# Game Theory

# Literature

- Game theory with economic applications
  - Scott Bierman and Luis Fernandez
- Auctions: Theory and practice
  - Paul Klemperer
- Fun and Games
  - Ken Binmore
- A Course in Game Theory
  - Martin Osborne and Ariel Rubinstein

# What is it about?

- game = interaction
  - traffic
  - supermarket
  - NS, union and board
  - student teacher
  - lawyers and judges
  - George and Osama
  - marriage and career

# What is it about?

- Terms from chess and bridge
- logic and systematics of interaction
- analysis distinguishes irrational choices
  - strategic interaction is hard
  - Because of circular reasoning
    - J and M play a game; J’s strategy depends on his prediction of M’s strategy, that depends on its turn of her expectation of that of J.

# Surprise and paradox

- Does it make sense to
  - Vote for your least favorite candidate?
  - Throw a coin as general?
  - Bid maximal with the worst possible cards?
  - Throw away some goods before starting the negotiation about the rest?
  - Sell your house to the second highest bidder?

# Surprise and paradox

- Does it make sense to
  - Vote for your least favorite candidate? yes!
  - Throw a coin as general? yes!
  - Bid maximal with the worst possible cards? yes!
  - Throw away some goods before starting the negotiation about the rest? yes!
  - Sell your house to the second highest bidder? yes!

# Strategic voting

- Boris, Horace and Maurice determine who can become member of the Dead Poet Society
  - proposal: admit Alice
  - amendment: Bob instead of Alice
  - First vote about amendment then about admission of extra member

# Strategic voting

| Boris | Horace | Maurice |
|---|---|---|
| Alice | Nobody | Bob |
| Nobody | Alice | Alice |
| Bob | Bob | Nobody |

# Strategic voting

- First between A, B
  - Winner

Maurice

| |
|---|
| Bob |
| Alice |
| Nobody |

Horace

| |
|---|
| Nobody |
| Alice |
| Bob |

Boris

| |
|---|
| Alice |
| Nobody |
| Bob |

# Strategic voting

- First between A, B
  - Winner Alice
- Then between A and N
  - winner

Maurice

| |
|---|
| Bob |
| Alice |
| Nobody |

Horace

| |
|---|
| Nobody |
| Alice |
| Bob |

Boris

| |
|---|
| Alice |
| Nobody |
| Bob |

# Strategic voting

- First between A, B
  - Winner Alice
- Then between A and N
  - winner Alice
- strategic voting H:

Horace

| |
|---|
| Nobody |
| Alice |
| Bob |

Maurice

| |
|---|
| Bob |
| Alice |
| Nobody |

Boris

| |
|---|
| Alice |
| Nobody |
| Bob |

# Strategic voting

- First between A, B
  - Winner Alice
- Then between A and N
  - winner Alice
- strategic voting H:
  - First vote for Bob!
  - result...

| Maurice |
|---|
| Bob |
| Alice |
| Nobody |

| Horace |
|---|
| Nobody |
| Alice |
| Bob |

| Boris |
|---|
| Alice |
| Nobody |
| Bob |

# Strategic voting

- First between A, B
  - Winner Alice
- Then between A and N
  - winner Alice
- strategic voting H:
  - First vote for Bob!
  - result… … B, N

| Maurice | Horace | Boris |
|---|---|---|
| Bob | Nobody | Alice |
| Alice | Alice | Nobody |
| Nobody | Bob | Bob |

# Strategic voting

- First between A, B
  - Winner Alice
- Then between A and N
  - winner Alice
- strategic voting H:
  - First vote for Bob!
  - result... ... B, N
- M anticipates: votes for A

Maurice

| |
|---|
| Bob |
| Alice |
| Nobody |

Horace

| |
|---|
| Nobody |
| Alice |
| Bob |

Boris

| |
|---|
| Alice |
| Nobody |
| Bob |

# History

- Von Neumann en Morgenstern
  - *The Theory of Games and Economic Behaviour* (1944)
- ideas from economics and mathematics
- first optimism, afterwards depression
- Revival since 1970's

# History

- Von Neumann and Morgenstern
- strategic, non-cooperative games
  - Easy case: zero-sum
- coalitional, cooperative approach
- Revival after work of
  - Nash, Aumann, Shapley, Selten, Harsanys

# Game Theory

- strategic games
  - strictly competitive
  - Nash equilibria
- extensive games
  - (subgame perfect) equilibria
  - value of a game

# Game Theory

- theory of decision makers

# Game Theory

- theory of decision makers
  - are rational:

# Game Theory

- theory of decision makers
  - are rational:
  - aware of alternatives
  - form expectations
  - have preferences
  - optimize after deliberation

- set A of actions;
- set C consequences;
- g: A → C
  - consequence functio
- preference relation ≥ on A
  - sometimes: utility function
  - u: C → R

# Game Theory

- theory of decision makers
  - are rational
  - reason strategically

- Players anticipate th
  knowledge and expectations of the behavior of other decision-makers

# Strategic Games

- Definition
  - Finite set $N$ (**players**)
  - set $A_i$ (**actions**) for every player i
  - **preference relation** $\geq_i$ for every player i

# Strategic Games

- Definition $G = \langle N, (A_i), (\geq_i) \rangle$
  - Finite set $N$ (**players**)
  - set $A_i$ (**actions**) for every player i
  - **preference relation** $\geq_i$ for every player i

# Strategic Games

- Definition $G = \langle N,(A_i),(u_i)\rangle$
  - Finite set $N$ (**players**)
  - set $A_i$ (**actions**) for every player i
  - **preference relation** $\geq_i$ for every player i
  - $u_i$ is **utility function**: $A_i \rightarrow R$ with
    - $a \geq_i b \Leftrightarrow u_i(a) \geq u_i(b)$
    - Also called **payoff-function**
    - NOT really the same!

# Representation strategic game

- $N = \{1,2\}$
- $A_1 = \{T,B\}$
- $A_2 = \{L,R\}$
- $u_1(T) = w_1$,
- $u_2(L) = w_2$, etc

| | L | R |
|---|---|---|
| T | $w_1,w_2$ | $x_1,x_2$ |
| B | $y_1,y_2$ | $z_1,z_2$ |

# Representation strategic game

- Interpretation
  - One time
  - simultaneous
  - independent
  - utilities are known
  - Choice of other is not known

| | L | R |
|---|---|---|
| T | $w_1,w_2$ | $x_1,x_2$ |
| B | $y_1,y_2$ | $z_1,z_2$ |

# Example: BoS

- $N = \{1,2\}$
- $A_1 = \{B,S\}$
- $A_2 = \{B,S\}$
- $u_1$, $u_2$ see figure
  - B: Bach
  - S: Strawinsky
- Battle of the Sexes

| | B | S |
|---|---|---|
| B | 2,1 | 0,0 |
| S | 0,0 | 1,2 |

# Profiles

- $A_1, A_2, \ldots, A_n$ are the sets of actions
- $(a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \ldots \times A_n$ is called a **profile**
- notation: $(x)_i$
- $x_{i-1} \in A_1 \times A_2 \times \ldots \times A_{i-1} \times A_{i+1} \times \ldots \times A_n$
- $(x_{i-1}, x_i) = (x)_i$
- focus on i, given the profile of the others

# Profiles: example

- $A_1$, $A_2$ ,… , $A_7$ are bids ($\in$ R)
- ($a_1$, $a_2$ ,… , $a_7$) is a concrete bid
- notation: $(x)_7 = (25,22,20,12,0,27,22) = a^*$
- $x_{6\text{-}1} \in (25,22,20,12,0,22)$
- $(x_{6\text{-}1}, x_6) = ((25,22,20,12,0,22),27)$
  - $(x_{6\text{-}1}, x'_6) = ((25,22,20,12,0,22),26)$ would have been better for player 6, given the profile of the others

# Nash equilibrium

- John Nash
- equilibrium (“solution”)
  - Every player is rational
  - Every player plays optimal
  - No reason to deviate individually
  - Not algorithmic

# Nash equilibrium (ctd)

- steady state interpretation
  - From economics
  - Game is model for family of situations
  - Game theory studies regularities
  - Players “know” the equilibrium
- deductive interpretation
  - game is “one-shot event”
  - players “deduce” the equilibrium
  - Point of departure is rationality

# Nash equilibrium (definition)

- Given $G = \langle N,(A_i),(\geq_i)\rangle$
- $a^* \in A = A_1 x\ A_2\ x\ \ldots\ x\ A_n$ is Nash equilibrium if
- $\forall i \in N\ \forall a_i \in A_i\ (a^*_{i-1}, a^*_i) \geq_i (a^*_{i-1}, a_i)$

# Nash equilibrium (definition)

- Given $G = \langle N,(A_i),(\geq_i)\rangle$
- $a^* \in A = A_1 x\ A_2\ x \ldots x\ A_n$ is Nash equilibrium if
- $\forall i \in N\ \forall a_i \in A_i\ (a^*_{i-1}, a^*_i) \geq_i (a^*_{i-1}, a_i)$
  - So: no player i can improve in $a^*$, if all the other players keep playing $a^*_{i-1}$

# Nash equilibrium (alternative)

- define for every $a_{-i} \in A_{-i}$ the **best response for i**, $B_i(a_{-i})$
- $B_i(a_{-i}) = \{a_i \in A_i \mid \forall a'_i \in A_i \; (a_{-i}, a_i) \geq_i (a_{-i}, a'_i)\}$
- $a^*$ is N.eq if $\forall i \in N \; a^*_i \in B_i(a^*_{-i})$

# Example: BoS

- $N = \{1,2\}$
- $A_1 = \{B,S\}$
- $A_2 = \{B,S\}$
- $u_1$, $u_2$ see figure
  - B: Bach
  - S: Strawinsky
- Battle of the Sexes

| | B | S |
|---|---|---|
| B | 2,1 | 0,0 |
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# Example: BoS (N.eq)

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# Example: BoS (N.eq)

- $N = \{1,2\}$
- $A_1 = \{B,S\}$
- $A_2 = \{B,S\}$
- $u_1$, $u_2$ see figure
  - B: Bach
  - S: Strawinsky
- two equilibria:
- (bach,bach) and
- (strawinsky, strawinsky)

| | B | S |
|---|---|---|
| B | 2,1 | 0,0 |
| S | 0,0 | 1,2 |

# Ex: coordination game

- Mozart or Mahler?
- Equal preferences

| | Mo | Ma |
|---|---|---|
| Mo | 2,2 | 0,0 |
| Ma | 0,0 | 1,1 |

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- Mozart or Mahler?
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| | Mo | Ma |
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| Mo | 2,2 | 0,0 |
| Ma | 0,0 | 1,1 |

# Ex: coordination game

- Mozart or Mahler?
- Equal preferences
- two equilibria:
- (Mozart,Mozart) and
- (Mahler,Mahler)
- N.eq right concept?

| | Mo | Ma |
|---|---|---|
| Mo | 2,2 | 0,0 |
| Ma | 0,0 | 1,1 |

# Ex: prisoner's dilemma

- C: cooperate and don't testify
- D: testify against the other

| | C | D |
|---|---|---|
| C | 0,0 | -2,3 |
| D | 3,-2 | -1,-1 |

# Ex: prisoner's dilemma

- C: cooperate and don't testify
- D: testify against the other
- Although cooperating would be better, both players prefer to testify

| | C | D |
|---|---|---|
| C | 0,0 | -2,3 |
| D | 3,-2 | -1,-1 |

# Ex: prisoner's dilemma

- Z: silent, B: confess
- Although being silent would be better, both players prefer to confess

| | Z | B |
|---|---|---|
| Z | 3,3 | 0,4 |
| B | 4,0 | 1,1 |

# Ex: hawk-dove

- preference:
  - hawkish if other is dovish
  - dovish if other is hawkish

| | D | H |
|---|---|---|
| D | 3,3 | 1,4 |
| H | 4,1 | 0,0 |

# Ex: hawk-dove

- preference:
  - hawkish if other is dovish
  - dovish if other is hawkish
- N.eq: (Dove,Hawk)

| | D | H |
|---|---|---|
| D | 3,3 | 1,4 |
| H | 4,1 | 0,0 |

# Ex: hawk-dove

- preference:
  - hawkish if other is dovish
  - dovish if other is hawkish
- N.eq: (Dove,Hawk)
- and (Hawk,Dove)

| | D | H |
|---|---|---|
| D | 3,3 | 1,4 |
| H | 4,1 | 0,0 |

# Ex: Matching Pennies

- Head and Tail
- If pennies match then 2 pays 1, if they differ then 1 pays 2

| | H | T |
|---|---|---|
| H | 1,-1 | -1,1 |
| T | -1,1 | 1,-1 |

# Ex: Matching Pennies

- Head and Tail
- If pennies match then 2 pays 1, if they differ then 1 pays 2
- no equilibrium!
- Game is strict competitive

| | H | T |
|---|---|---|
| H | 1,-1 | -1,1 |
| T | -1,1 | 1,-1 |