SCSG

- Strictly Competitive Strategic Game
- if $G = \langle\{1,2\},(A_i),(\geq_i)\rangle$,
- and $\forall a,b \in A$: $a \geq_1 b \Leftrightarrow b \geq_2 a$

SCSG

- Strictly Competitive Strategic Game
- if $G = \langle \{1,2\},(A_i),(\geq_i)\rangle$,
- and $\forall a,b \in A$: $a \geq_1 b \Leftrightarrow b \geq_2 a$
- zero-sum game if:
  - $\forall x\ u_1(x) + u_2(x) = 0$

Finding a strategy: maxminimizer

- let $G = \langle\{1,2\},(A_i),(\geq_i)\rangle$ be a SCSG
- action $x^* \in A_1$ is maxminimizer for 1:
- $\forall x \in A_1 \; \min_{y \in A_2} u_1(x^*,y) \geq_1 \min_{y \in A_2} u_1(x,y)$

maxminimizers

- solve $\max_x \min_y$ for 1 $u_1(x,y)$ =
  - max{
  - $\min\{u_1(x,y) | y \in A_2\}$
  - $| x \in A_1\}$ =

…	y_1	y_2	…	…	…	y_k
x_1	…	…	…	…	…	…
x_2	…	…	…	…	…	…
…	…	…	…	…	…	…
	…	…	…	…	…	…
…	…	…	…	…	…	…
x_n	…	…	…	…	…	…

maxminimizers

- solve $\max_x \min_y$ for 1 $u_1(x,y)$ =
  - max{
  - $\min\{u_1(x,y) | y \in A_2\}$
  - $| x \in A_1\}$ =

x_1: $\min_y u(x_1,y) = a_1$

…	y_1	y_2	…	…	…	y_k
x_1	…	…	…	a_1	…	…
x_2	…	…	…	…	…	…
…	…	…	…	…	…	…
	…	…	…	…	…	…
…	…	…	…	…	…	…
x_n	…	…	…	…	…	…

maxminimizers

- solve $\max_x \min_y$ for 1 $u_1(x,y)$ =
  - max{
  - $\min\{u_1(x,y) | y \in A_2\}$
  - $| x \in A_1\}$ =

x_1: $\min_y u(x_1,y)$ = a_1

x_2: $\min_y u(x_2,y)$ = a_2

…	y_1	y_2	…	…	…	y_k
x_1	…	…	…	a_1	…	…
x_2	a_2	…	…	…	…	…
…	…	…	…	…	…	…
	…	…	…	…	…	…
…	…	…	…	…	…	…
x_n	…	…	…	…	…	…

maxminimizers

- solve $\max_x \min_y$ for 1 $u_1(x,y)$ =
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  - $\min\{u_1(x,y) | y \in A_2\}$
  - $| x \in A_1\}$ =

x_1: $\min_y u(x_1,y) = a_1$

x_2: $\min_y u(x_2,y) = a_2$

..: .... = ..

x_n: $\min_y u(x_n,y) = a_n$

…	y_1	y_2	…	…	…	y_k
x_1	…	…	…	a_1	…	…
x_2	a_2	…	…	…	…	…
…	…	a	…	…	…	…
	…	…	a	…	…	…
…	…	…	…	a	…	…
x_n	…	a_n	…	…	…	…

maxminimizers

- solve $\max_x \min_y$ for 1 $u_1(x,y)$ =
  - max{
  - $\min\{u_1(x,y) | y \in A_2\}$
  - $| x \in A_1\}$ =

…	y_1	y_2	…	…	…	y_k
x_1	…	…	…	a_1	…	…
x_2	a_2	…	…	…	…	…
…	…	a	…	…	…	…
x^*	…	…	a^*	…	…	…
…	…	…	…	a	…	…
x_n	…	a_n	…	…	…	…

x_1: $\min_y u(x_1,y) = a_1$

x_2: $\min_y u(x_2,y) = a_2$

..: .... = ..

x_n: $\min_y u(x_n,y) = a_n$

max = a^* for x^*

maxminimizers

- solve $\max_x \min_y$ for 1 $u_1(x,y)$ =
  - max{
  - $\min\{u_1(x,y) | y \in A_2\}$
  - $| x \in A_1\}$ =

…	y_1	y_2	y_3	y_4	y_5
x_1	2,-2	2,-2	3,-3	1,-1	1,-1
x_2	3,-3	5,-5	4,-4	6,-6	4,-4
x_3	5,-5	2,-2	4,-4	3,-3	3,-3
x_4	6,-6	8,-8	5,-5	7,-7	6,-6
x_5	3,-3	5,-5	4,-4	2,-2	3,-3
x_6	4,-4	3,-3	7,-7	5,-5	4,-4

maxminimizers

- solve $\max_x \min_y$ for 1 $u_1(x,y)$ =
  - max{
  - $\min\{u_1(x,y) | y \in A_2\}$
  - $| x \in A_1\}$ =

x_1: $\min_y u(x_1,y) = 1$

…	y_1	y_2	y_3	y_4	y_5
x_1	2,-2	2,-2	3,-3	1,-1	1,-1
x_2	3,-3	5,-5	4,-4	6,-6	4,-4
x_3	5,-5	2,-2	4,-4	3,-3	3,-3
x_4	6,-6	8,-8	5,-5	7,-7	6,-6
x_5	3,-3	5,-5	4,-4	2,-2	3,-3
x_6	4,-4	3,-3	7,-7	5,-5	4,-4

maxminimizers

- solve $\max_x \min_y$ for 1 $u_1(x,y)$ =
  - max{
  - $\min\{u_1(x,y) | y \in A_2\}$
  - $| x \in A_1\}$ =

x_1: $\min_y u(x_1,y)$ = 1

x_2: $\min_y u(x_2,y)$ = 3

…	y_1	y_2	y_3	y_4	y_5
x_1	2,-2	2,-2	3,-3	1,-1	1,-1
x_2	3,-3	5,-5	4,-4	6,-6	4,-4
x_3	5,-5	2,-2	4,-4	3,-3	3,-3
x_4	6,-6	8,-8	5,-5	7,-7	6,-6
x_5	3,-3	5,-5	4,-4	2,-2	3,-3
x_6	4,-4	3,-3	7,-7	5,-5	4,-4

maxminimizers

- solve $\max_x \min_y$ for 1
  $u_1(x,y)$ =
  - max{
  - $\min\{u_1(x,y) | y \in A_2\}$
  - $| x \in A_1\}$ =

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x_3	5,-5	2,-2	4,-4	3,-3	3,-3
x_4	6,-6	8,-8	5,-5	7,-7	6,-6
x_5	3,-3	5,-5	4,-4	2,-2	3,-3
x_6	4,-4	3,-3	7,-7	5,-5	4,-4

x_1: $\min_y u(x_1,y)$ = 1

x_2: $\min_y u(x_2,y)$ = 3

..: .... = ..

x_n: $\min_y u(x_n,y)$ = 3

max = 5 for $x^* = x_4$

maxminimizers

- solve $\max_x \min_y$ for 1 $u_1(x,y)$ =
  - max{
  - min{$u_1(x,y) | y \in A_2$}
  - $| x \in A_1$} = 5
- solve $\max_x \min_y$ for 2 $u_2(x,y)$ =
  - max{
  - min{$u_1(x,y) | x \in A_1$}
  - $| y \in A_2$} =

…	y_1	y_2	y_3	y_4	y_5
x_1	2,-2	2,-2	3,-3	1,-1	1,-1
x_2	3,-3	5,-5	4,-4	6,-6	4,-4
x_3	5,-5	2,-2	4,-4	3,-3	3,-3
x_4	6,-6	8,-8	5,-5	7,-7	6,-6
x_5	3,-3	5,-5	4,-4	2,-2	3,-3
x_6	4,-4	3,-3	7,-7	5,-5	4,-4

maxminimizers

- solves $\max_x \min_y$ for 1 $u_1(x,y)$ =
  - max{
  - $\min\{u_1(x,y) | y \in A_2\}$
  - $| x \in A_1\}$ = 5
- solves $\max_x \min_y$ for 2 $u_2(x,y)$ =
  - max{
  - $\min\{u_1(x,y) | x \in A_1\}$
  - $| y \in A_2\}$ = -6!
- Equilibrium (6,-6)

…	y_1	y_2	y_3	y_4	y_5
x_1	2,-2	2,-2	3,-3	1,-1	1,-1
x_2	3,-3	5,-5	4,-4	6,-6	4,-4
x_3	5,-5	2,-2	4,-4	3,-3	3,-3
x_4	6,-6	8,-8	5,-5	7,-7	6,-6
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x_6	4,-4	3,-3	7,-7	5,-5	4,-4

maxminimizer

- let $G = \langle \{1,2\},(A_i),(\succeq_i)\rangle$ be a SCSG
- action $x^* \in A_1$ is maxminimizer for 1:
- $\forall x \in A_1 \ \min_{y \in A_2} u_1(x^*,y) \succeq_1 \min_{y \in A_2} u_1(x,y)$
- action $y^* \in A_2$ is maxminimizer for 2:
- $\forall y \in A_2 \ \min_{x \in A_1} u_2(x,y^*) \succeq_2 \min_{x \in A_1} u_2(x,y)$

maxminimizer

- action $x^* \in A_1$ is maxminimizer for 1:
- $\forall x \in A_1 \min_{y \in A_2} u_1(x^*,y) \geq_1 \min_{y \in A_2} u_1(x,y)$

- solves $\max_x \min_y u_1(x,y)$ for 1
- solves $\max_y \min_x u_1(x,y)$ for 2
- ‘maximizes minimum that i can guarantuee’

maxminimizers

- solves $\max_x \min_y u_1(x,y)$ for 1 =
- $\max\{ \qquad |x \in A_1\}$ =

maxminimizers

- solves $\max_x \min_y u_1(x,y)$ for 1 =
- $\max\{ \min\{u_1(x,y) | y \in A_2\} | x \in A_1\}$ =

maxminimizers

- solves $\max_x \min_y u_1(x,y)$ for 1 =
- $\max\{ \min\{u_1(x,y) | y \in A_2\} | x \in A_1\}$ =
- $\max_{x \in A1} \min_{y \in A2} u_1(x,y)$ =

- $\max_x \min_y u_1(x,y)$
- The solution x* is called the 'security strategy' for 1

Lemma

- assume $u_1 = -u_2$
- then:
  - $\max_{Y \in A2} \min_{X \in A1} u_2(x,y) =$
  - $- \min_{Y \in A2} \max_{X \in A1} u_1(x,y)$

Equilibria and maxminimizers

- if (x*,y*) is a N.eq for G, then
  - x* is a maxminimizer for 1;
  - y* is a maxminimizer for 2

	y_1	y_2	y_3
x_1	2,-2	1,-1	3,-3
x_2	5,-5	3,-3	4,-4
x_3	1,-1	-2,2	-1,1

Equilibria and maxminimizers

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Equilibria and maxminimizers

- if (x*,y*) is a N.eq for G, then
  - x* is a maxminimizer for 1;
  - y* is a maxminimizer for 2
  - $\max_x \min_y u_1(x,y) =$
  - $\min_y \max_x u_1(x,y) = u_1(x^*,y^*)$

Equilibria and maxminimizers

- if (x^*, y^*) is a N.eq for G, then
  - x^* is a maxminimizer for 1;
  - y^* is a maxminimizer for 2
  - $\max_x \min_y u_1(x,y)$ =
  - $\min_y \max_x u_1(x,y)$
  - $= u_1(x^*, y^*) = 3$

	y_1	y_2	y_3
x_1	2,-2	1,-1	5,-5
x_2	5,-5	3,-3	4,-4
x_3	1,-1	-2,2	-3,3

Equilibria and maxminimizers

- if (x*,y*) is a N.eq for G, then
  - x* is a maxminimizer for 1;
  - y* is a maxminimizer for 2

c	y_1	y_2	y_3
x_1	2,-2	1,-1	6,-6
x_2	7,-7	3,-3	4,-4
x_3	1,-1	5,-5	-3,3

Equilibria and maxminimizers

- if (x*,y*) is a N.eq for G, then
  - x* is a maxminimizer for 1;
  - y* is a maxminimizer for 2

c	y_1	y_2	y_3
x_1	2,-2	1,-1	6,-6
x*	7,-7	3,-3	4,-4
x_3	1,-1	5,-5	-3,3

Equilibria and maxminimizers

- if (x^*,y^*) is a N.eq for G, then
  - x^* is a maxminimizer for 1;
  - y^* is a maxminimizer for 2

c	y_1	y^*	y_3
x_1	2,-2	1,-1	6,-6
x^*	7,-7	3,-3	4,-4
x_3	1,-1	5,-5	-3,3

Equilibria and maxminimizers 2

- if
  - $\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y)$
  - x* and y* are maxminimizers
- then
  - (x*,y*) is Nash equilibrium

security level vs equilibria

- Consider
  cooperative game G

	L	R
l	2,2	0,0
r	1,1	1,1

security level vs equilibria

- consider cooperative game G
- (2,2) seems the optimal solution

	L	R
l	2,2	0,0
r	1,1	1,1

security level vs equilibria

- consider cooperative game G
- (2,2) seems the optimal solution
- security strategy of 1 is r, gives 1!
- Nash equilibria?

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l	2,2	0,0
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security level vs equilibria

- consider cooperative game G
- (2,2) seems the optimal solution
- security strategy of 1 is r, gives 1!
- Nash equilibria?

	L	R
l	2,2	0,0
r	1,1	1,1

bimatrix games

- m x n matrix
- 1 has strategies s_1 and s_2, 2 has t_1, t_2 and t_3
- payoff $\pi_1(s_i,t_j) = i*j$
  - $\pi_2(s_i,t_j) = (i-2)(j-2)$

	t_1	t_2	t_3
s_1	1,1	2,0	3,-1
s_2	2,0	4,0	6,0

bimatrix games

- m x n matrix
- 1 has strategies s_1 and s_2, 2 has t_1, t_2 and t_3
- Nash equilibrium (σ,τ):
  - $\forall s,t\ \pi_1(\sigma,\tau) \geq \pi_1(s,\tau)$
  - $\forall s,t\ \pi_2(\sigma,\tau) \geq \pi_2(\sigma,t)$

	t_1	t_2	t_3
s_1	1,1	2,0	3,-1
s_2	2,0	4,0	6,0

domination

- strategy s_d of 1 strongly dominates s_i:
- $\forall t\ \pi_1(s_d,t) > \pi_1(s_i,t)$

domination

- strategy s_d of 1 strongly dominates s_i:
- $\forall t\ \pi_1(s_d,t) > \pi_1(s_i,t)$
- s_2 strongly dominates s_1

	t_1	t_2	t_3
s_1	1,1	2,0	3,-1
s_2	2,0	4,0	6,0

domination

- strategy s_d of 1 strongly dominates s_i:
- $\forall t\ \pi_1(s_d,t) > \pi_1(s_i,t)$
- And weak if:
  - $\forall t\ \pi_1(s_d,t) \geq \pi_1(s_i,t)$
  - $\exists t\ \pi_1(s_d,t) > \pi_1(s_i,t)$

	t_1	t_2	t_3
s_1	1,1	2,0	3,-1
s_2	2,0	4,0	6,0

domination

- strategy s_d of 1 strongly dominates s_i:
- $\forall t \ \pi_1(s_d,t) > \pi_1(s_i,t)$
- And weak if:
  - $\forall t \ \pi_1(s_d,t) \geq \pi_1(s_i,t)$
  - $\exists t \ \pi_1(s_d,t) > \pi_1(s_i,t)$
- t_1 weakly dominates t_2

	t_1	t_2	t_3
s_1	1,1	2,0	3,-1
s_2	2,0	4,0	6,0

iterated elimination

- s_2 strongly dominates s_1

	t_1	t_2	t_3
s_1	1,1	2,0	3,-1
s_2	2,0	4,0	6,0

iterated elimination

- s_2 strongly dominates s_1

	t_1	t_2	t_3
s_1	1,1	2,0	3,-1
s_2	2,0	4,0	6,0

iterated elimination

- s_2 strongly dominates s_1
- t_1 weakly dominates t_2

	t_1	t_2	t_3
s_1	1,1	2,0	3,-1
s_2	2,0	4,0	6,0

iterated elimination

- s_2 strongly dominates s_1
- t_1 weakly dominates t_2

	t_1	t_2	t_3
s_1	1,1	2,0	3,-1
s_2	2,0	4,0	6,0

iterated elimination

- s_2 strongly dominates s_1
- t_1 weakly dominates t_2
- In the new game t_1 and t_3 are not weakly dominated
- (s_2,t_1) and (s_2,t_3) N.eq!

	t_1	t_2	t_3
s_1	1,1	2,0	3,-1
s_2	2,0	4,0	6,0

	t_1	t_2	t_3
s_1	1,1	2,0	3,-1
s_2	2,0	4,0	6,0

Order of elimination

	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

Order of elimination

	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
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BE	3,3	3,3
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Order of elimination

	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

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AE	2,0	1,1
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BE	3,3	3,3
BF	3,3	3,3

	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

equilibrium gone!

elimination: conclusions

- strict strategies: no problem
- With weakly dominated strategies:
  - subgame perfect equilibrium can be lost
  - Order of elimination matters

Order of elimination

	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

equilibrium gone!

	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

Example: BoS

- $N = \{1,2\}$
- $A_1 = \{B,S\}$
- $A_2 = \{B,S\}$
- u_1, u_2 see figure
  - B: Bach
  - S: Strawinsky
- Battle of the Sexes

	B	S
B	2,1	0,0
S	0,0	1,2

No dominant strategies

(two) Nash equilibira

ex: coordination game

- Mozart of Mahler?
- Equal preferences

	Mo	Ma
Mo	2,2	0,0
Ma	0,0	1,1

No dominant strategy

two Nash equilibria

ex: prisoner's dilemma

- C: cooperate and be silent
- D: testify against the other

	C	D
C	0,0	-2,3
D	3,-2	-1,-1

ex: prisoner's dilemma

- C: cooperate and be silent
- D: testify against the other
- D dominates C

	C	D
C	0,0	-2,3
D	3,-2	-1,-1

ex: prisoner's dilemma

- C: cooperate and be silent
- D: testify against the other
- D dominates C
- D dominates C

	C	D
C	0,0	-2,3
D	3,-2	-1,-1

ex: prisoner's dilemma

- C: cooperate and be silent
- D: testify against the other
- D dominates C
- D dominates C
- Gives Nash equilibrium (-1,-1)

	C	D
C	0,0	-2,3
D	3,-2	-1,-1

ex: hawk-dove

- preferences:
  - hawkish if other is dovish
  - dovish if other is hawkish

	D	H
D	3,3	1,4
H	4,1	0,0

no dominant strategy

(two) Nash equilibria

ex: Matching Pennies

- Head and Tail
- If matching 2 pays one euro to 1, otherwise 1 pays 2

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

no dominant strategy

No Nash equilibrium