Continuous Simulation

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Why do we do this?

So far, we saw discrete event simulations. This is useful when our problem is like a queue of events, sorted by the simulation time at which they should occur.

Things are not always that simple.

Example: predator-prey model

Consider an environment consisting of two populations, predators and prey. We are interested in both the predator and prey population size. However, these populations interact.

What is continuous simulation?

Definition of continuous simulation

Continuous simulation concerns the modeling over time of a system by a representation in which state variables change continuously with respect to time.

Typically, we use differential equations, that give relationships for the rates of change of the state variables with time.

So, how do we solve these systems of differential equations?

- in very easy cases: analytically;
- otherwise: numerically.

An easy predator-prey model

Let the prey population at time t be given by $x(t)$, and the predator population by $y(t)$. Assume that, in the absence of predators, the prey will grow exponentially according to $x' = ax$ for a certain $a > 0$. We also assume that the death rate of the prey due to interaction is proportional to $x(t)y(t)$, with a positive proportionality constant. So:

$$x'(t) = a\,x(t) - b\,x(t)\,y(t)$$

Without prey, predators will die exponentially according to $y' = -cy$ for a certain $c > 0$. Their birth strongly depends on both population sizes, so we finally find for a certain $d > 0$:

$$y'(t) \;=\; -c\,y(t) + d\,x(t)\,y(t)$$

Finding solutions

$$\begin{cases} x'(t) = a\,x(t) - b\,x(t)\,y(t) \\ y'(t) = -c\,y(t) + d\,x(t)\,y(t) \end{cases}$$

We immediately see (I hope) that both $(e^{at}, 0)$ and $(0, e^{-ct})$ are solutions of $(x(t), y(t))$.

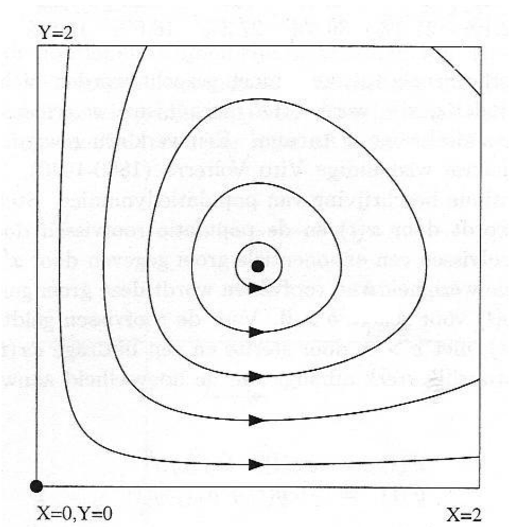
From this system we find that for every solution we must have

$$x'\left(\frac{c}{x} - d\right) + y'\left(\frac{a}{y} - b\right) = 0$$

Integrating both sides gives us

$$c \log x(t) - dx(t) + a \log y(t) - by(t) = \text{constant}$$

Solutions for $a = b = c = d = 1$



So why do we need numerical methods?

The given model is considered very simple. Why?

- the integrating we did two slides ago is possible;
- rest of the world has no influence;
- no randomness involved.

Usually we cannot find closed-form solutions for the system of differential equations.

How do we deal with this problem? Solve numerically!

Recursion for numerical solutions

Consider any system of first order differential equations $\vec{y}' = \vec{f}(t, \vec{y})$. Let $\vec{y}(t)$ be a solution with given initial value $\vec{y}(t_0)$ at time t_0. Choose a fixed and small enough time interval Δt. We now look at the solution $\vec{y}(t)$ from time t_0 at times $t_0, t_1 = t_0 + \Delta t, t_2 = t_0 + 2\Delta t, \ldots$. We obtain the sequence of vectors $\vec{y}(t_0), \vec{y}(t_1), \vec{y}(t_2), \ldots$. If Δt is small enough, we know that

$$\frac{\vec{y}(t + \Delta t) - \vec{y}(t)}{\Delta t} \approx \vec{y}'(t)$$

So, for the sequence of vectors $\vec{y}(t_n)$ we find an approximating recursion

$$\frac{\vec{y}(t_{n+1}) - \vec{y}(t_n)}{\Delta t} \approx \vec{y}'(t_n) = \vec{f}(t_n, \vec{y}(t_n))$$

which we can rewrite to

$$\vec{y}(t_{n+1}) \approx \vec{y}(t_n) + \vec{f}(t_n, \vec{y}(t_n))\Delta t.$$

Application of the previous slide

We now turn the whole thing around. Choose $\vec{y}_0 = \vec{y}(t_0), \Delta t$ and consider the recursion

$$\vec{y}_{n+1} = \vec{y}_n + \vec{f}(t_n, \vec{y}_n)\Delta t.$$

The question is: how good does the sequence $\vec{y}_0, \vec{y}_1, \vec{y}_2, \ldots$ approximate $\vec{y}(t_0), \vec{y}(t_1), \vec{y}(t_2), \ldots$?

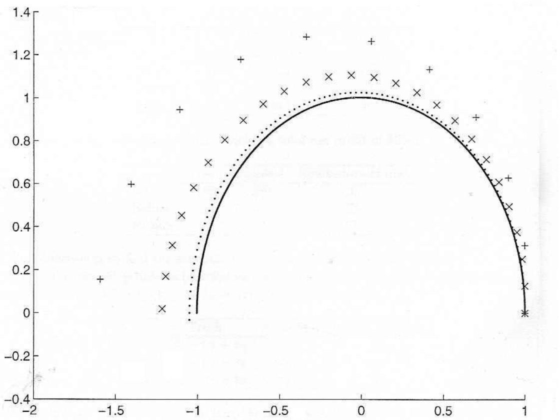
Illustration: upper half of the unit circle

Let $x' = -y,\ y' = x,\ x(0) = 1,\ y(0) = 0,\ 0 \leq t \leq \pi$. This system of differential equations describes the upper half of the unit circle.

On the next slide, we see some approximations.

Results

Below are the approximations with $\Delta t = \pi/10(+),\ \pi/25(\times),\ \pi/100(\cdot),\ \pi/1000(\cdot)$.



A little more on numerical methods

The shown method is called Euler's method, for obvious reasons. One can prove that the precision of this method is proportional to Δt. There are more ingenious variants, where the precision is proportional to a power of Δt. This power is called the order of the method.

The best known of these methods are

- Runge-Kutta methods (fourth-order);
- methods of Adams (can be extended to any order).

The image about the predator-prey model was plotted using a variant of the last methods; the method of Adams-Bashford.

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Why do we do this?

We have discussed continuous systems whose process of evolution depends on differential equations. Such a system contains a number of parameters that must be estimated (for instance, the $a, b, c, d > 0$ in the predator-prey model). Usually point estimates are calculated and used in the model. These estimates typically have uncertainty associated with them.

We can incorporate uncertainty in our differential equations. This is done by using fuzzy numbers as estimates of the unknown parameters.

Fuzzy sets (1/3)

In a classical set, an element is either a member of the set or not. Fuzzy sets are defined in terms of classical sets.

Definition: fuzzy set

A fuzzy set A on a classical set X is defined as:

$$A = \{(x, \mu_A(x)) | x \in X\}$$

The membership function $\mu_A(x)$ quantifies the grade of membership of the elements x of the fundamental set X. For the functional values of $\mu_A(x)$, we have the following properties:

- $\forall_{x \in X} \;\; \mu_A(x) \geq 0$;
- $\sup_{x \in X} \{\mu_A(x)\} = 1$.

Fuzzy sets (2/3)

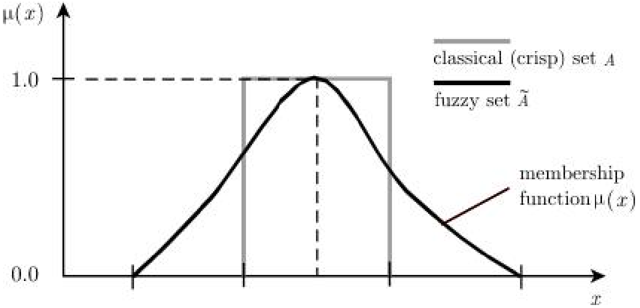
If the fuzzy set B is defined as $B = \{(3, 0.3), (4, 0.7), (5, 1), (6, 0.4)\}$, it is standard fuzzy notation to write

$$B = \{0.3/3, 0.7/4, 1/5, 0.4/6\}$$

Any value with a membership grade of zero does not appear in the expression of the set.

Classical sets are called crisp sets, to distinguish between them and fuzzy sets.

Fuzzy sets (3/3)



Fuzzy sets are used in fuzzy logic, which is an extension of Multi-valued logic. This is used for “approximated reasoning”. Apart from that, fuzzy sets also form the basis for fuzzy numbers.

Fuzzy numbers (at last)

Definition: fuzzy numbers

A fuzzy number is a convex, normalized fuzzy set $A \subseteq \mathbb{R}$ whose membership function is at least segmentally continuous and has the functional value $\mu_A(x)$ at precisely one element. This element is called the vertex.

Usually we will be using triangular, or triangular shaped fuzzy numbers. A triangular fuzzy number is is defined by three numbers $m < n < p$, where the base of the triangle is on the interval $[m, p]$ and the vertex is at $x = n$. We write $\bar{N} = (m/n/p)$ for triangular fuzzy number $\bar{N}$. A triangular shaped fuzzy number, written $\bar{N} \approx (m/n/p)$, has curves for its side instead of straight line segments.

Alpha-cuts

Let $\bar{N}$ be a fuzzy number. For $0 < \alpha \leq 1$, the alpha-cut of $\bar{N}$, written as $\bar{N}[\alpha]$, is defined as $\{x|\bar{N}(x) \geq \alpha\}$.

$\bar{N}[0]$ is defined as the closure of the union of $\bar{N}[\alpha]$ for $\alpha \in (0, 1]$.

Simulation of fuzzy continuous systems

Solving fuzzy differential equations was considered before. One could only fuzzify the initial values, because the fuzzy solution became too difficult to obtain when more parameters became fuzzy.

Instead, we can apply our knowledge about continuous simulation. Then, we can fuzzify more parameters. For instance, in the predator-prey model we may use fuzzy numbers $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{x}_0$, and $\bar{y}_0$.

Types of fuzzy estimators

We will consider only two methods of fuzzy estimators:

- expert opinion;
- from data using confidence intervals.

More information on fuzzy estimators can be found in J.J. Buckley, Fuzzy Statistics, Springer-Verlag, Heidelberg, Germany, 2004.

Fuzzy estimators based on expert opinion

We want to estimate the value for a certain parameter b. First, assume that we have only one expert. Let b_1 be the smallest possible value for b, let b_3 be the largest possible value for b, and let b_2 be the most likely value. We can ask the expert to give values for b_1, b_2, b_3, and we construct the triangular fuzzy estimator $\overline{b} = (b_1/b_2/b_3)$ for b.

Now suppose we have N experts. We still want to construct a triangular fuzzy estimator $\overline{b} = (b_1, b_2, b_3)$. The easiest way to do this is to ask the experts for their b_{1i}, b_{2i}, b_{3i} for all $1 \leq i \leq N$, and then take average of each component.

Fuzzy estimators based on data

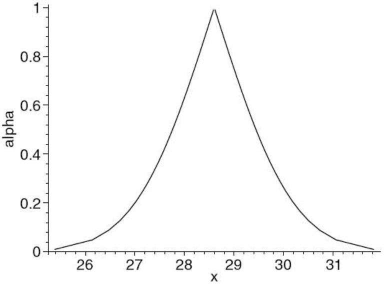
Let X be a random variable with probability density function $f(x;\theta)$ for single parameter θ. Assume that θ is unknown, and must be estimated from a random sample $X_1, \ldots, X_n$. In statistics, we have learned to construct point estimators θ^* and $(1-\beta)100\%$confidence intervals for θ. β is usually set to $0.10, 0.05$, or 0.01.

The trick is to find all $(1-\beta)100\%$ confidence intervals for $0.01 \leq \beta < 1$ (starting at 0.01 is arbitrary). To this we add the interval $[\theta*, \theta*]$ for the 0% confidence interval. Now we place these intervals on top of each other, to produce a triangular shaped fuzzy number $\bar{\theta}$ whose α-cuts are the confidence intervals. To make it a complete fuzzy number, we will drop the graph of $\bar{\theta}$ straight down.

Example of a data-based fuzzy estimator

Consider $X \sim N(\mu, \sigma^2)$, with σ known and μ unknown. Suppose the mean of a random sample from $N(\mu, \sigma^2)$ turns out to be $\bar{x}$. We know that $\bar{x} \sim N(\mu, \sigma^2/n)$, so $(\bar{x} - \mu)/(\sigma/\sqrt{n}) \sim N(0, 1)$. So, $P(-z_{\beta/2} \leq \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \leq z_{\beta/2}) = 1 - \beta$.

This leads to the $(1 - \beta)100\%$ confidence interval for μ:
$[\theta_1(\beta), \theta_2(\beta)] =$
$[\bar{x} - z_{\beta/2}\sigma/\sqrt{n}\ ,$
$\bar{x} + z_{\beta/2}\sigma/\sqrt{n}]$



Example 1: Queueing model

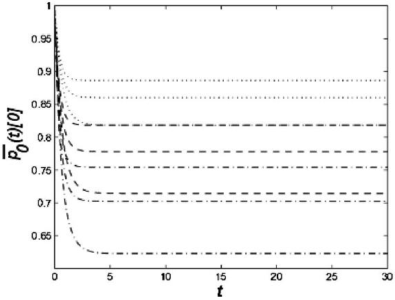
Consider a queueing model with c identical and parallel servers, capacity M. Assume customers arrive $\sim \exp(\lambda)$, and service times are distributed $\sim \exp(\lambda)$. Let $p_i(t)$ be the probability of i customers in the system at time $t > 0$, for $0 \leq i \leq M$.

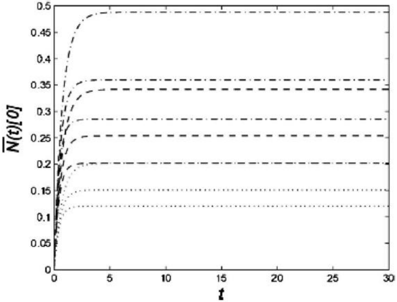
For simplicity we assume that $M = 4$ and $c = 2$. We have the following system of differential equations:

$$\begin{aligned}
p_0'(t) &= -\lambda p_0(t) + \mu p_1(t) \\
p_1'(t) &= \lambda p_0(t) - [\lambda + \mu] p_1(t) + 2\mu p_2(t) \\
p_2'(t) &= \lambda p_1(t) - [\lambda + 2\mu] p_2(t) + 2\mu p_3(t) \\
p_3'(t) &= \lambda p_2(t) - [\lambda + 2\mu] p_3(t) + 2\mu p_4(t) \\
p_4'(t) &= \lambda p_3(t) - 2\mu p_4(t)
\end{aligned}$$

Queueing model results

We used $\bar{\lambda} = (0.3/0.5/0.7)$ and $\bar{\mu} = (1.5/2/2.5)$, both triangular, with time in hours.





Example 2: Bungee jumping model

A bungee jumper jumps from 240 ft above the ground. The length of the unstretched bungee cord is 90 feet. The differential equation of motion is

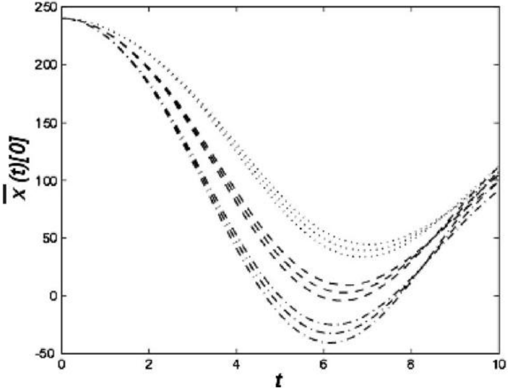
$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = mg - F(x) - R(v)$$

where $F(x) = kx$ is the force on the jumper exerted by the bungee cord, and $R(v) = cv$ is air resistance with velocity v. Notice that $v = x'(t)$.

A cord we plan to use has $k = 2.5$ pounds per foot. Is this cord good enough?

Bungee jumping model results

We used for the weight ($= m \cdot g$) the fuzzy value (120/170/220) and for c, from expert opinion, (1.0/1.2/1.4).



Example 3: Infectious disease model (1/2)

Suppose we have a population of N people and a certain contagious disease $\mathcal{D}$ infecting this population. The population is split up into three groups:

$x(t)$	those uninfected with $\mathcal{D}$ but may become so;
$y(t)$	those who are presently infected with $\mathcal{D}$ and can spread the disease;
$z(t)$	those who have been infected with $\mathcal{D}$, but cannot spread the disease.

We assume that the initial conditions are crisp, and known to be $x_0 = 950, y_0 = 50, z_0 = 0$, so $N = 1000$.

Example 3: Infectious disease model (2/2)

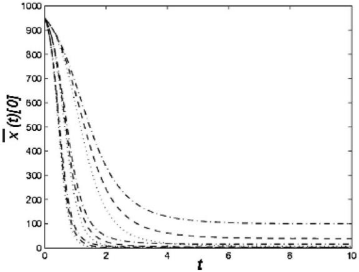
We will always have $N = x(t) + y(t) + z(t)$. We can use this equation to avoid using a third differential equation. Our model becomes:

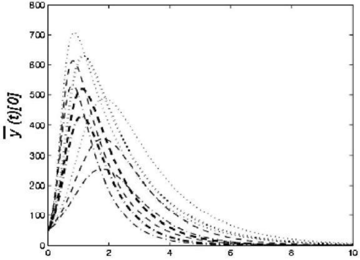
$$\begin{aligned} x'(t) &= -kxy \\ y'(t) &= kxy - cy \\ z(t) &= N - x - y \end{aligned}$$

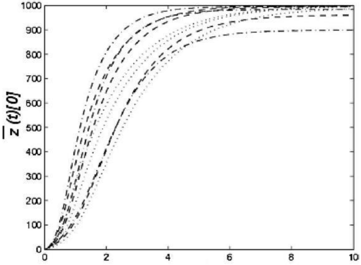
The constants k and c depend on the type of disease, the season, whether or not the population has been vaccinated against $\mathcal{D}$, etcetera. Experts are asked to estimate k and c.

Infectious disease model results

We obtained the fuzzy numbers $\bar{c} = (0.6/0.9/1.2)$ and $\bar{k} = (0.003/0.005/0.007)$.







Sources

The main source of the first part of this presentation were the lecture notes “Modellen en Simulatie” by Frits Beukers. These notes are written in Dutch, and available at the student administration of the Mathematics department, Utrecht University.

The second part of this presentation used the following sources:

- (hardly) `http://en.wikipedia.org/wiki/Fuzzy_set`
- (extensively) Jowers et al.,“Simulating continuous fuzzy systems”. To be published in the forthcoming “Information Sciences Special Issue: Advances in Fuzzy Logic”. Available online since April 18, `http://www.sciencedirect.com`.

Thank you for your attention

Are there any questions?