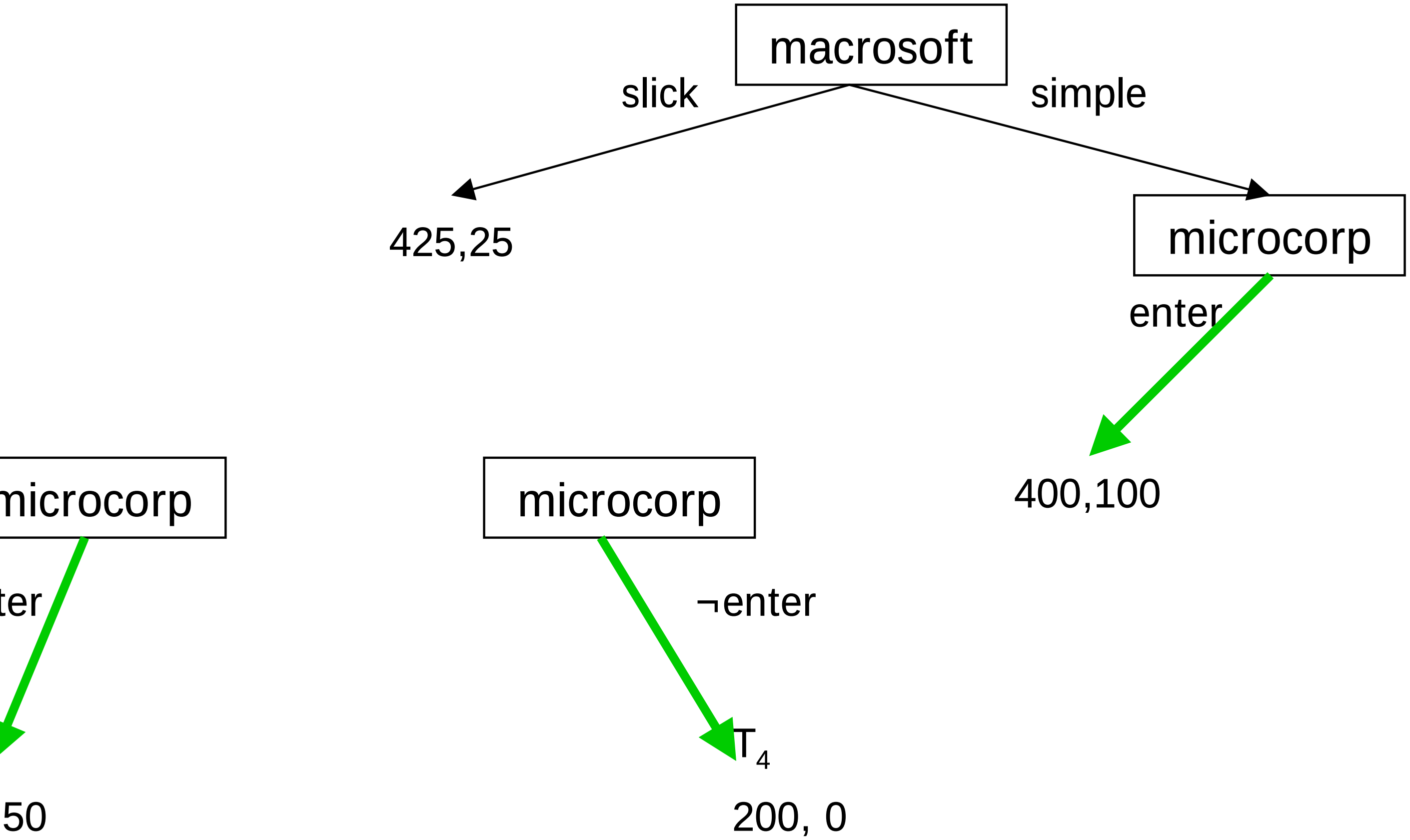
# Nash Equilibrium



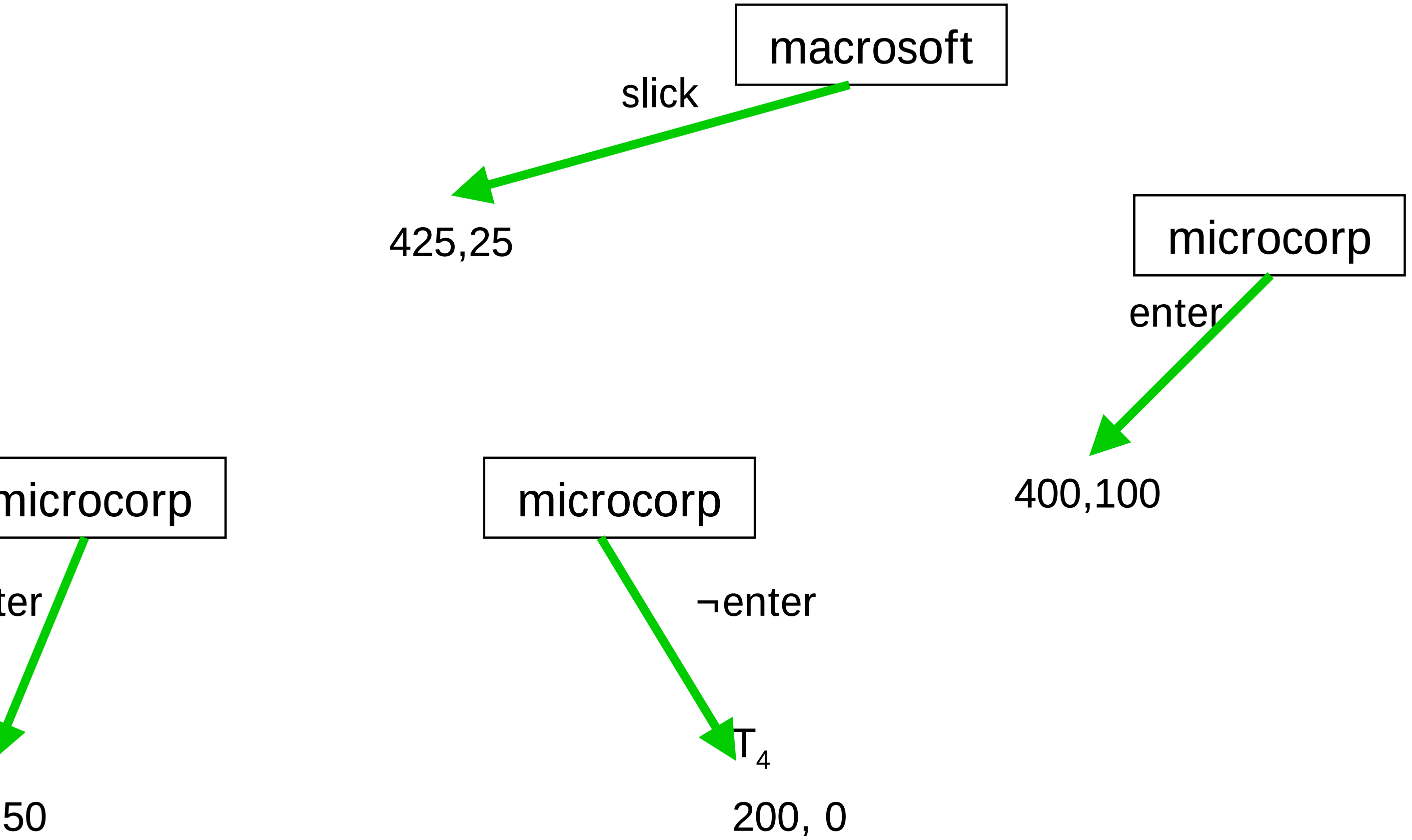
# Nash Equilibrium



# Nash Equilibrium



# Nash Equilibrium



# Endogenous uncertainty

- Mixed strategy games
- Player does not know the strategy the other player will play
- (in exogenous uncertainty the players do not know the state of the world)
- We assume players will maximize the expected payoffs

# Mixed strategy games

- Player strategies: $(S_1{:}p_1,\ldots,S_k{:}p_k)$
- With two players:
  - Player 1: $(S_1{:}p_1,\ldots,S_k{:}p_k)$
  - Player 2: $(T_1{:}q_1,\ldots,T_l{:}q_l)$
  - Player 1's payoff when it adopts $S_i$ and player 2 adopts $T_j$ is $V_{ij}$
- Player 1: $EV=\sum^{k}_{i=1}\sum^{l}_{j=1} V_{ij}.p_i.q_j$

# Battle of the sexes

Nash equilibria

| | | Rhett | |
|---|---|---|---|
| | | Saloon | Dinner club |
| Scarlett | Saloon | 5,12 | 0,0 |
| | Dinner club | 0,0 | 10,4 |

# Battle of the sexes

| | | Rhett | |
|---|---|---|---|
| | | Saloon ($p_r$) | Dinner club ($1-p_r$) |
| Scarlett | Saloon ($p_s$) | 5,12 | 0,0 |
| | Dinner club ($1-p_s$) | 0,0 | 10,4 |

# Battle of the sexes

$Ev_s(p_s,p_r) = 5.p_s.p_r + 0.P_s.(1-p_r) + 0.(1-p_s).p_r + 10.(1-p_s).(1-p_r)$

$= 10.(1-p_r) + 15.p_s.(p_r-2/3)$

$Ev_r(p_s,p_r) = 12.p_s.p_r + 0.P_s.(1-p_r) + 0.(1-p_s).p_r + 4.(1-p_s).(1-p_r)$

$= 4.(1-p_s) + 16.p_r.(p_s-1/4)$

Equilibrium: $p^*_r = 2/3$ and $p^*_s = 1/4$

Mixed strategy Nash equilibrium:

{(Saloon:1/4,dinner club:3/4),(saloon:2/3,dinner club:1/3)}

# Battle of the sexes

Mixed strategy Nash equilibrium

| | | Rhett | |
|---|---|---|---|
| | | Saloon (2/3) | Dinner club (1/3) |
| Scarlett | Saloon(1/4) | 5,12 | 0,0 |
| | Dinner club (3/4) | 0,0 | 10,4 |

# Bayesian games

- Players know their own payoffs
- Players do not know the payoffs of the other players
- Players have common belief about the distribution of the other players’ payoff
- Players’ types can be stochastically dependent (e.g. in auction for products whose resale value is approx. known)

# Static Bayesian game

1. List of players
2. List of moves for each player
   1. A move profile is a list of moves, one for each player
3. List of possible types of each player
   1. A type profile is a list of types, one for each player
4. List of probabilities of each type profile ($P[t_1,\ldots t_n]$)
5. List of payoffs $U_i$ depends on move profile and type profile $U_i(m_1,\ldots,m_n,t_1,\ldots,t_n)$

# Example: entry deterrence game

*Expansion costs low*

**incumbent**

| | | expand | ¬ expand |
|---|---|---|---|
| **ntrant** | enter | -1,2 | 1,1 |
| | ¬enter | 0,4 | 0,3 |

*Expansion costs high*

**incumbent**

| | expand | ¬ expand |
|---|---|---|
| enter | -1,-1 | 1,1 |
| ¬enter | 0,0 | 0,3 |

# Deterrence game

- Players:{entrant, incumbent}
- Moves:{[enter,¬enter],[expand,¬expand]}
- Types:{[normal],[low cost, high cost]}
  - Type profiles:{[normal,low cost],[normal,high cost]}
- Probabilities:

  {[normal,low cost]:2/3,[normal,high cost]:1/3}
- Payoffs given in matrices.
  - E.g. $U_{entrant}$(enter,expand,normal,high cost)=-1

# Conditional expected utility

- A pure strategy for i consists of a move as a function o the player's type $S_i(t_i)$

- The conditional expected utility of player i of type $t_i$ when the players adopt strategy profile $S=\{S_1(t_1),\dots,S_n(t_n)\}$ is: $EU_i(S,t_i)=\sum_{t-i}U_i(S_1(t_1),\dots,S_n(t_n),t_1,\dots,t_n).P_i[t_{-i}|t_i]$

If the types are stochastically independent

$P_i[t_{-i}|t_i]=P_i[t_1,\dots,t_{i-1},t_{i+1},\dots,t_n]$

# Bayesian Nash equilibrium

A strategy profile $S^*=\{S^*_1(t_1),\ldots,S^*_n(t_n)\}$ is a Bayesian Nash equilibrium

iff

$\forall i,\ \forall t_i,\ \forall \check{S}_i(t_i)$:

$$EU_i(S^*_I,t_i) \geq EU_i(S^*_1(t_1),\ldots,\check{S}_i(t_i),\ldots,S^*_n(t_n),t_i)$$

# Example: entry deterrence game

We represent the strategy of the incumbent with the pair (x,y) where x is the incumbent's strategy if he has low costs and y is his strategy when he has high costs

| | | **incumbent** | | | |
|---|---|---|---|---|---|
| | | (expand, expand) | (expand, ¬expand) | (¬expand, expand) | (¬ expand, ¬expand) |
| **entrant** | enter | ((-1),(2,-1)) | ((-1/3),(2,1)) | ((1/3),(1,-1)) | ((1),(1,1)) |
| | ¬enter | ((0),(4,0)) | ((0),(4,3)) | ((0),(3,0)) | ((0),(3,3)) |

Nash equilibrium

# entry deterrence game

*Expansion costs low (2/3)*

**incumbent**

| **entrant** | | expand | ¬ expand |
|---|---|---|---|
| | enter | -1,2 | 1,1 |
| | ¬enter | 0,4 | 0,3 |

*Expansion costs high (*

**incumbent**

| | expand | ¬ expand |
|---|---|---|
| enter | -1,-1 | 1,1 |
| ¬enter | 0,0 | 0,3 |

2/3 2/3 1/3

**incumbent**

| **entrant** | | (expand, expand) | (expand, ¬expand) | (¬expand, expand) | (¬ expand, ¬expand) |
|---|---|---|---|---|---|
| | enter | ((-1),(2,-1)) | ((-1/3),(2,1)) | ((1/3),(1,-1)) | ((1),(1,1)) |
| | ¬enter | ((0),(4,0)) | ((0),(4,3)) | ((0),(3,0)) | ((0),(3,3)) |

Nash equilibrium

# Expected utility hypothesis

There exists a function U() over the final outcomes $\{T_1,\ldots,T_n\}$ such that a lottery $L_1=(T_1:p_{11},\ldots,T_n:p_{1n})$ is preferred to a lottery $L_2=(T_2:p_{21},\ldots,T_n:p_{2n})$ iff

$$\sum_k p_{1k}.U(T_k) \geq \sum_k p_{2k}.U(T_k)$$

$U(T_i)$ is called a von Neumann-Morgenstern utility (VNMU)

# Expected utility hypothesis

- Seems very simple and innocent but the indicated utility function has to be a cardinal measure
- Suppose $U_i$ is a VNMU then any other VNMU $V_i$ that respects the preferences of i can be written as $a+bU_i$
- This implies that the marginal utilities of $V_i$ and $U_i$ are equal!

# Expected utility hypothesis

The expected utility hypothesis implies the following three axioms:

1. There is a complete and transitive preference ordering over the elements of the set L of lotteries constructed from the finite set of outcomes X.

$\Rightarrow$ The elements of X can be ordered

# Expected utility hypothesis

Let $L(u)=(T_b:u,T_w:1-u)$

2. L(u) is preferred to L(v) iff u>v.
3. For every outcome T in X there exists a unique number, U(T) called the normalized VNMU of T, such that i is indifferent between the certain outcome T and the lottery L(U(T)).

# Attitudes towards risk

- A person is called risk averse if he has a diminishing marginal (expected) utility
- A person is called risk loving if he has an increasing marginal (expected) utility

# Attitudes towards risk

Example:

Assume $L_1$=(\$10:0.5,-\$10:0.5) and assume $L_2$=(\$1000:0.5,-\$1000:0.5) and assume the person is risk averse

Let $y_1 = y_0 - \$1000$

$y_2 = y_0 - \$10$

$y_3 = y_0 + \$10$

$y_4 = y_0 + \$1000$

Then

$$\frac{U(y_4)-U(y_3)}{y_4-y_3} < \frac{U(y_2)-U(y_1)}{y_2-y_1}$$

⇒ $0.5\,U(y_4) + 0.5\,U(y_1) < 0.5\,U(y_2) + 0.5\,U(y_3)$

⇒ $L_2 < L_1$

# Certainty equivalent

If a players preferences over lotteries satisfy the expected utility hypothesis the player can assign to every lottery a value called the certainty equivalent o CE(L).

$$U(CE(L)) = EU(L)$$

If $L= (T_1:p_1,...,T_n:p_n)$ its expected payoff is $E(L) = \sum_i p_i.T$ and the variance is given by $V(L)= \sum_I (T_i-E(L))^2.p_i$ then

$$CE(L) \approx E(L) + \rho.V(L)$$

Where $\rho$ is called the risk aversion coefficient. It is positive if a person is risk-loving and negative if a person is risk aversive.