



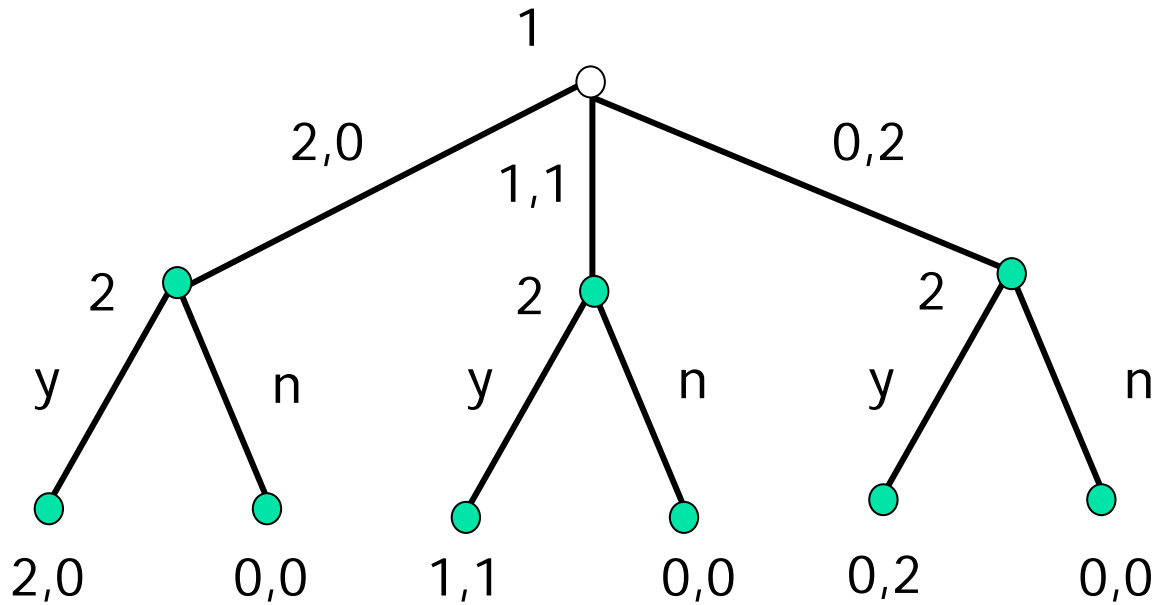
# Game Theory

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## Equilibria in Extensive Games

# Extensive games

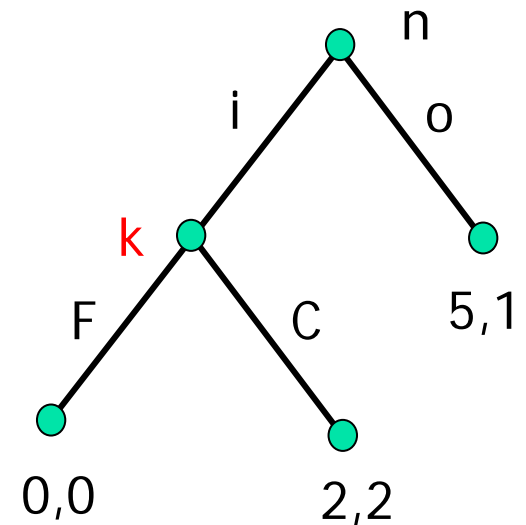
## Example I: Dividing goods



# Extensive games

## Example II: Chain store game

- Chain store  $k$  and  $n$  competitors
- Each competitor can fight (i) or cooperate (o)
- If so,  $k$  can choose between cooperate (c) or fight (f)





# extensive games: definitions

---

- extensive games:  $G = \langle N, H, P, (\succeq_i) \rangle$ 
  - $N$ : set of players
  - $H$  histories:  $\emptyset, (a^k)_{k=1..K}$  (possibly infinite)
    - closed under prefixes
    - terminals  $Z$ : without successor or infinite
  - $P: H \setminus Z \rightarrow N$  player that has to move
  - $\succeq_i$ : preference relation on  $Z$



# extensive games: definitions

---

- extensive games:  $G = \langle N, H, P, (\geq_i) \rangle$ 
  - $H$  histories:  $\emptyset, (a^k)_{k=1..K}$  (possibly infinite)
    - closed under prefixes
    - terminals  $Z$ : without successor or infinite
    - $h \in H, a$  actie  $\Rightarrow (h, a) \in H$
  - $H$  is finite  $\Rightarrow G$  is **finite**
  - $H$  only contains finite  $h \Rightarrow G$  has **finite horizon**

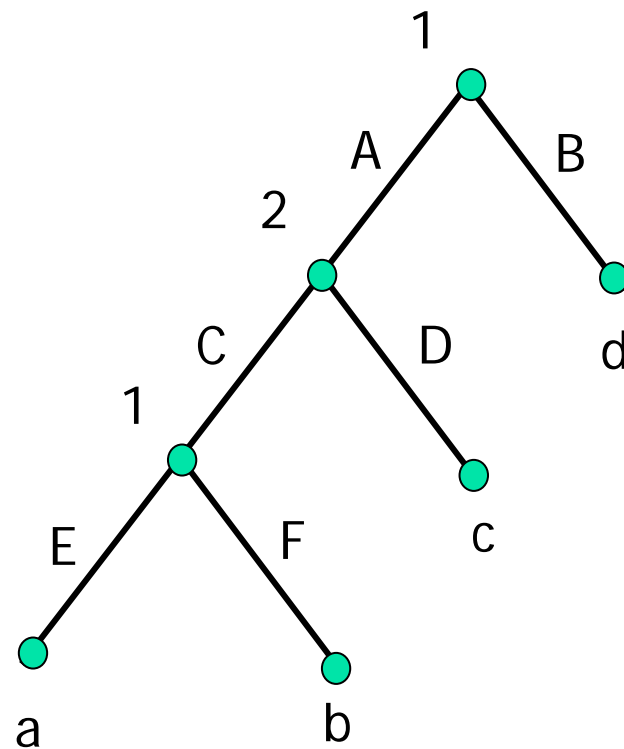


# Subgame perfect solutions

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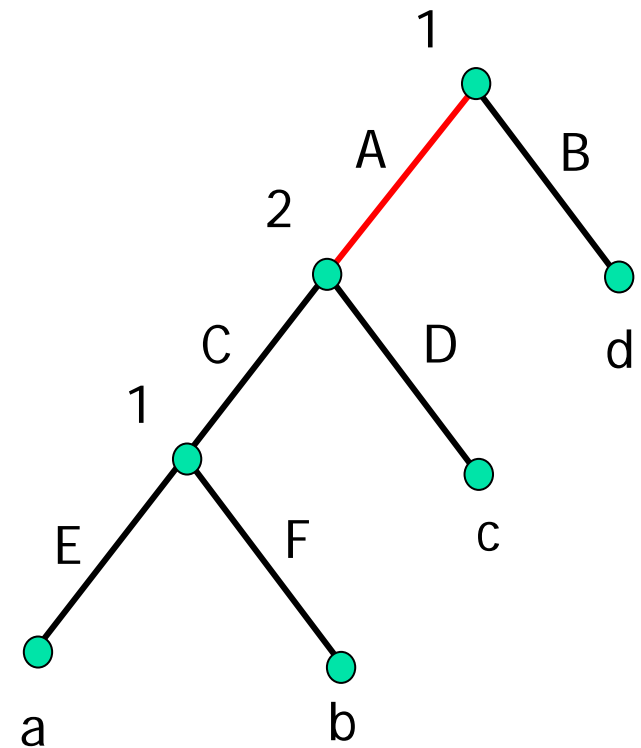
- extensive games:  $\Gamma = \langle N, H, P, (\succeq_i) \rangle$ 
  - $N$ : set of players
  - $H$  histories:  $\emptyset, (a^k)_{k=1..K}$  (possibly infinite)
  - $P: H \setminus Z \rightarrow N$  player moving
  - $\succeq_i$ : preference relation on  $Z$
- subgames:  $\Gamma(h) = \langle N, H|_h, P|_h, (\succeq_i|_h) \rangle$ 
  - All continuations of  $h$

# Subgames



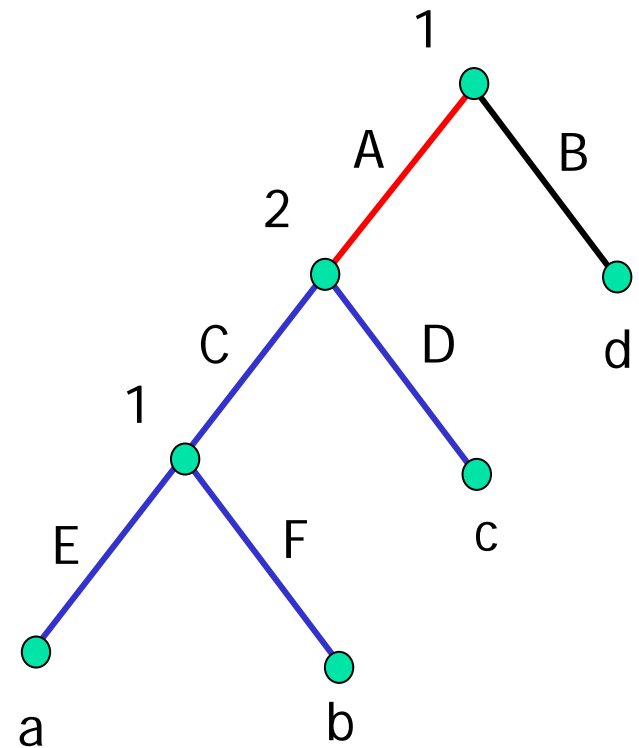
# Subgames

- history  $h$



# Subgames

- history  $h$
- subgame  $\Gamma(h)$





# Extensive games strategies

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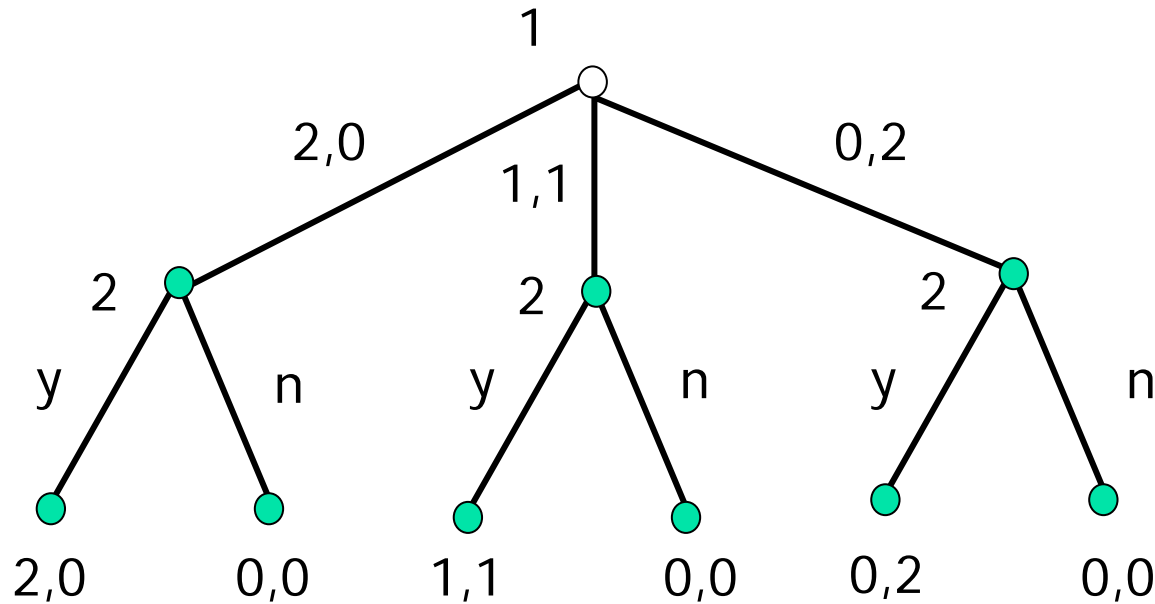
A **strategy** for a player  $i$  is a tuple that indicates which action  $i$  will choose for ALL choice points of  $i$  in the game.

A **profile** is a combination of strategies of all players

# Extensive games strategies

Strategy player 1:  
(2,0)

Strategy player 2:  
nnn



Some Profiles:

- ((2,0)yyy)
- ((2,0)yyn)
- ((1,1),nyy)
- ((1,1),nyn)
- ((0,2),nny)



# subgame perfect N.-eq

---

- let  $\Gamma = \langle N, H, P, (\geq_i) \rangle$  extensive game
  - $s^*$  is N.-eq if  $\forall i \forall s_i O(s_{-i}^*, s_i^*) \geq_i O(s_{-i}^*, s_i)$



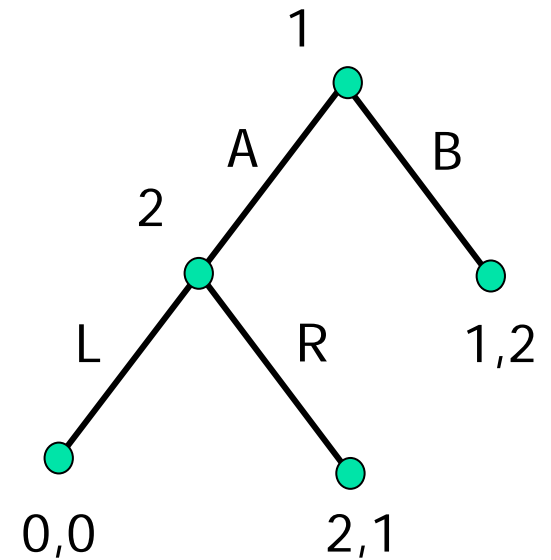
# subgame perfect N.-eq

---

- zij  $\Gamma = \langle N, H, P, (\geq_i) \rangle$  extensive
  - $s^*$  is N.-eq if  $\forall i \forall s_i O(s_{-i}^*, s_i^*) \geq_i O(s_{-i}^*, s_i)$
- $s^*$  is subgame perfect N.-eq if
  - $\forall i \forall h \in H \setminus Z (P(h)=i \Rightarrow$ 
    - $O_h(s_{-i}^* | h, s_i^* | h) \geq_{i|h} O(s_{-i}^* | h, s_i))$
    - for all strategies  $s_i$  for  $i$  in  $\Gamma(h)$
  - $s^* | h$  is N.-eq for all  $\Gamma(h)$

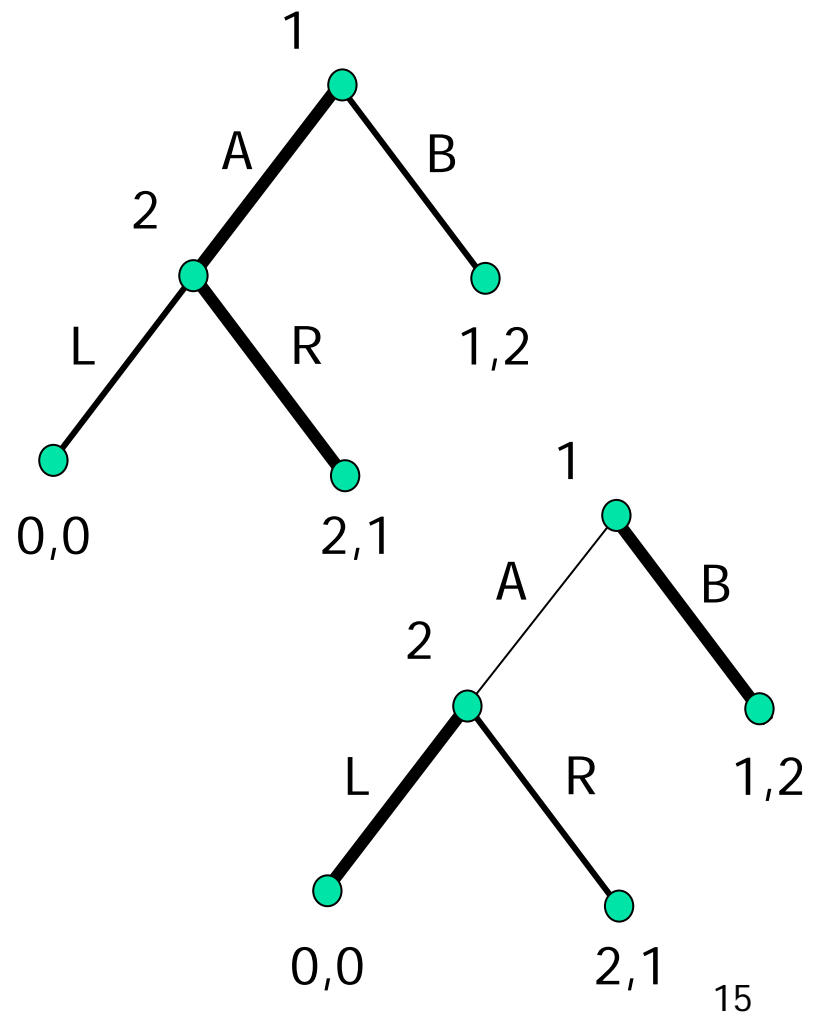
# Equilibria: example

- Nash equilibria?



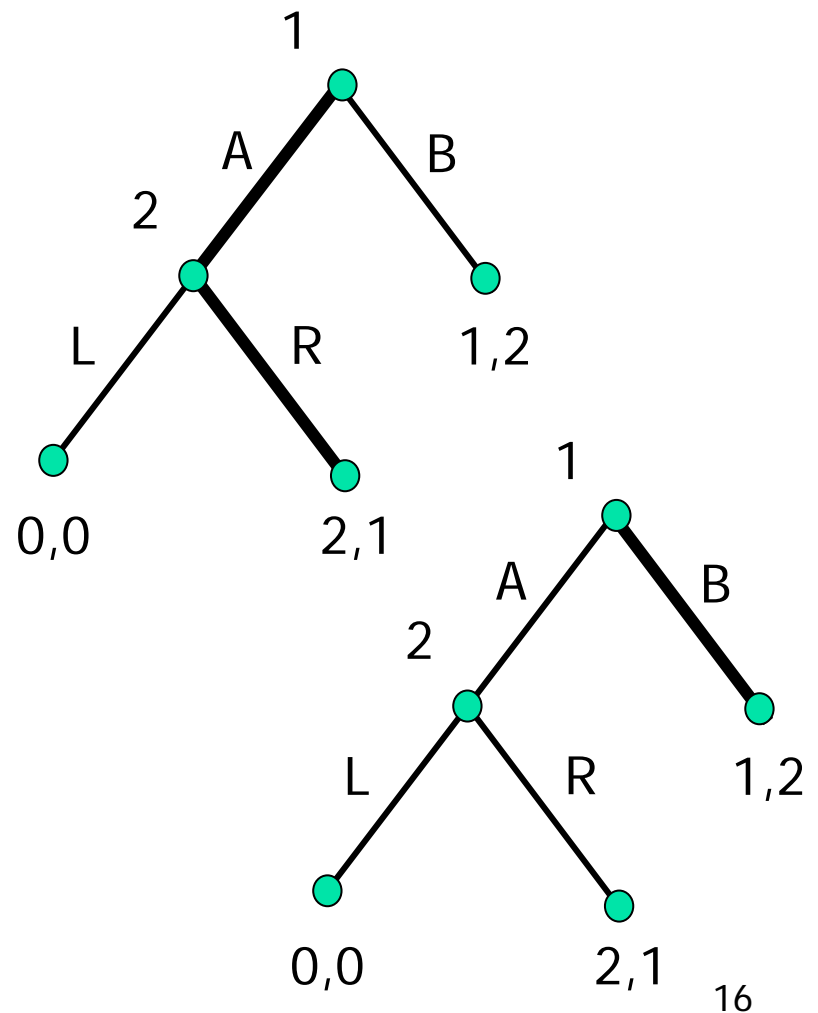
# equilibria (ctd)

- so: (A,R) and (B,L)
- interpretation (B,L):
- assuming that 2 plays L after A, B is the better choice for 1
- intuitive?
- what is optimal for 1?



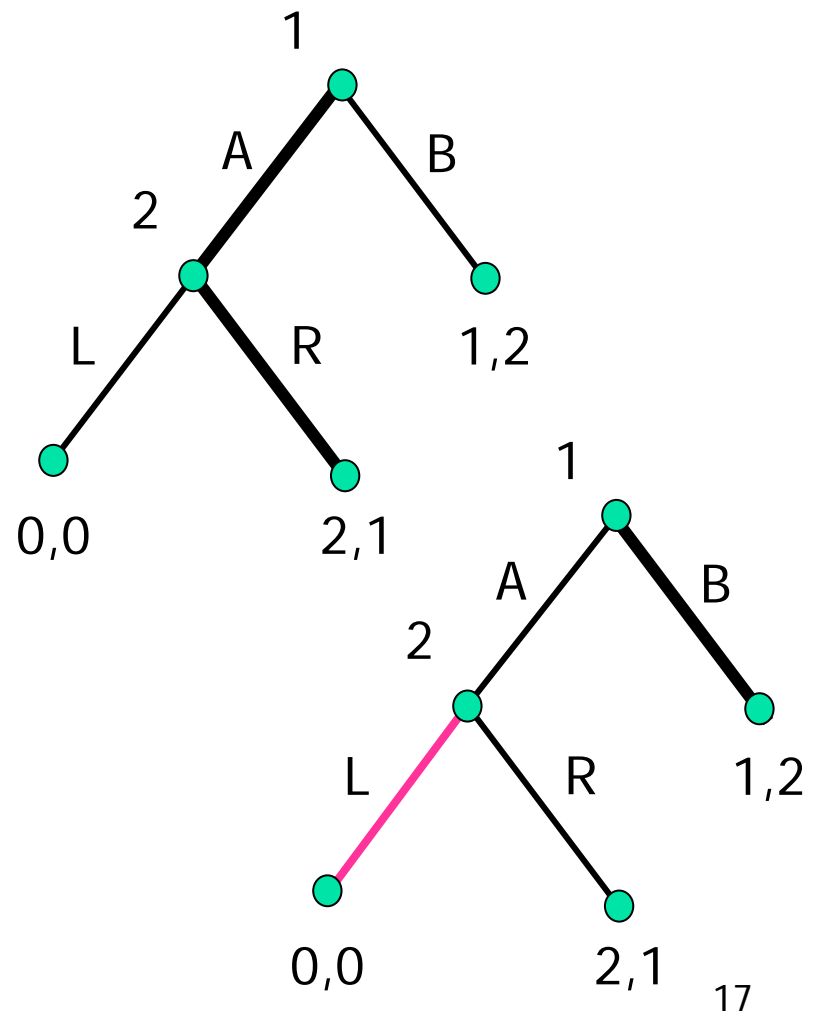
# equilibria (ctd)

- so: (A,R) and (B,L)
- interpretation (B,L):
- assuming that 2 plays L after A, B is the better choice for 1
- AR is the only subgame perfect equilibrium



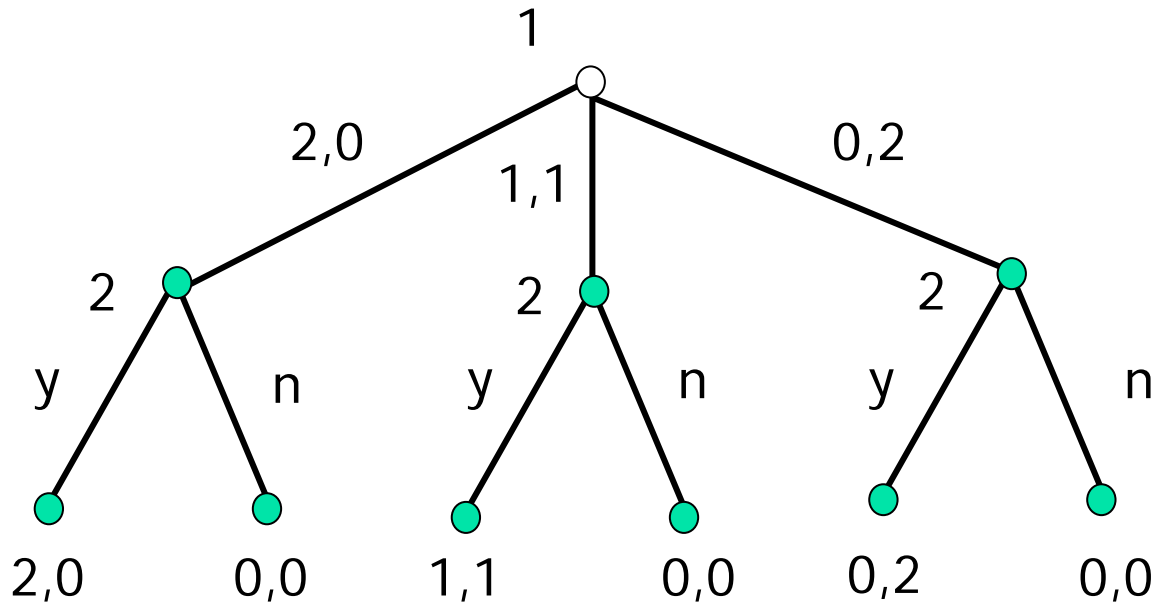
# equilibria (ctd)

- so: (A,R) and (B,L)
- interpretation (B,L):
- assuming that 2 plays L after A, B is the better choice for 1
- AR is the only subgame perfect
- equilibrium BL is not!

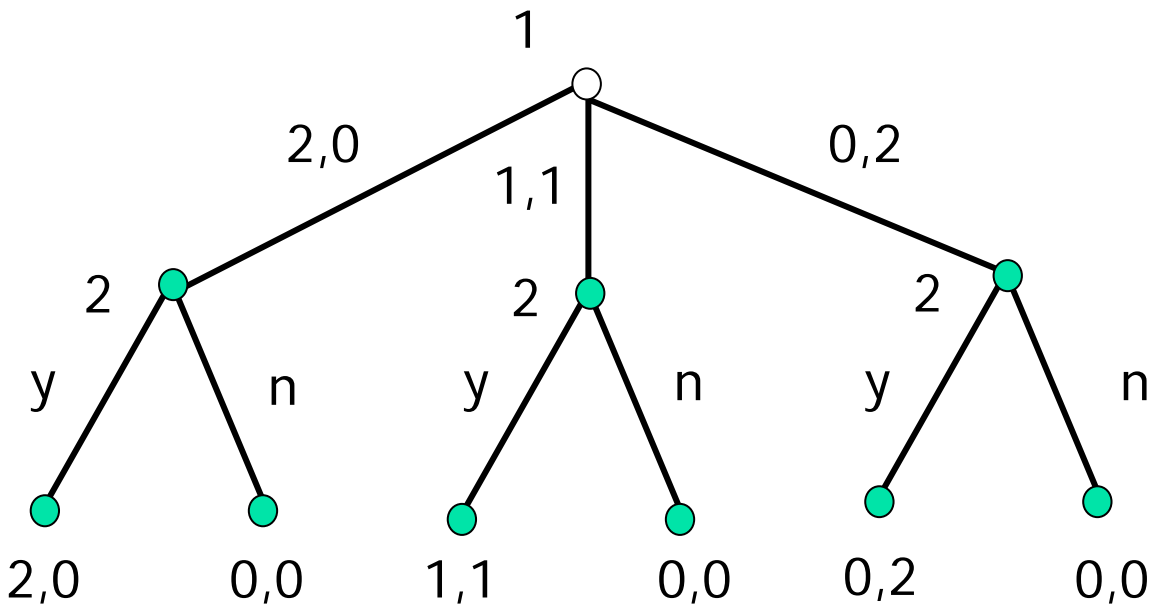


# Dividing goods

Nash equilibria:



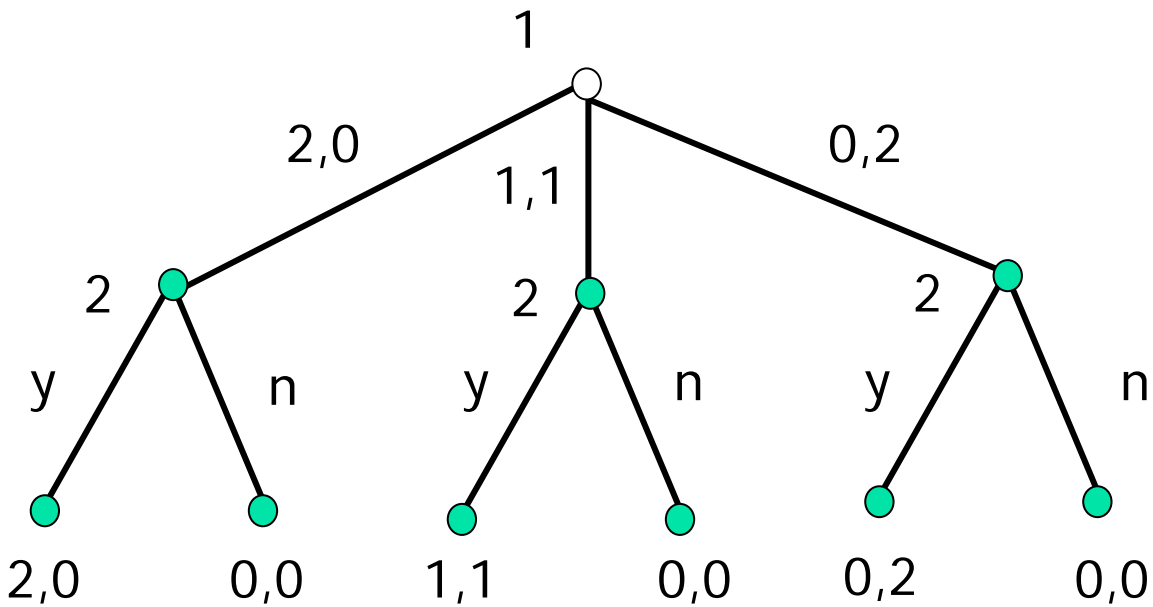
# Dividing goods



Nash equilibria:

$((2,0)yyy)$

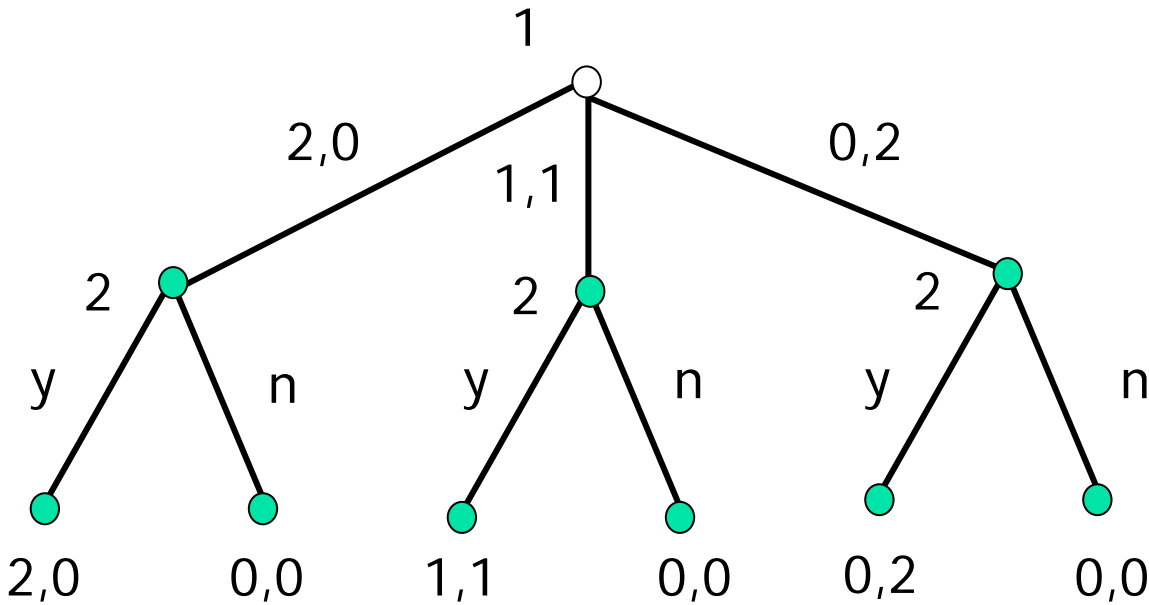
# Dividing goods



Nash equilibria:

- $((2,0)yyy)$
- $((2,0)yyn)$
- $((2,0)yny)$
- $((2,0)ynn)$

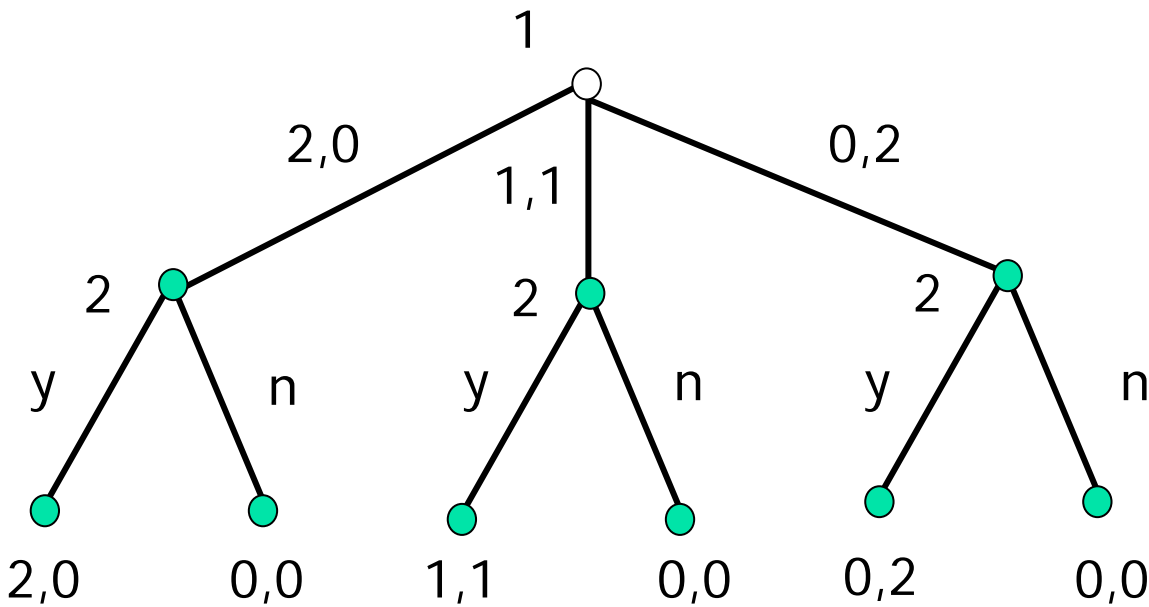
# Dividing goods



Nash equilibria:

- $((2,0)yyy)$
- $((2,0)yyn)$
- $((2,0)yny)$
- $((2,0)ynn)$
- $((1,1),nyy)$
- $((1,1),nyn)$

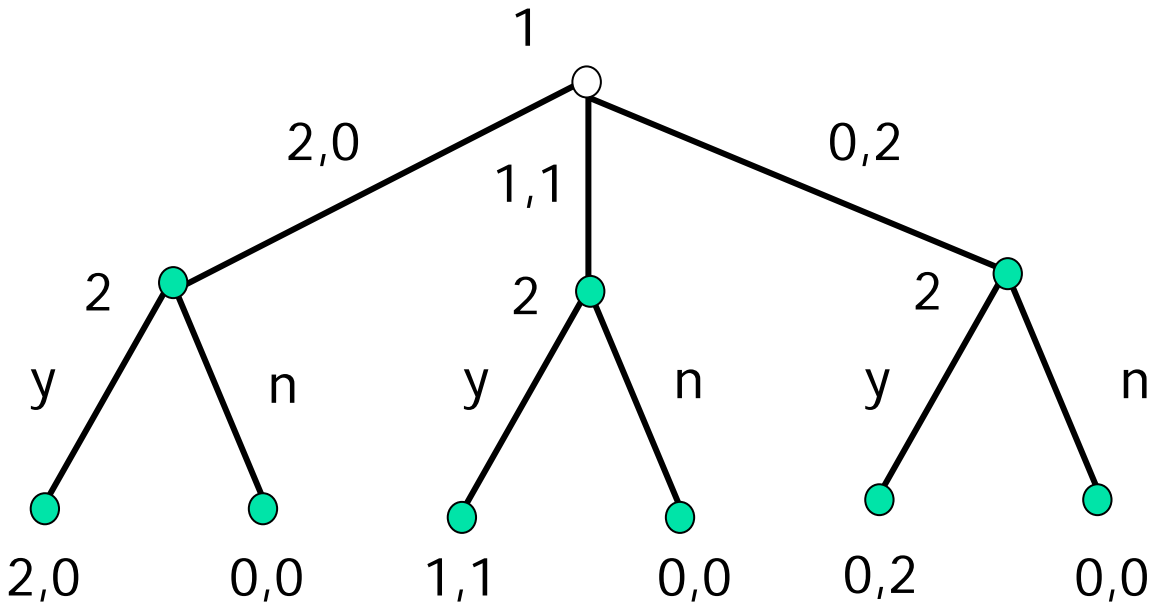
# Dividing goods



Nash equilibria:

- $((2,0), yyy)$
- $((2,0), yyn)$
- $((2,0), yny)$
- $((2,0), ynn)$
- $((1,1), nyy)$
- $((1,1), nyn)$
- $((0,2), nny)$

# Dividing goods

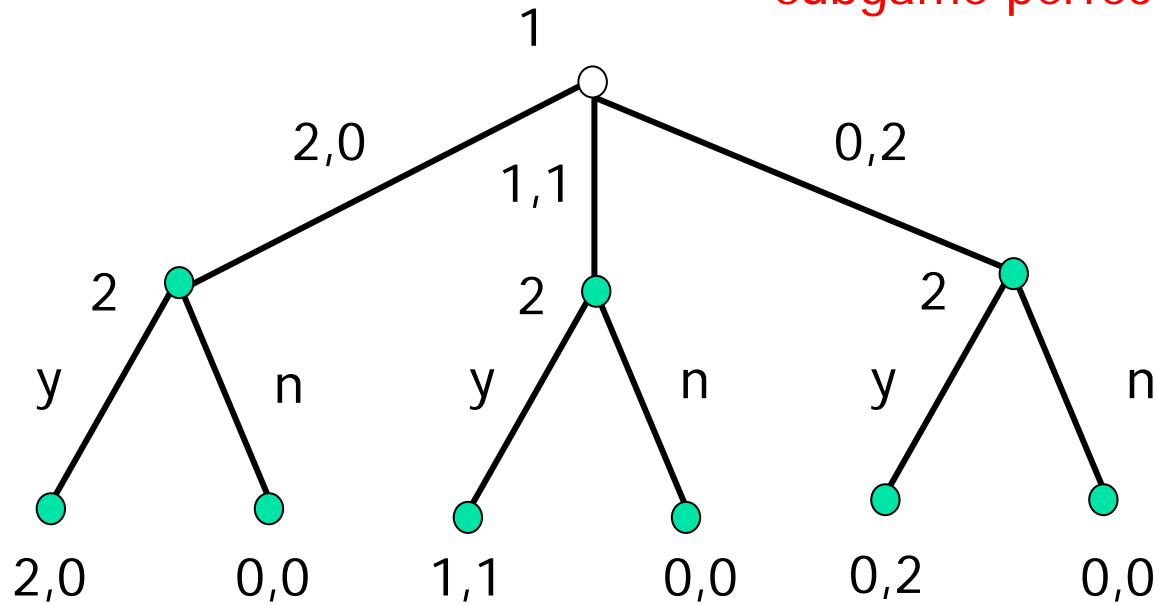


Nash equilibria:

$((2,0)yyy)$	2,0
$((2,0)yyn)$	2,0
$((2,0)yny)$	2,0
$((2,0)ynn)$	2,0
$((1,1),nyy)$	1,1
$((1,1),nyn)$	1,1
$((0,2),nny)$	0,2
$((2,0),nny)$	0,0
$((2,0),nnn)$	0,0

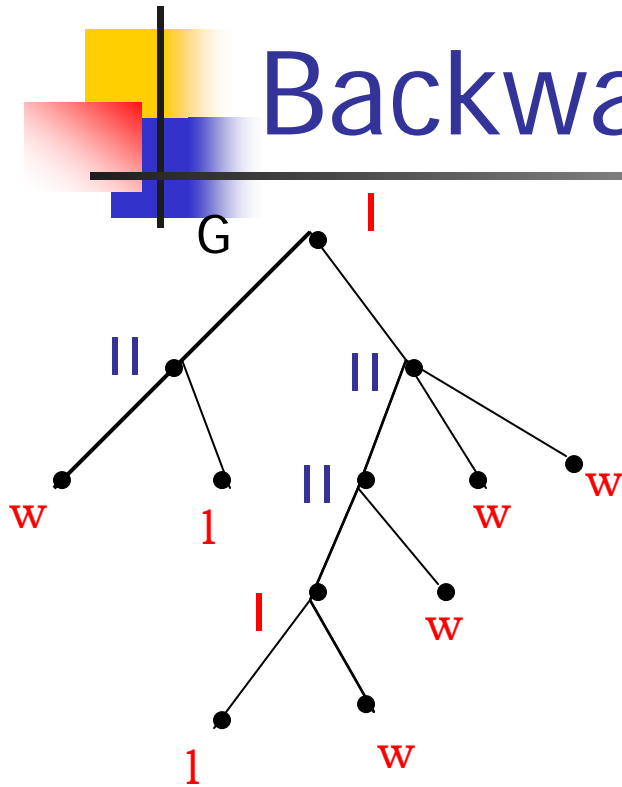
# Dividing goods

subgame perfect Nash equilibria:

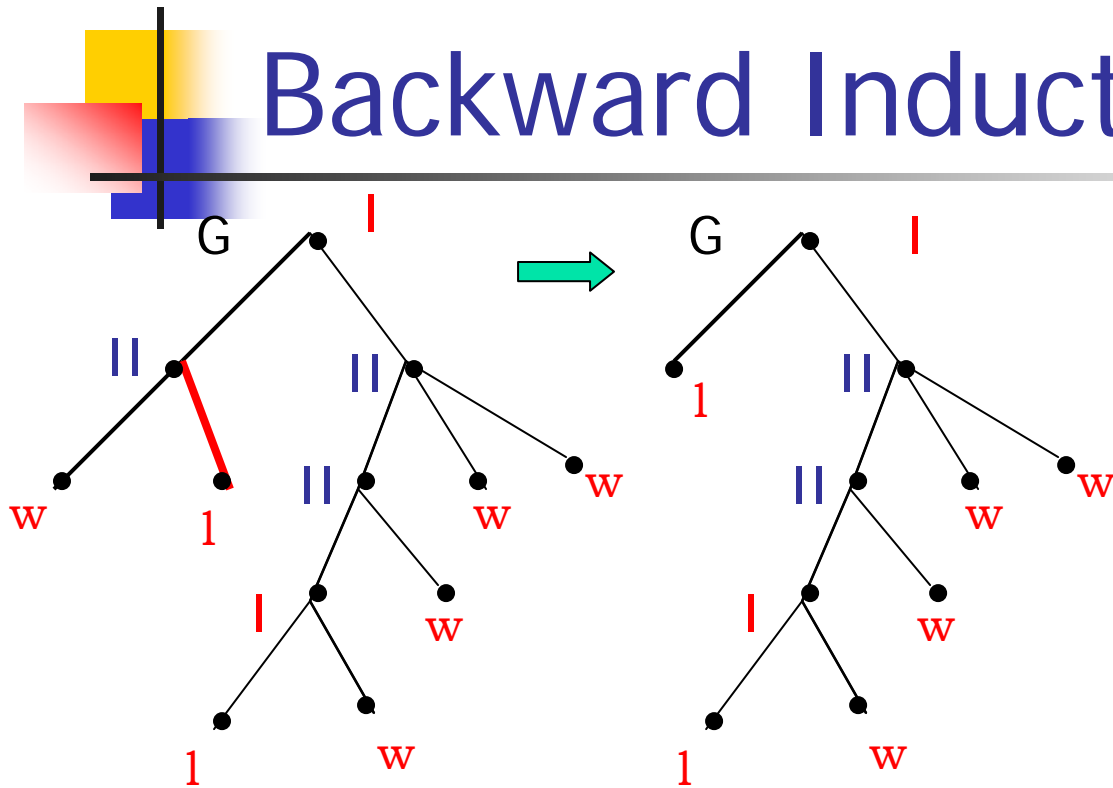


$((2,0), yyy)$	2,0
$((2,0), yyn)$	2,0
$((2,0), yny)$	2,0
$((2,0), ynn)$	2,0
$((1,1), nyy)$	1,1
$((1,1), nyn)$	1,1
$((0,2), nny)$	0,2
$((2,0), nny)$	0,0
$((2,0), nnn)$	0,0

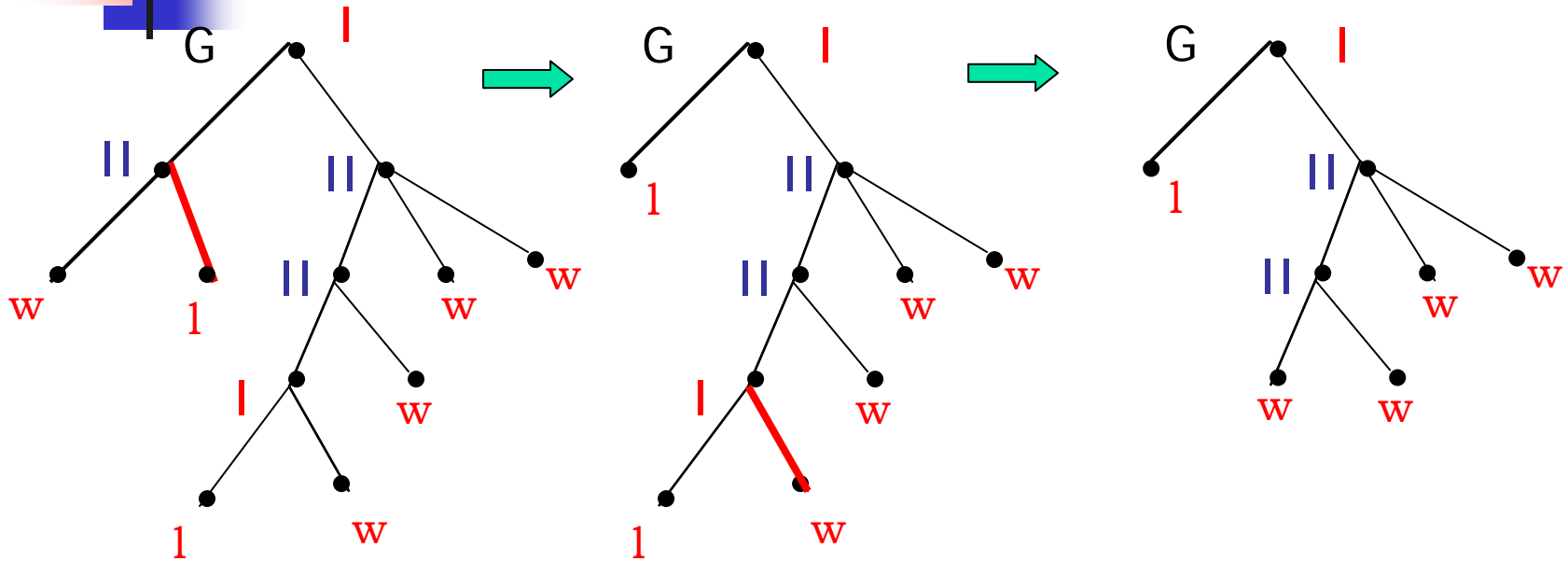
# Backward Induction



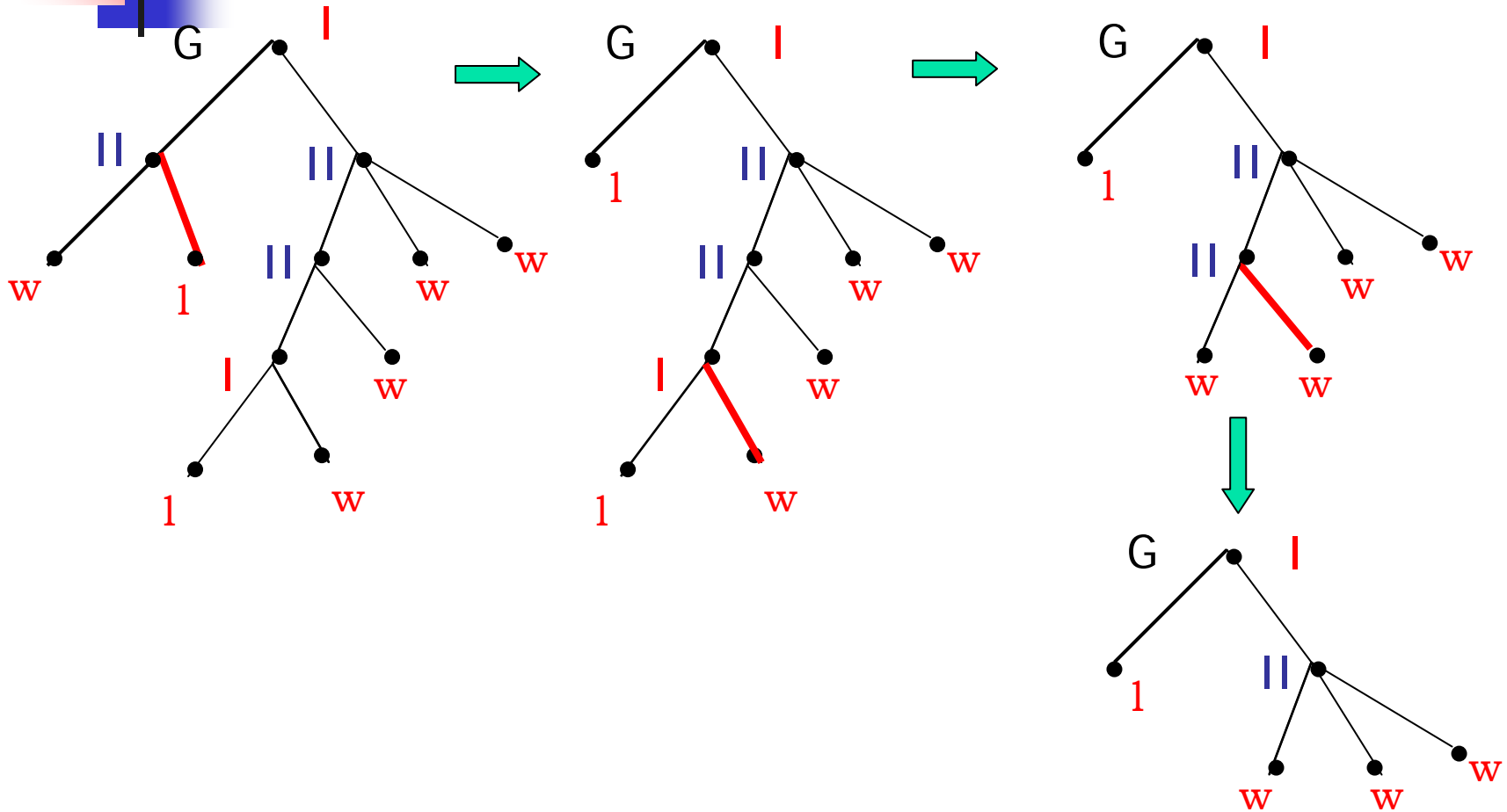
# Backward Induction



# Backward Induction

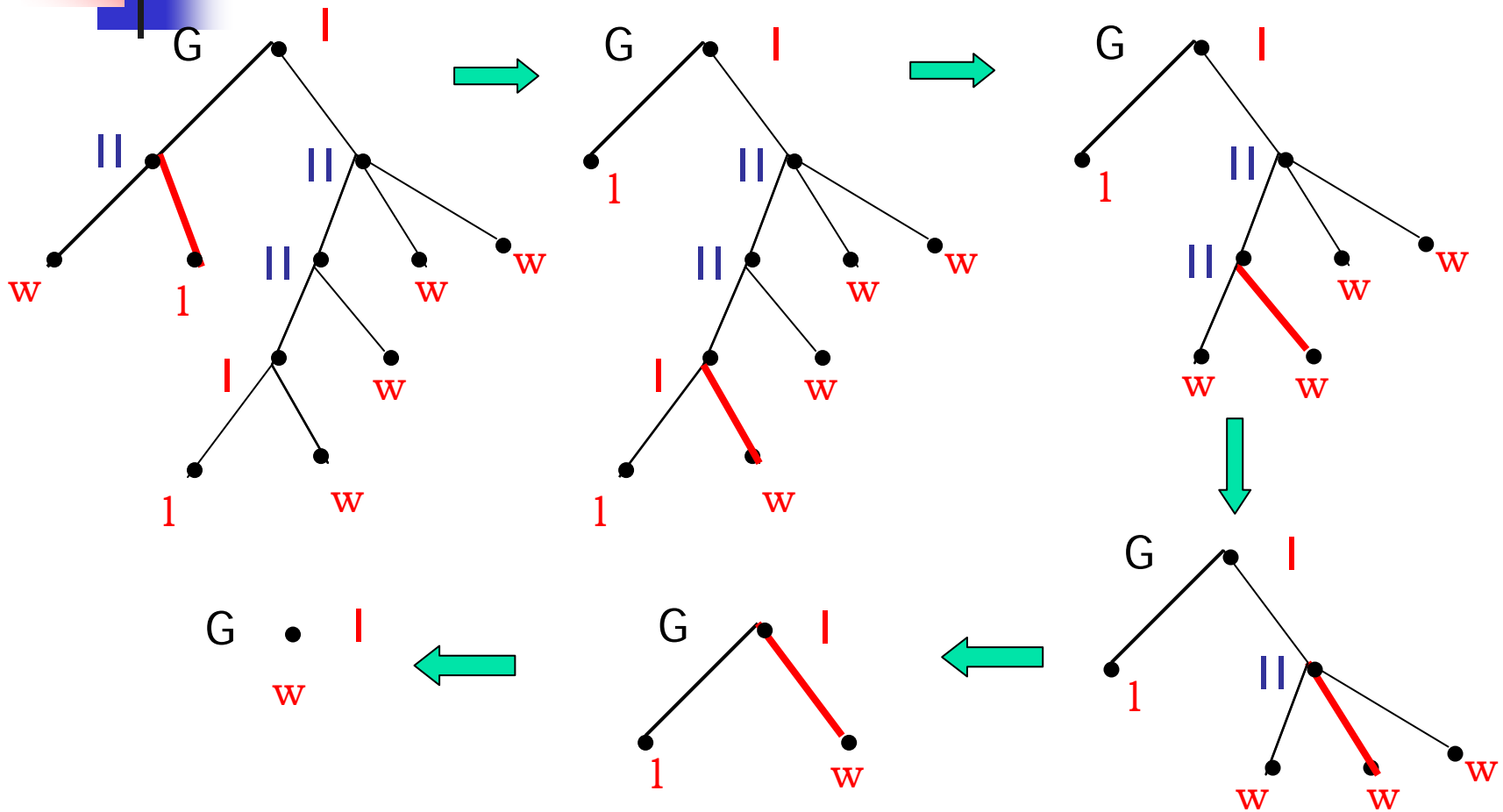


# Backward Induction

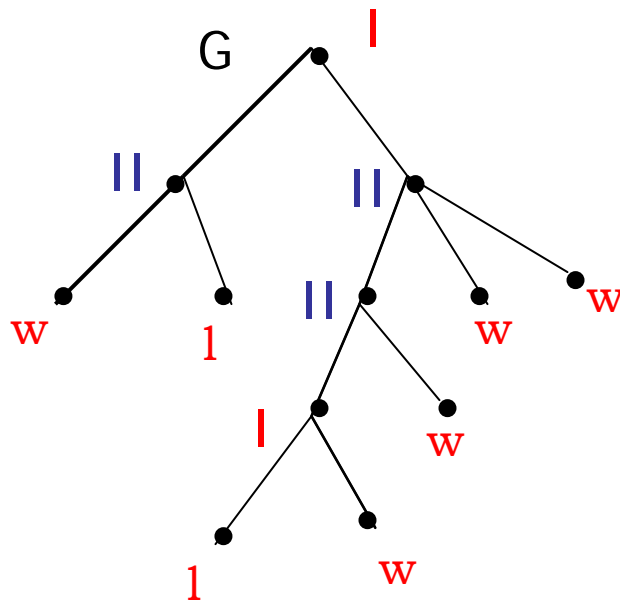




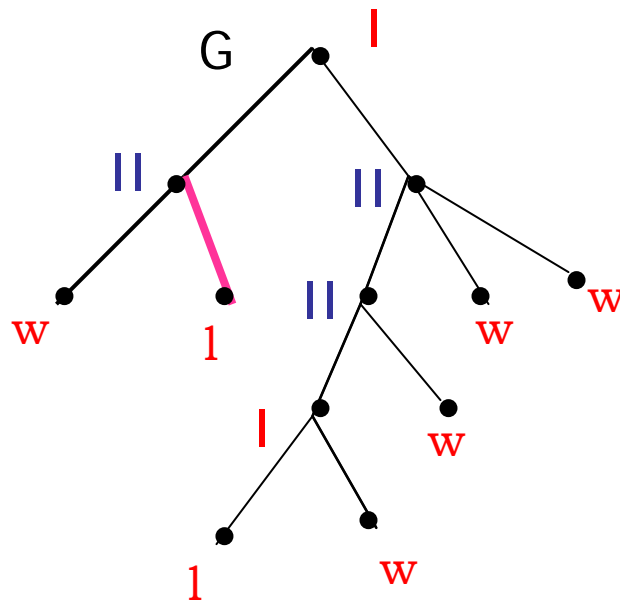
# Backward Induction



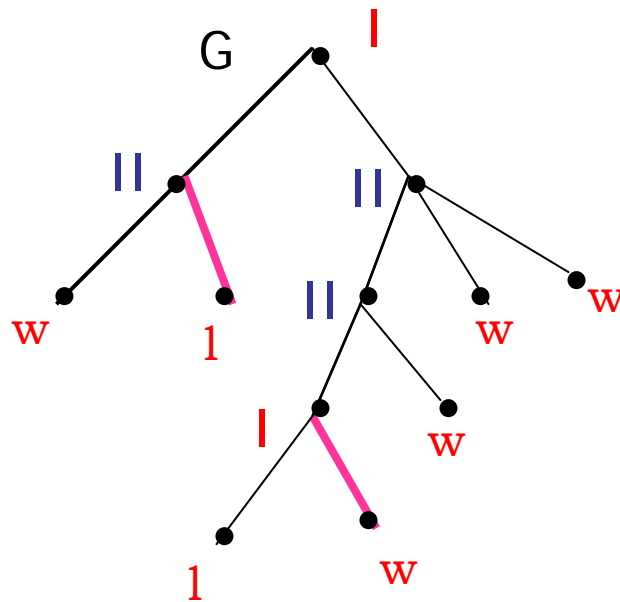
# backward induction



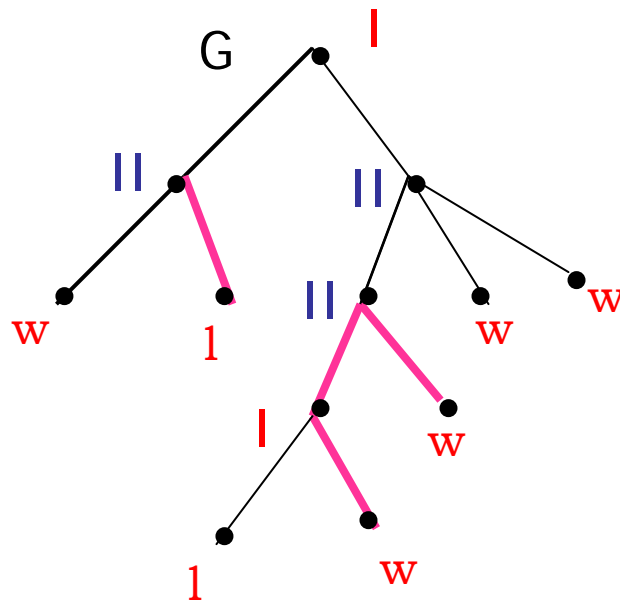
# backward induction



# backward induction



# backward induction









# I can win

---

	LLL	LLR	LML	LMR	LRL	LRR	RLL	RLR	RML	RMR	RRL	RRR
ll	w	w	w	w	w	w	l	l	l	l	l	l
lr	w	w	w	w	w	w	l	l	l	l	l	l
rl	l	w	w	w	w	w	l	w	w	w	w	w
rr	w	w	w	w	w	w	w	w	w	w	w	w

again: rr is winning strategy,  
because that row only contains w



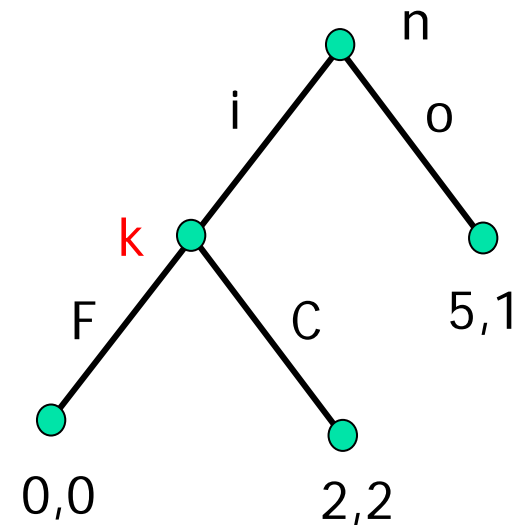
# Chain store game

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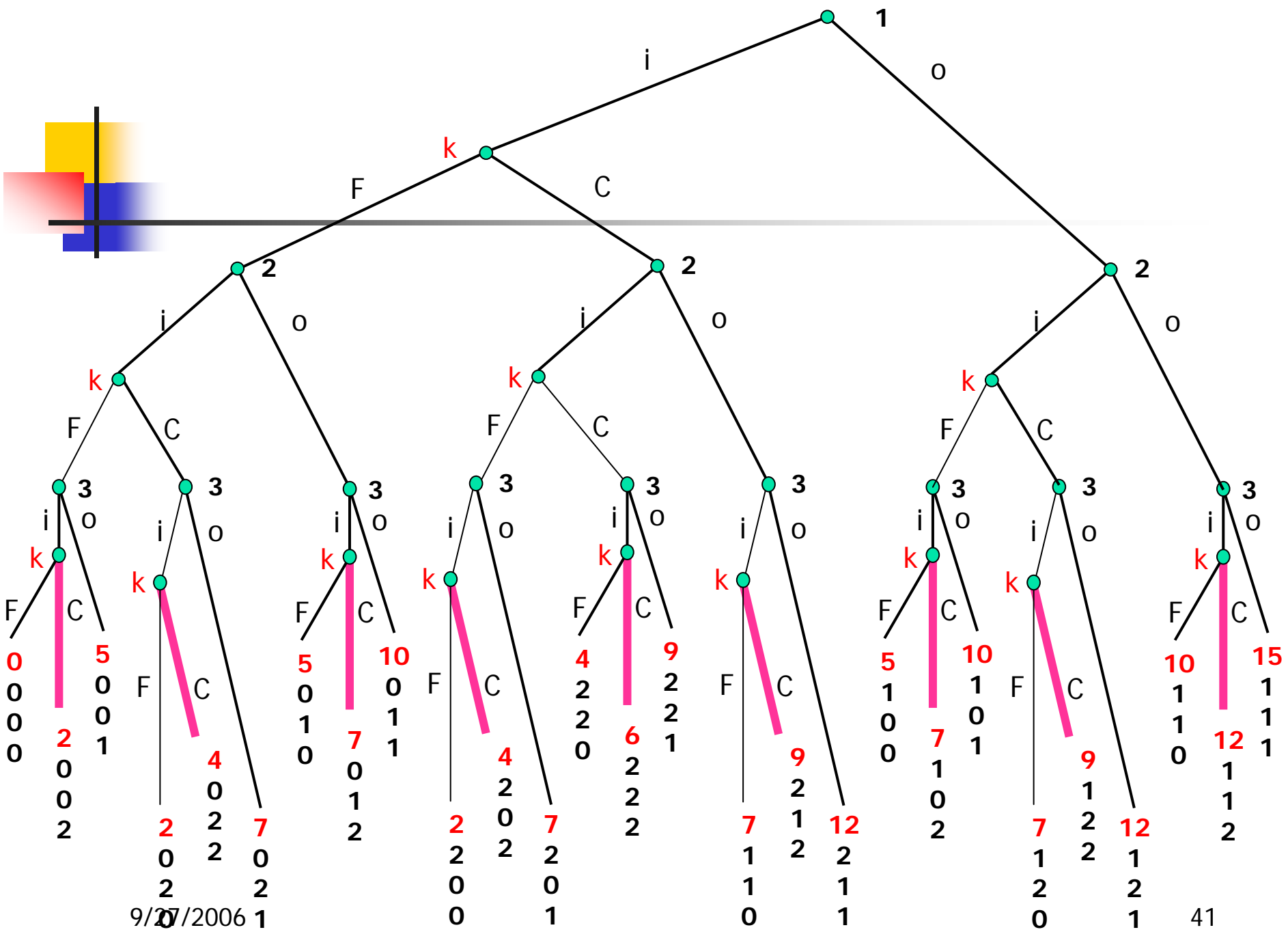
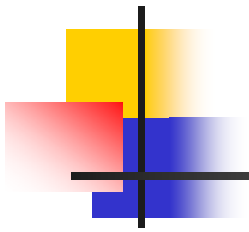
- Chain store **k** and  $n$  competitors
- Each competitor can fight (i) or cooperate (o)
- If so,  $k$  can choose between cooperate (c) or fight (f)

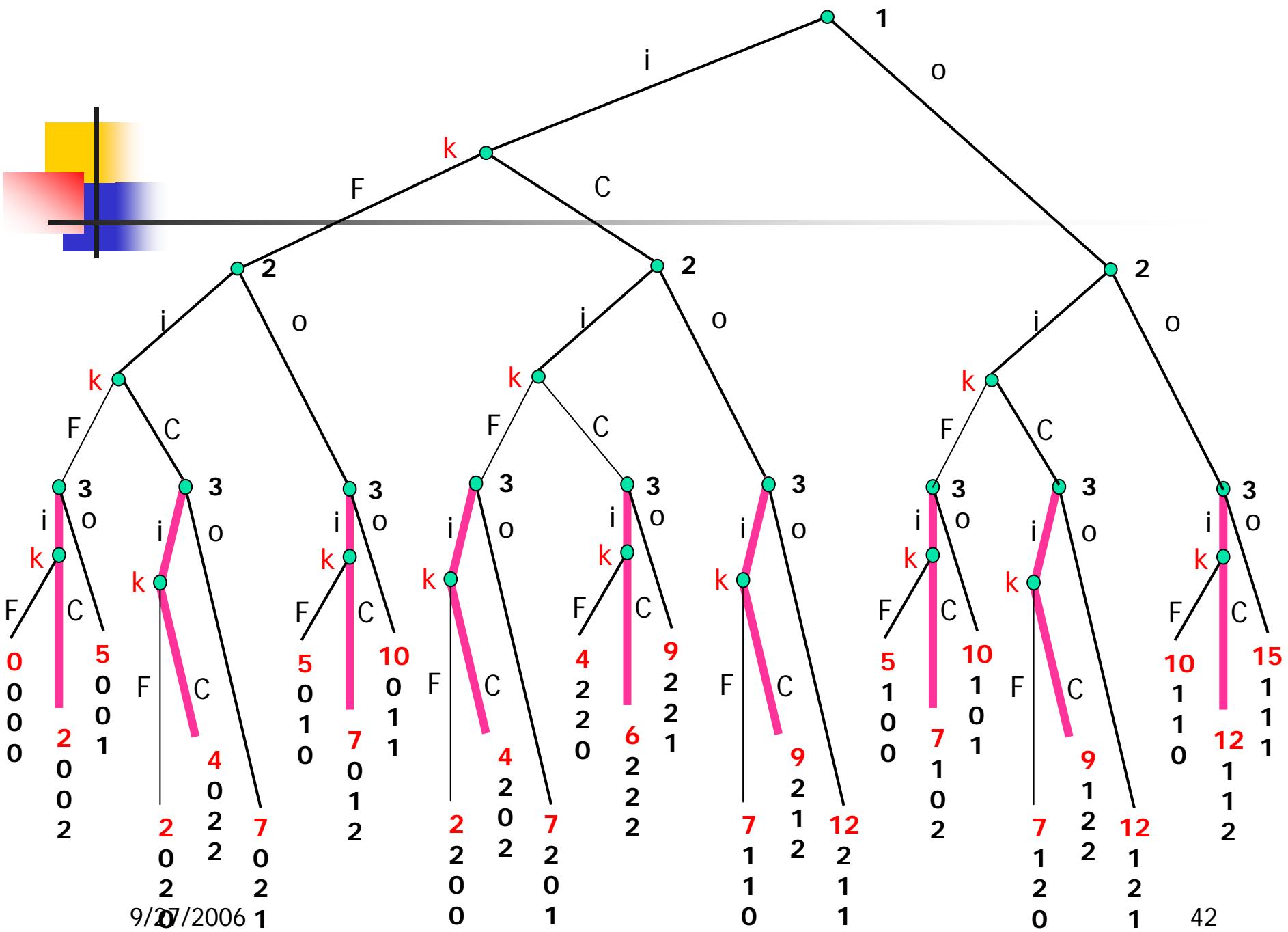
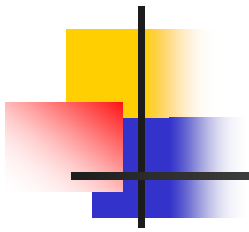
# Chain store game

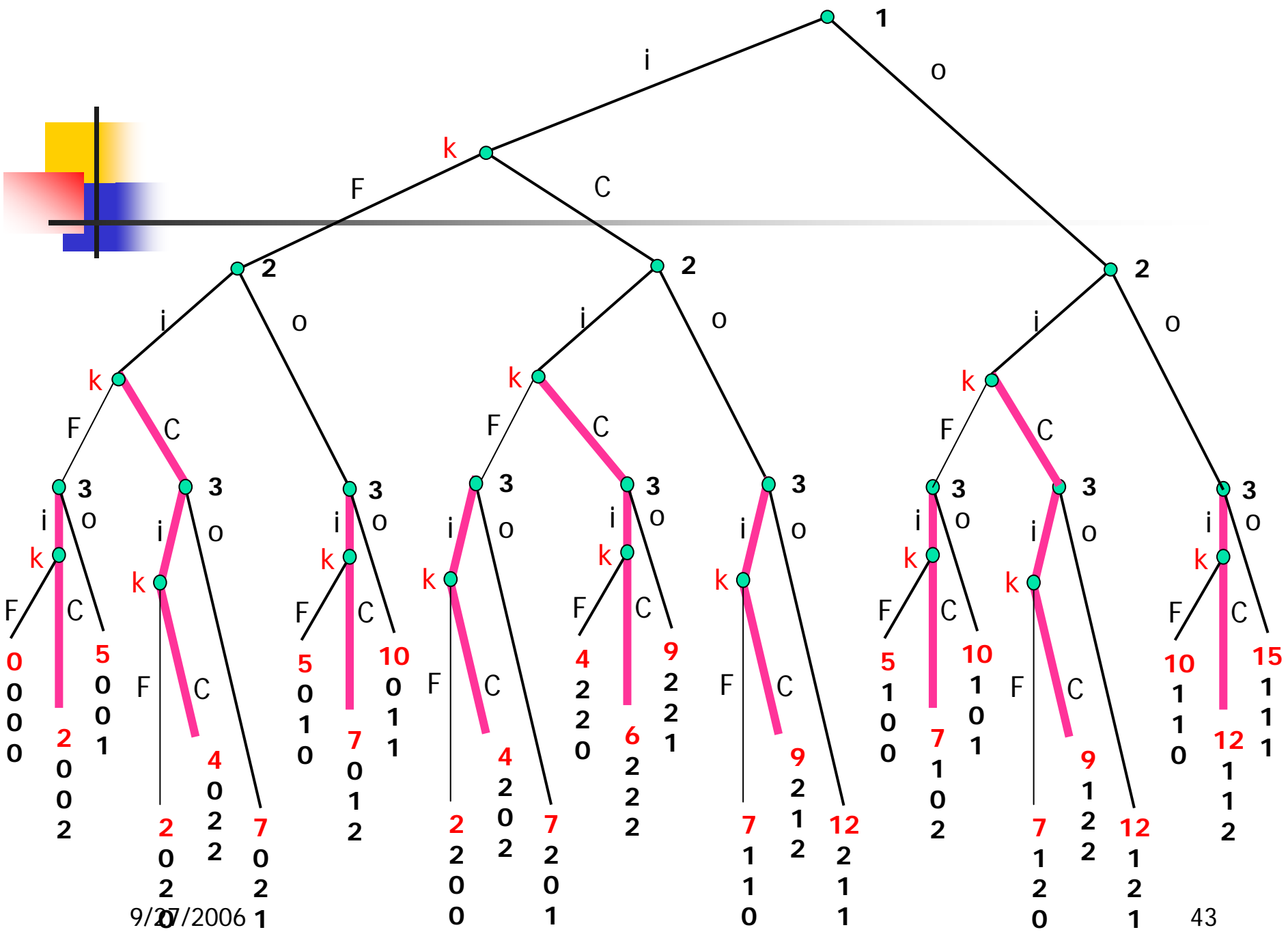
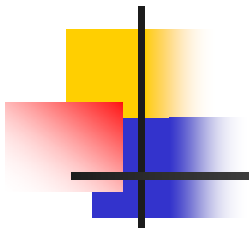
- Chain store **k** and  $n$  competitors
- Each competitor can fight (i) or cooperate (o)
- If so,  $k$  can choose between cooperate (c) or fight (f)

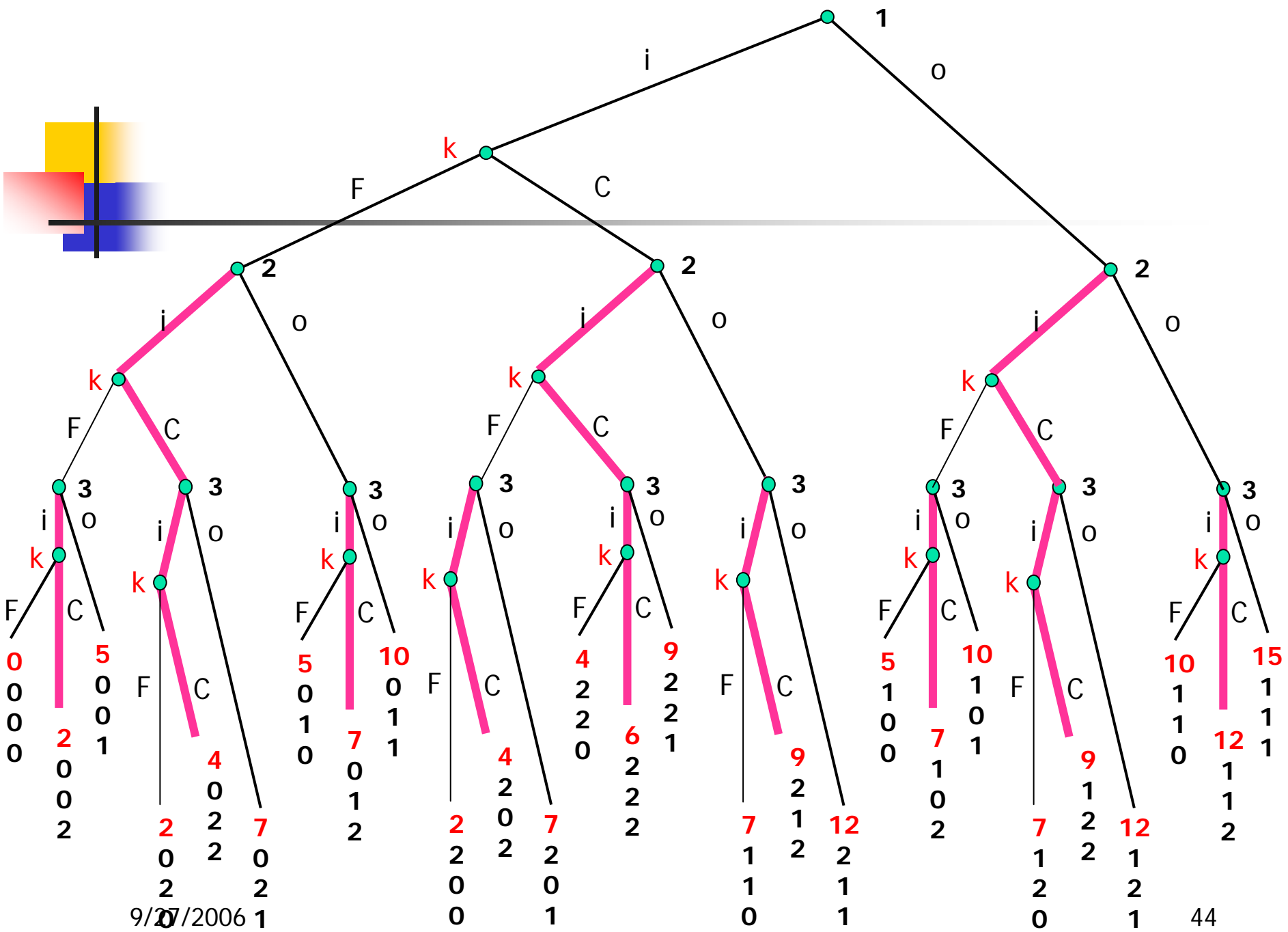
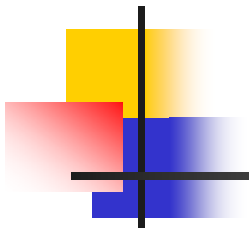


















# Chain store game

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- subgame perfect equilibrium:
  - all shops play  $i$ , chain store chooses  $C$
- not realistic, if many stores still have to choose
- solution: shops should be left uncertain about motives of  $k$

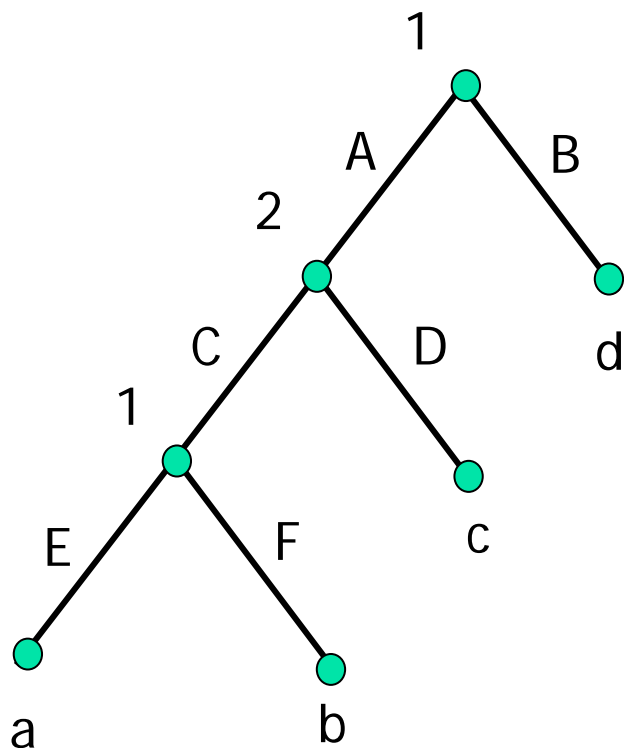


# From extensive to strategic

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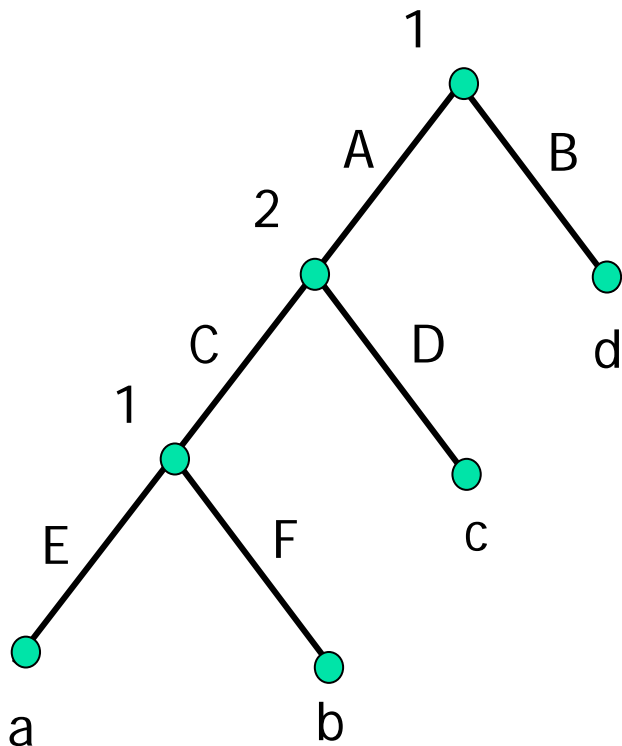
- let  $G = \langle N, H, P, (\geq_i) \rangle$  extensive game
- the strategic form is  $G' = \langle N, (S_i), (\geq'_i) \rangle$ :
  - $S_i$ : set of strategies for  $G$
  - $s \geq'_i s' \Leftrightarrow O(s) \geq_i O(s')$ ,
    - $s, s' \in S_1 \times \dots \times S_n$ ,

# From extensive to strategic



	C	D
AE	a	c
AF	b	c
BE	d	d
BF	d	d

# From extensive to strategic



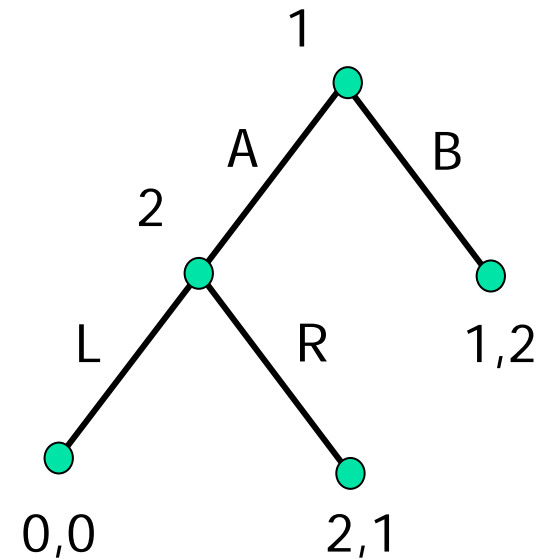
	C	D
AE	a	c
AF	b	c
B	d	d

	C	D
AE	a	c
AF	b	c
BE	d	d
BF	d	d

reduced strategic form

# Equilibria: example

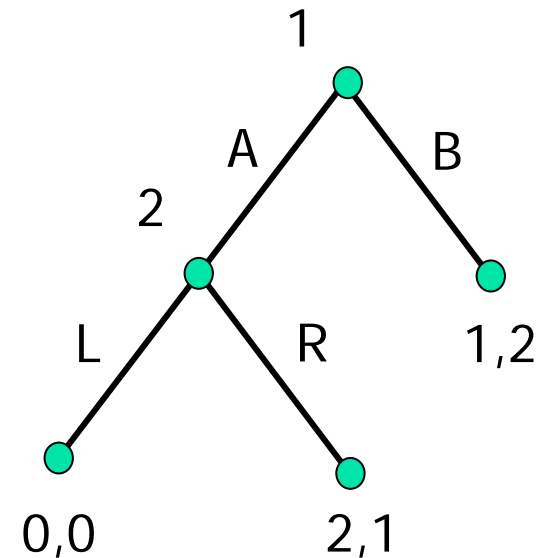
- Nash equilibria?



# Equilibria: example

- Nash equilibria?
- via strategic form:

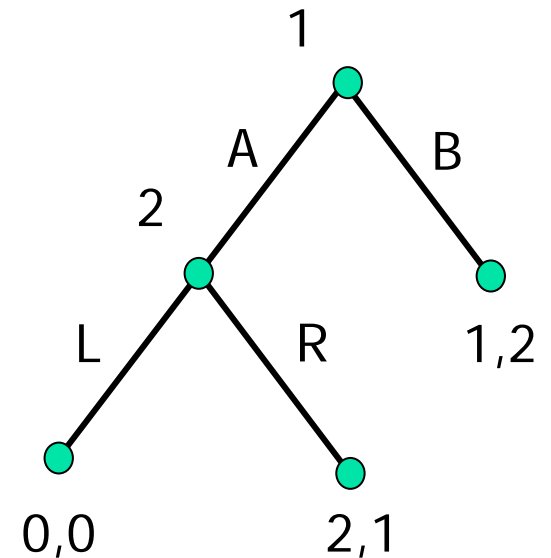
	L	R
A	0,0	2,1
B	1,2	1,2



# Equilibria: example

- Nash equilibria?
- via strategic form:

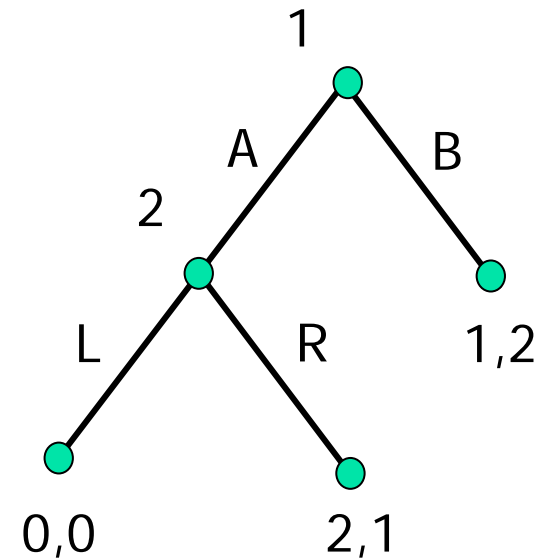
	L	R
A	0,0	2,1
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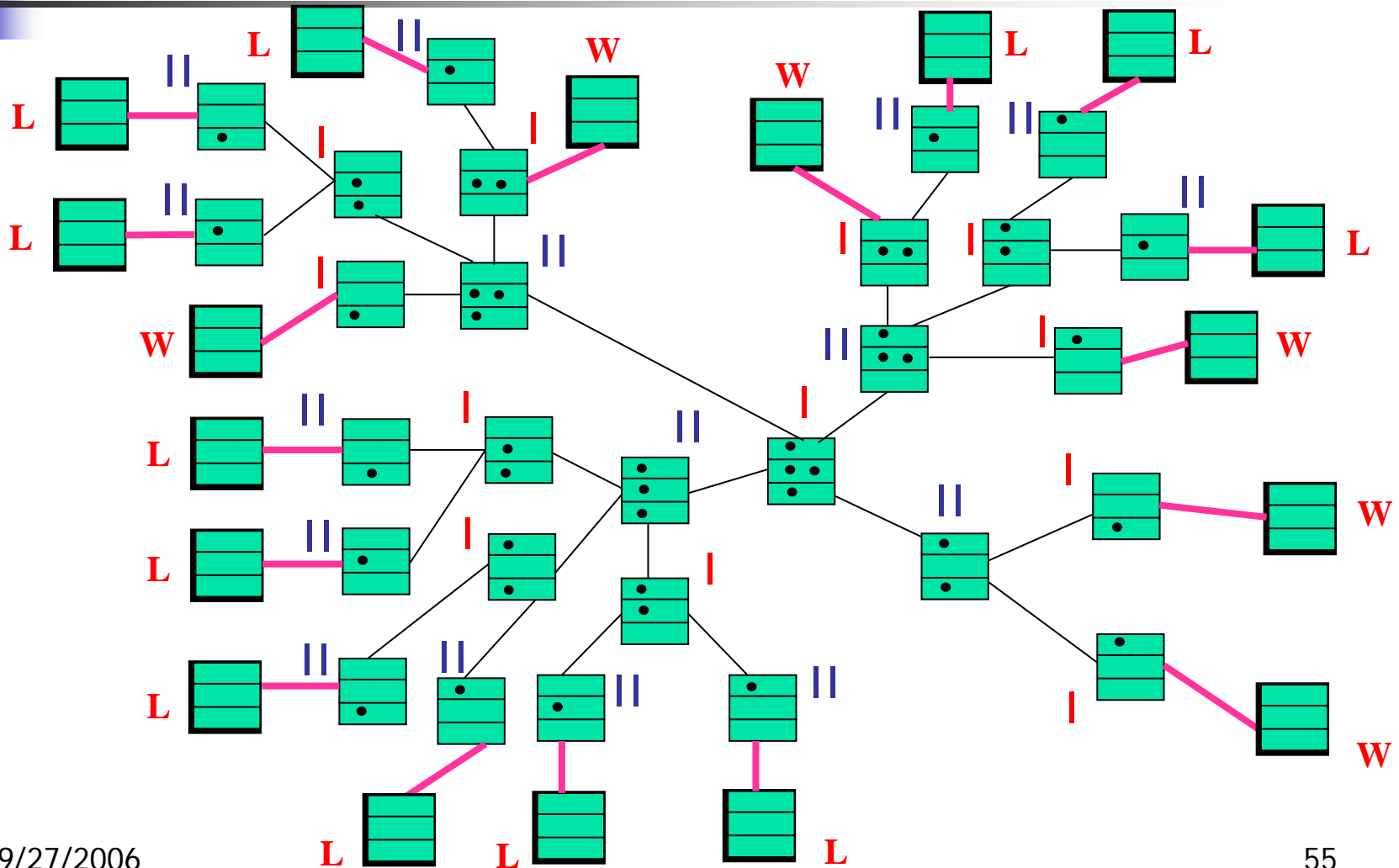
# Equilibria: example

- Nash equilibria?
- via strategic form:

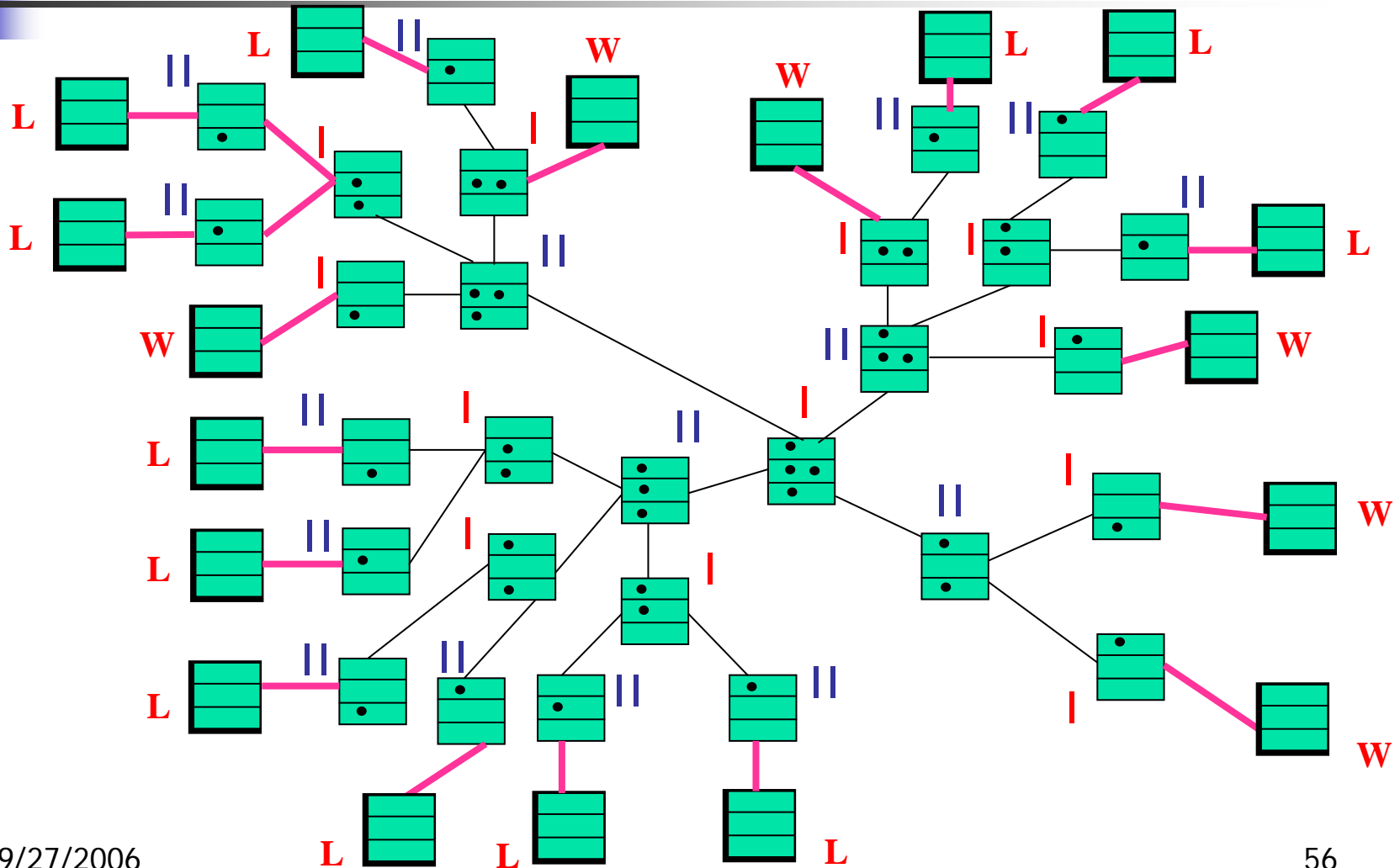
	L	R
A	0,0	2,1
B	1,2	1,2



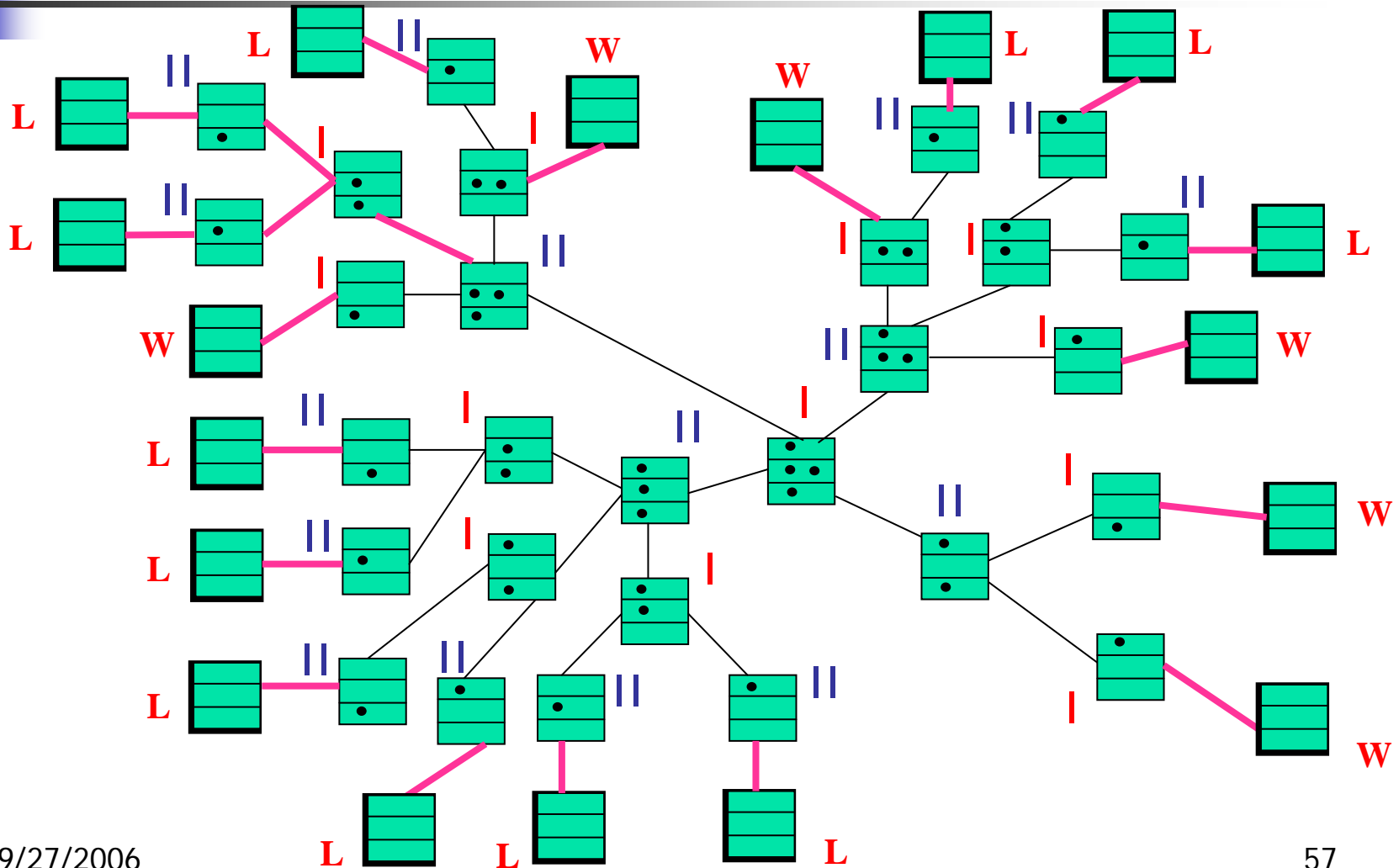
# backward induction: Nim



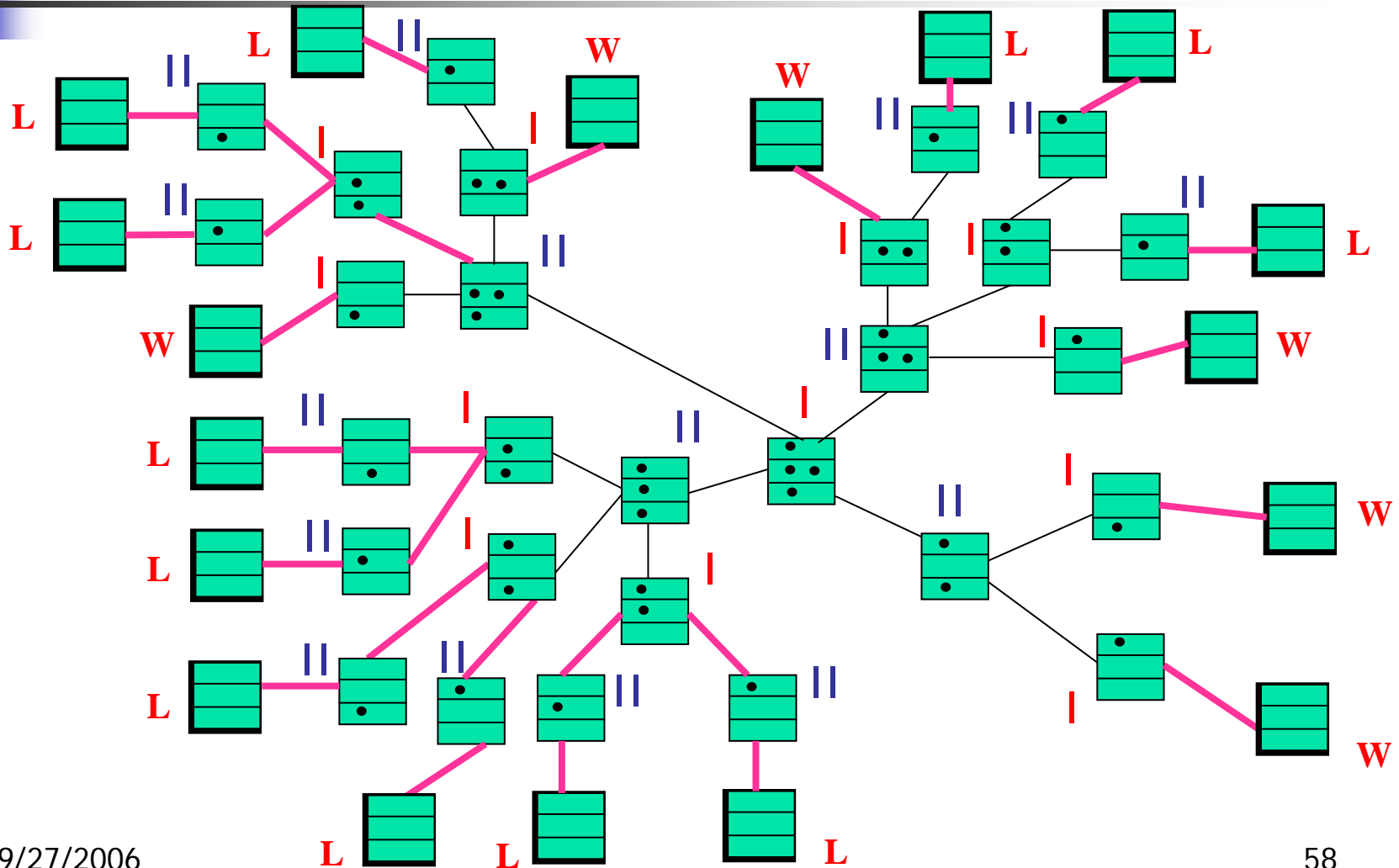
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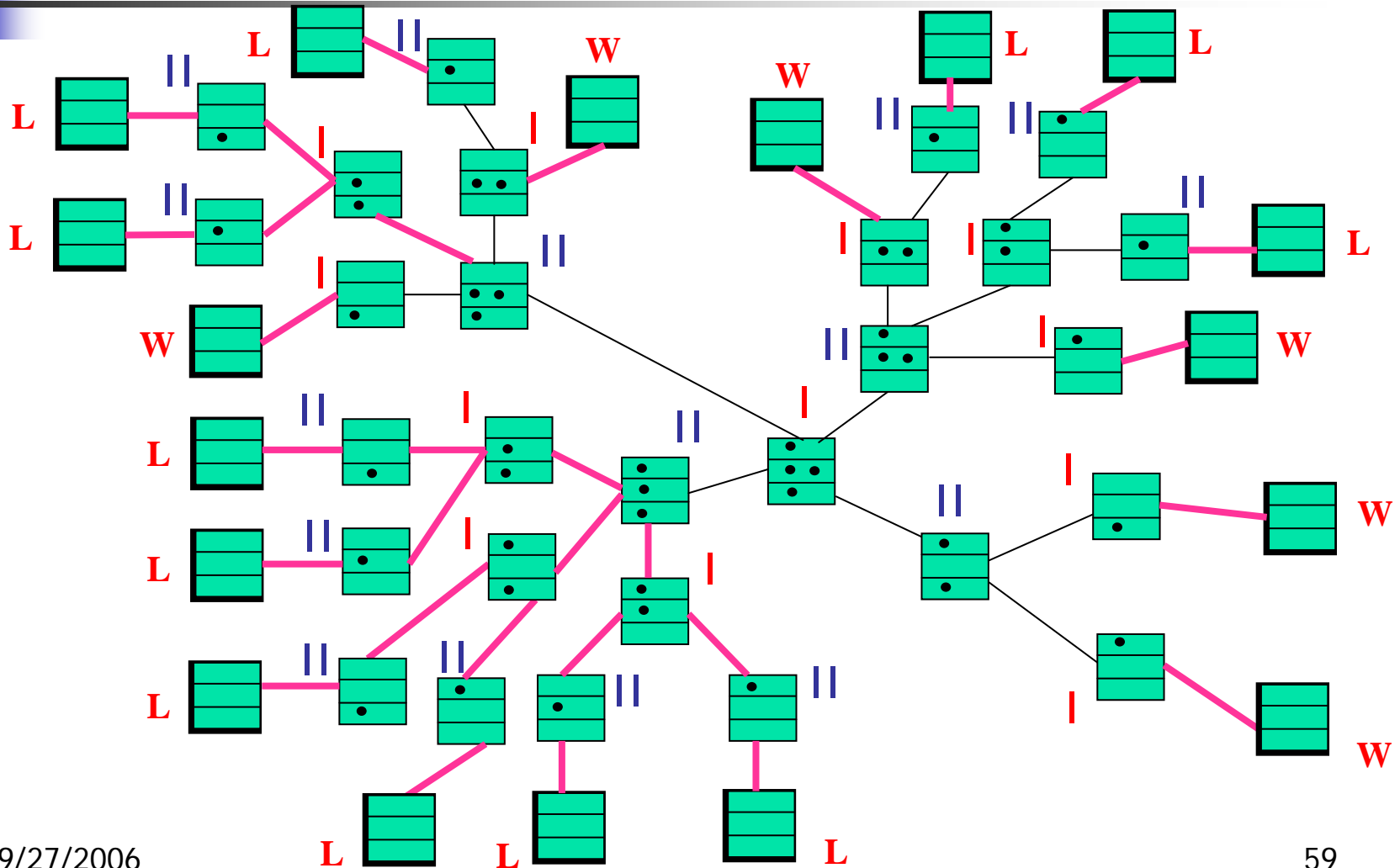
# backward induction: Nim



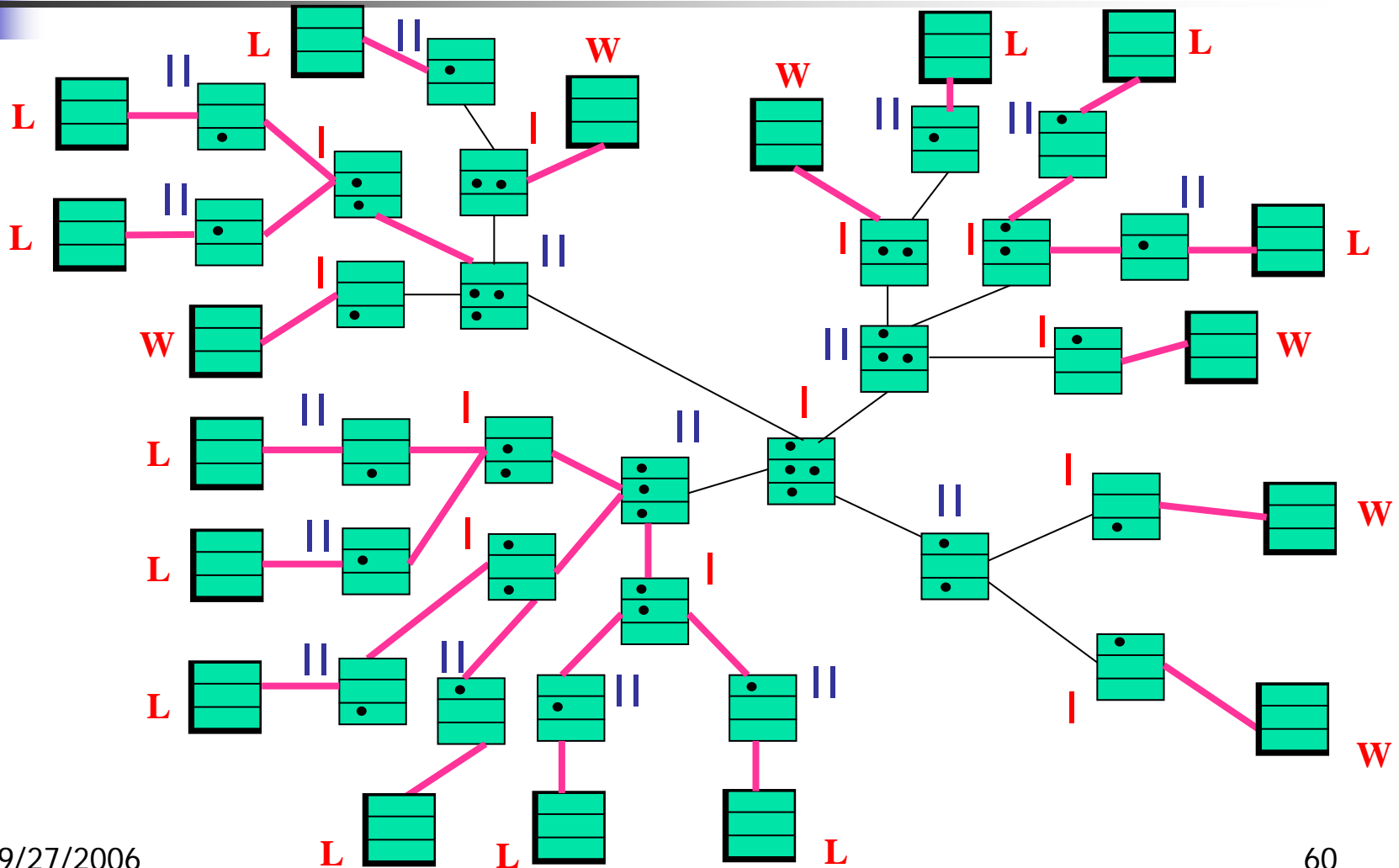
# backward induction: Nim



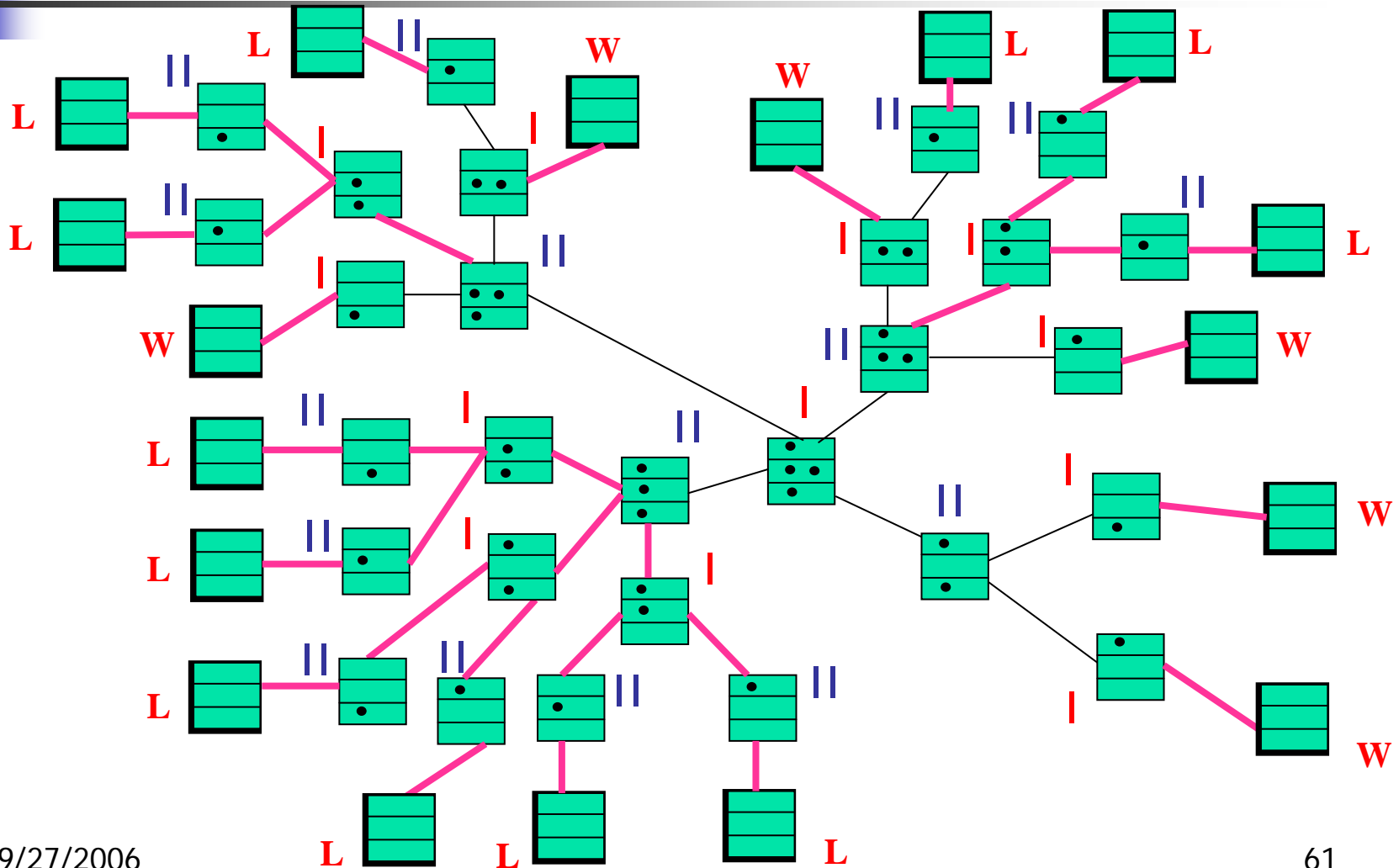
# backward induction: Nim



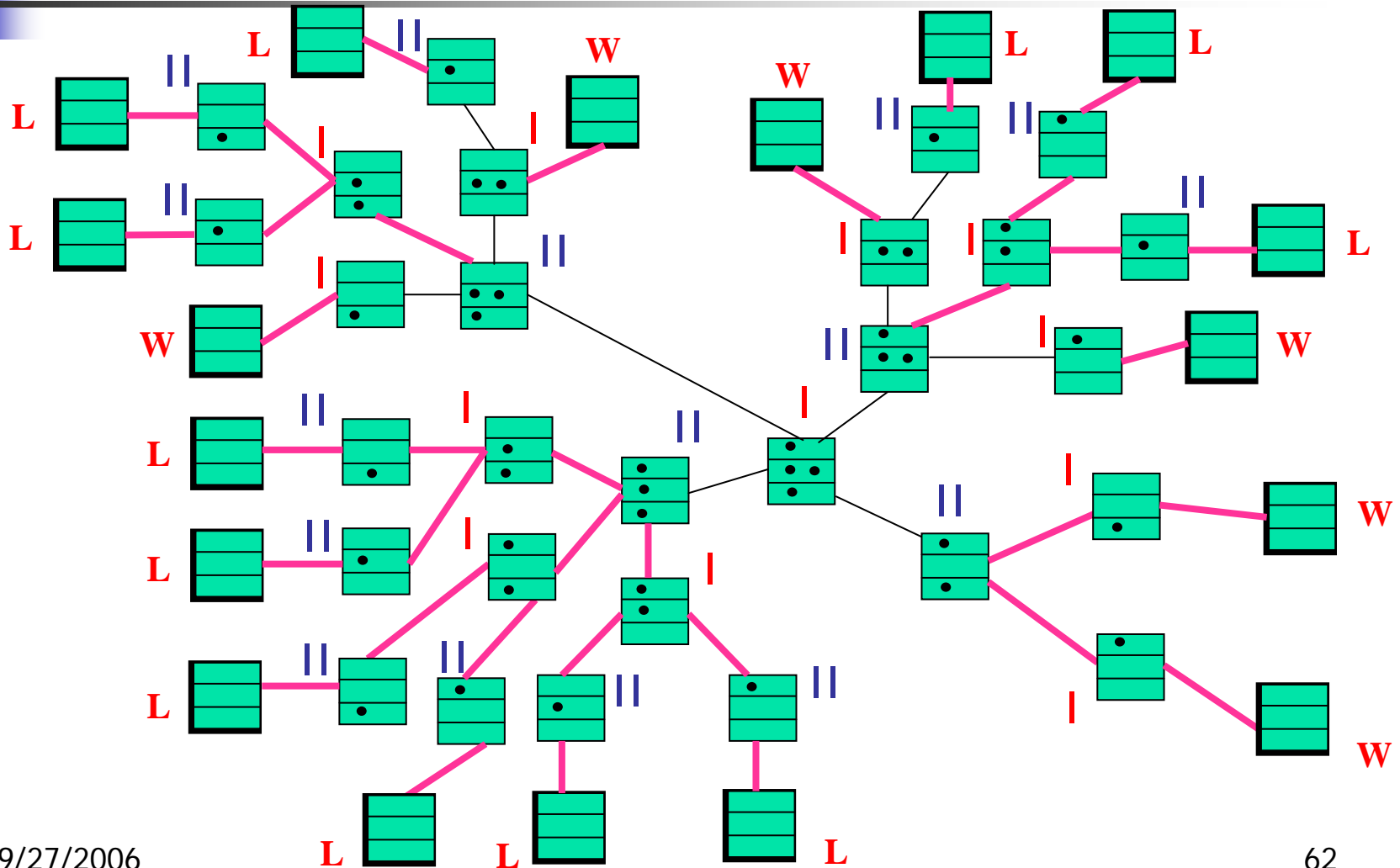
# backward induction: Nim



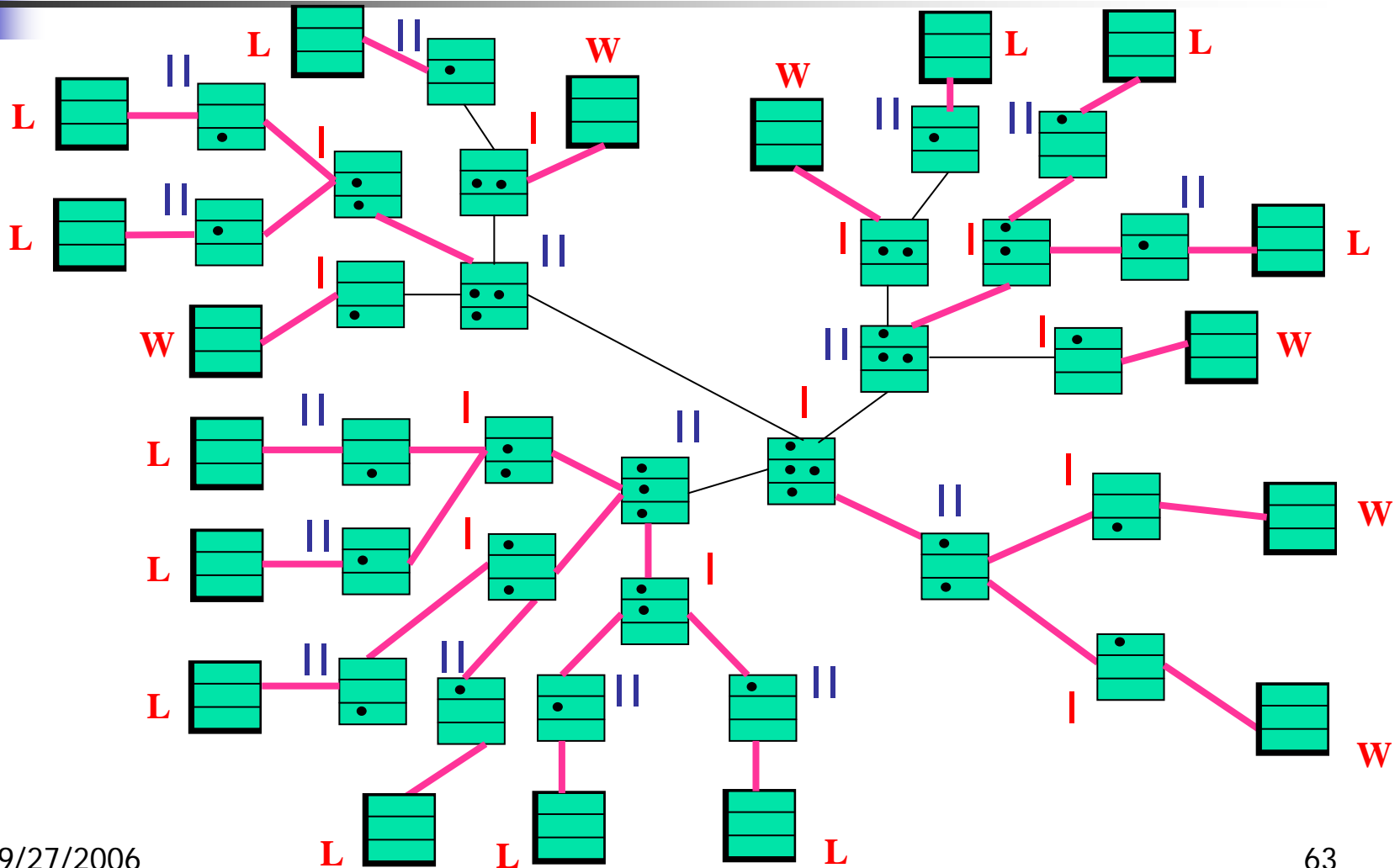
# backward induction: Nim



# backward induction: Nim

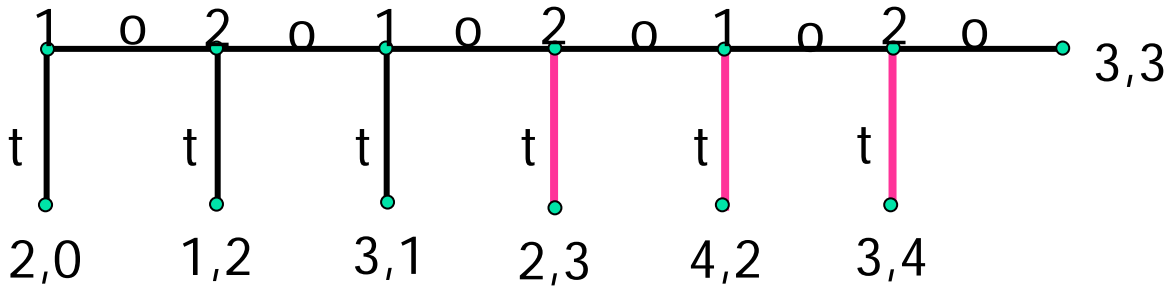
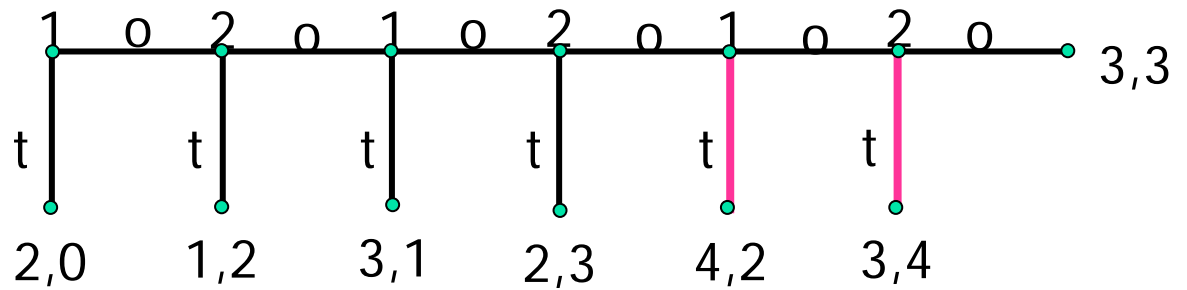
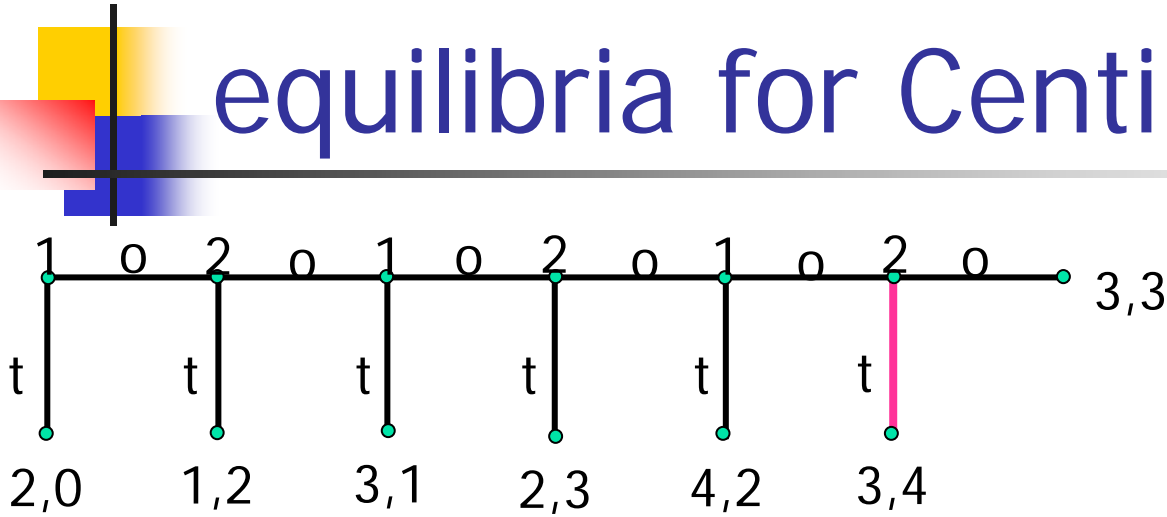


# backward induction: Nim





# equilibria for Centipede







# Strategic voting

---

- Boris, Horace and Maurice determine who can become member of the Dead Poet Society
  - proposal: admit Alice
  - amendment: Bob instead of Alice
  - First vote about amendment then about admission of extra member



# Strategic voting

---

Boris

Alice
Nobody
Bob

Horace

Nobody
Alice
Bob

Maurice

Bob
Alice
Nobody



# Strategic voting

---

- First between A, B
  - Winner

Horace		Maurice	
Nobody	Bob	Bob	Alice
Alice	Alice	Alice	Nobody
Bob	Bob	Nobody	Bob

# Strategic voting

- First between A, B
  - Winner Alice
- Then between A and N
  - winner

Horace		Maurice	
Nobody	Alice	Bob	Alice
Alice	Bob	Alice	Nobody
Bob		Nobody	Bob

# Strategic voting

- First between A, B
  - Winner Alice
- Then between A and N
  - winner Alice
- strategic voting H:

Horace		Maurice	
Nobody	Alice	Bob	Alice
Alice	Bob	Alice	Nobody
Bob			
Boris			
Alice	Nobody		
Bob			



# Strategic voting

- First between A, B
  - Winner Alice
- Then between A and N
  - winner Alice
- strategic voting H:
  - First vote for Bob!
  - result...

Horace

Nobody
Alice
Bob

Maurice

Bob
Alice
Nobody

Boris

Alice
Nobody
Bob



# Strategic voting

- First between A, B
  - Winner Alice
- Then between A and N
  - winner Alice
- strategic voting H:
  - First vote for Bob!
  - result... .. B, N

Maurice	
	Bob
	Alice
	Nobody

Horace	
	Nobody
	Alice
	Bob

Boris	
	Alice
	Nobody
	Bob



# Strategic voting

- First between A, B
  - Winner Alice
- Then between A and N
  - winner Alice
- strategic voting H:
  - First vote for Bob!
  - result... .. B, N
- M anticipates: votes for A

Horace

Nobody
Alice
Bob

Maurice

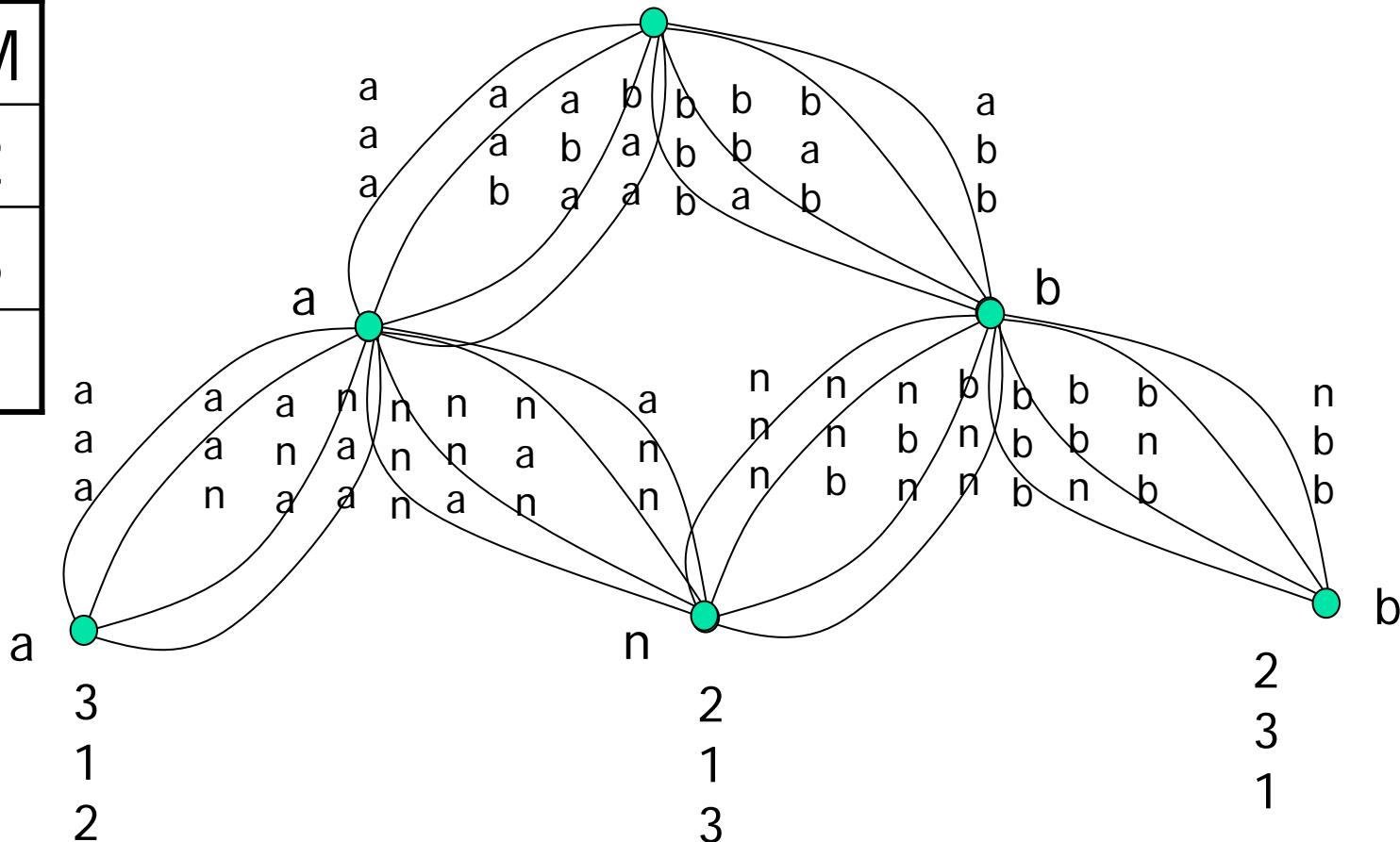
Bob
Alice
Nobody

Boris

Alice
Nobody
Bob

# Strategic voting: extensive

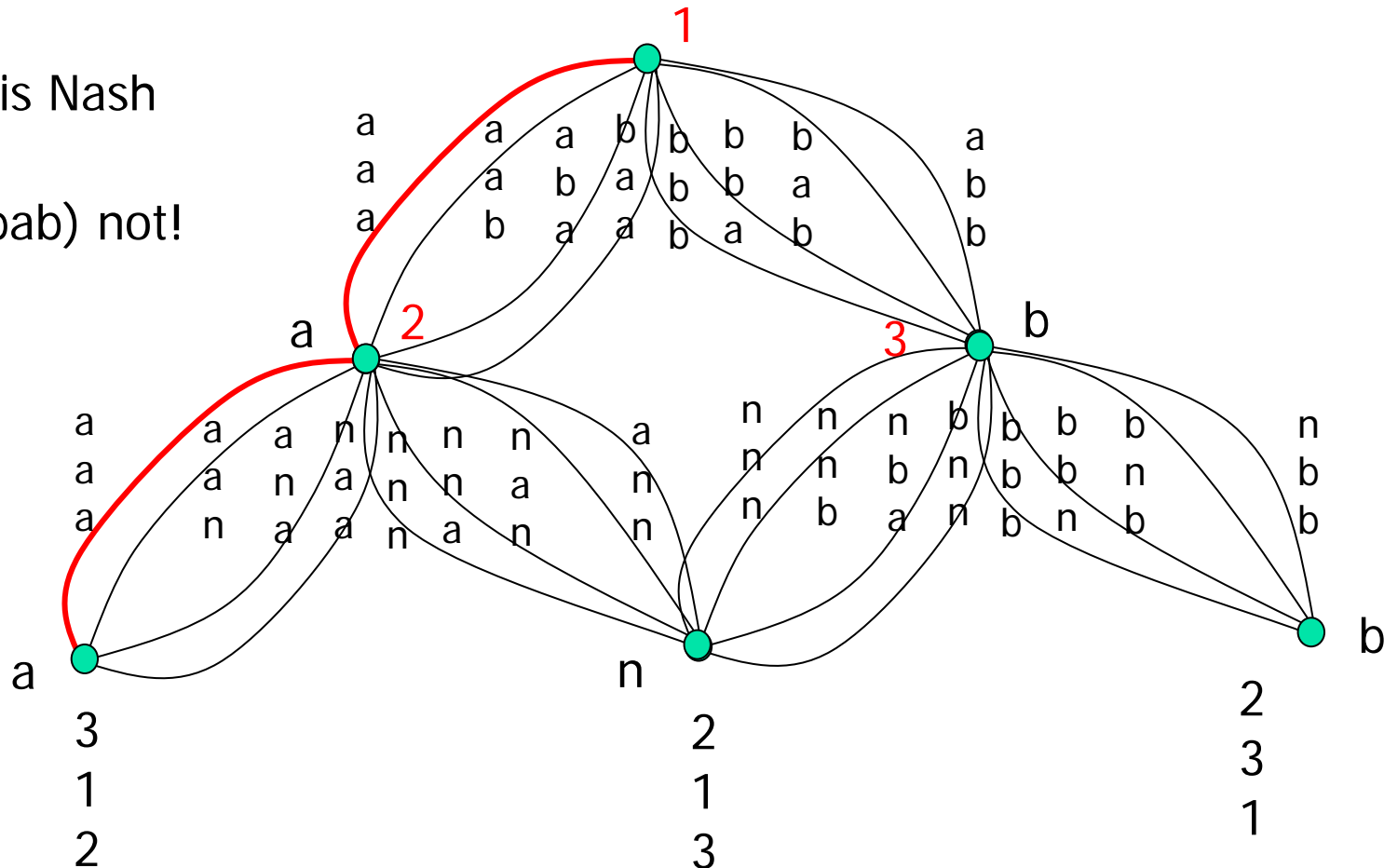
u	B	H	M
a	3	2	2
b	1	1	3
n	2	3	1



# Strategic voting: extensive

(aax,aay,aaz) is Nash

(aab,aab,bab) not!



# Strategic voting: extensive

(aax,aay,aaz) is Nash

(aab,aab,bab) not!

H can do better: bxb

