Automated Theorem Proving 4/4: Satisfiability Checkers, SAT/SMT

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Course Program Semantics and Verfication 2019, Utrecht University

October 2, 2019

Lecture Notes

"Automated Reasoning" by Gerard A.W. Vreeswijk.
Available for download on the course website.
My slides are largely based on them.

In This Course

- Propositional theorem proving (September 23), Chapter 2 of the lecture notes
- First-order theorem proving (September 25), Chapter 3 of the lecture notes
- Clause sets and resolution (September 30), Chapters 4 and 5 of the lecture notes
- Satisfiability checkers, SAT/SMT (today), Chapter 6 of the lecture notes, additional material

Recap: Clause Sets and Resolution

- Conjunctive Normal Form (CNF)
- Clause sets
- Conversion to $\leq$3CNF in linear time (Tseitin-derivative)
- Cleaning up and simplifying clause sets (one-literal rule, monotone variable fixing, tautology rule, subsumption, DPLL, classic DPP)
- Binary resolution, linear/unit/input resolution
- Semantic resolution, ordered resolution, semantic clash, hyperresolution, set-of-support resolution
- First-Order Resolution
- Normalization, Skolemization
- Equality, Demodulation, Paramodulation

Satisfiability Checkers

- Resolution can only be used to prove that a clause set is unsatisfiable.
- To be able to discover satisfiable formulas, we need satisfiability tests.
- Such tests try to find a satisfying assignment of ϕ.
- If such an assignment is found, the formula is proven satisfiable and the search can be stopped.
- Two popular satisfiability tests:
  1. Gradient of polynomial transforms of CNFs.
  2. Weighted variant of a greedy local search algorithm.

Gradient of Polynomial Transforms of CNFs

- Choose a proposition.
- Set to true if its number of unnegated occurrences is higher than its number of negated occurrences, otherwise set to false.
- Apply simplification rules.
- Repeat until a satisfiable assignments has been constructed.

Local Search

- Start with a random assignment of variables.
- Invert the truth values of a number of variables, so that the weight of satisfied clauses increases.
- If further inversion yields no improvement, the weights of the unsatisfied clauses are increased until an improving inversion comes into existence.
- Thus, "difficult" clauses receive large weights and are more likely to be satisfied in the end.
- Repeat until either a satisfiable assignment is constructed, or the number of inversions exceeds a certain maximum.
- For many satisfiable formulas the local search algorithm is sufficient for finding a model quickly.

GSAT

- Popular example of a local search algorithm.
- Given a clause set S, GSAT tries to find a model m such that $m \models S$ by performing a greedy local search within the space of possible models.

Algorithm 2 GSAT

Input: A clause set S, a value for MAX-FLIPS and a value for MAX-TRIES
Output: A model m, such that $m \models S$, if found

```
1: for i = 1 to MAX-TRIES do
2:   choose a random model m
3:   for j = 1 to MAX-FLIPS do
4:     return m if m ⊨ S
5:     p = a propositional variable such that a change in its truth assignment gives the largest
       increase in the total number of clauses of S that are satisfied by m
6:     m = m with the truth assignment of p reversed
7: return "no satisfying model found"
```

GSAT (cont'd)

- GSAT only explores potential solutions that are close to the one currently being considered (differing only in one variable).
- Clearly, GSAT could fail to find an assignment even if one exists, i.e. it is incomplete.
- Specific feature of GSAT: Chooses at random the variable whose assignment is to be changed from those that would give an equally good improvement.
- Thus, unlikely that the algorithm makes the sequence of changes over and over.
- Another characteristic: GSAT makes sidesteps (flips variables without increasing the number of satisfied clauses).
- Thus, it can move over "plateaus" to get to better spots where improvements are again possible.

GSAT for Non-Clausal Formulas

- GSAT can also be applied to non-clausal formulas.
- To show how, we first define the criterion for satisfaction on clause sets:

Definition

The *penalty* of a model m on a clause set S, written $pen(S, m)$, is equal to the number of clauses in S that are made false by m.

- Thus, the purpose of GSAT is to find a model m for S with penalty as low as possible.
- In fact, $pen(S, m) = 0$ means $m \models S$.

GSAT for Non-Clausal Formulas (cont'd)

- GSAT works by considering proposition variables that, when flipped, bring the penalty down as much as possible.
- Write

$$\Delta(S, m, p) =_{Def} pen(S, m \text{ with } p \text{ flipped}) - pen(S, m)$$

- GSAT tries to go downhill and searches for p in the direction where the slope $\Delta(S, m, p)$ is negative.

GSAT for Non-Clausal Formulas (cont'd)

- The penalty function *pen* may now be extended to a penalty function *Pen* on arbitrary formulas:
- For literals L, $Pen(L, m) = pen(m, L)$, that is,

$$Pen(m, L) = \begin{cases} 1 & \text{if } m \not\models L, \\ 0 & \text{otherwise.} \end{cases}$$

GSAT for Non-Clausal Formulas (cont'd)

Further, $Pen^{-}(m, L) = 1 - pen(m, L)$, and

$$\begin{aligned}
Pen(m, \neg\phi_1) &= Pen^{-}(m, \phi_1) \\
Pen(m, \phi_1 \wedge \phi_2) &= Pen(m, \phi_1) + Pen(m, \phi_2) \\
Pen(m, \phi_1 \vee \phi_2) &= Pen(m, \phi_1) \cdot Pen(m, \phi_2) \\
Pen(m, \phi_1 \supset \phi_2) &= Pen^{-}(m, \phi_1) \cdot Pen(m, \phi_2) \\
Pen(m, \phi_1 \equiv \phi_2) &= Pen^{-}(m, \phi_1) \cdot Pen(m, \phi_2) + Pen(m, \phi_1) \cdot Pen^{-}(m, \phi_2)
\end{aligned}$$

and

$$\begin{aligned}
Pen^{-}(m, \neg\phi_1) &= Pen(m, \phi_1) \\
Pen^{-}(m, \phi_1 \wedge \phi_2) &= Pen(m, \phi_1) \cdot Pen(m, \phi_2) \\
Pen^{-}(m, \phi_1 \vee \phi_2) &= Pen(m, \phi_1) + Pen(m, \phi_2) \\
Pen^{-}(m, \phi_1 \supset \phi_2) &= Pen(m, \phi_1) + Pen^{-}(m, \phi_2) \\
Pen^{-}(m, \phi_1 \equiv \phi_2) &= (Pen(m, \phi_1) + Pen^{-}(m, \phi_2)) \cdot (Pen^{-}(m, \phi_1) + Pen(m, \phi_2))
\end{aligned}$$

GSAT for Non-Clausal Formulas (cont'd)

It can be proven that, for every arbitrary formula ϕ,

$$\mathit{Pen}(m, \phi) = \mathit{pen}(m, \mathrm{CNF}(\phi))$$

where $\mathrm{CNF}(\phi)$ is a CNF-conversion of ϕ.

Pen and Pen^- can be computed in time linear to the length of ϕ.

So, plug in *Pen* into GSAT and apply GSAT to arbitrary formulas.
Result: NC-GSAT (very efficient implementations).

Shortcomings of GSAT

- GSAT often “wanders” through large plateaus of truth-assignments that show no variation.
- Can easily be misled into exploring the wrong part of the search space.
- Search is non-deterministic, so that trials are not reproducible.

Improvements of GSAT

- *Random Walk Strategy* (to escape from local minima):
    - With probability p, flip a variable that occurs in some unsatisfied clause.
    - With probability $1 - p$, follow the standard GSAT scheme, i.e., make the best possible local move.

  Upward moves (which would otherwise lead us astray) are now used to "repair" unsatisfied clauses.

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  - With probability p, flip a variable that occurs in some unsatisfied clause.
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  Upward moves (which would otherwise lead us astray) are now used to "repair" unsatisfied clauses.
- The *WalkSAT* algorithm takes this idea one step further and makes it the central component of the algorithm:
  - Randomly select an unsatisfied clause.
  - If the clause has a variable that can be flipped without breaking other clauses, that variable is flipped.
  - Else, with probability p we flip the variable that breaks the fewest clauses, and with probability $1 - p$ we flip a random variable in the selected clause.

Propositional Formula Checkers

Theorem proving amounts to verifying whether ψ follows from $\phi_1, \ldots, \phi_n$, for some $\phi_1, \ldots, \phi_n$ and ψ. There are two possibilities: either $\phi_1, \ldots, \phi_n \vdash \psi$ or $\phi_1, \ldots, \phi_n \nvdash \psi$.

1. If "$\vdash$", then $\phi_1 \wedge \ldots \wedge \phi_n \wedge \neg\psi$ is not satisfiable, which can be shown by means of a refutation method, such as resolution or the tableaux method.
2. If "$\nvdash$", then $\phi_1 \wedge \ldots \wedge \phi_n \wedge \neg\psi$ is satisfiable and this can be proven by finding a countermodel.

Propositional Formula Checkers (cont'd)

- Satisfiability as well as unsatisfiability can be expressed by an existential statement.
- ϕ is satisfiable if there *exists* a satisfying assignment for ϕ.
- ϕ is unsatisfiable if there *exists* a refutation (resolution refutation or tableau refutation) of ϕ.
- In general, writing down a resolution proof is harder than writing down a satisfying assignment.
- This non-symmetry is caused by the fact that the satisfiability problem (aka SAT) is NP-complete, and propositional provability is co-NP-complete (see page 37 in the lecture notes).

Propositional Formula Checkers (cont'd)

- Still, resolution is the most powerful resolution method.
- State-of-the-art theorem provers are based on the manipulation of clause sets (not refutation trees).
- Tableaux method: search for a countermodel coincides with the search for a refutation.
- Thus, a tableaux is *always* useful.
- Not true for resolution refutation.
- Resolution can only prove (1), but never (2).
- To prove (2), so-called model-checking techniques are used (trying to guesss countermodels).

Propositional Formula Checkers (cont'd)

- A *propositional formula checker (PFC)* is an ATP program that is able to prove valid formulas, and disprove invalid formulas.
- Tableau method is "complete enough" to form the basis for a PFC, resolution is not.
- Resolution needs to be supplemented with some sort of model checking.

SAT/SMT

Here we leave the Vreeswijk lecture notes. The following slides are based on mainly two sources:

1. Dennis Yurichev's "SAT/SMT by Example" (https://yurichev.com/writings/SAT_SMT_by_example.pdf)
2. The "Programming Z3" tutorial (https://theory.stanford.edu/~nikolaj/programmingz3.html)

SAT/SMT

- SAT/SMT solvers deal with huge systems of equations.
- (A lot of real world problems can be represented as problems of solving system of equations.)
- Difference: SMT solvers take systems in arbitrary format, while SAT solvers are limited to boolean equations in CNF.
- SMT solvers are frontends to SAT solvers.
- Some SMT solvers use external SAT solvers (e.g. STP uses MiniSAT or CryptoMiniSAT as backend).
- Some other SMT-solvers (like Z3) have their own SAT solvers.
- You can also say that SMT vs. SAT is like high-level programming languages vs. assembly languages. The latter can be much more efficient, but it's hard to program in it.

Example: MiniSat

- This example from [1] shows what working with the MiniSat solver can look like.
- The following CNF describes a 2-bit adder circuit:

$$(\neg aH \vee \neg bH) \wedge (aH \vee bH) \wedge (\neg aL \vee \neg bL) \wedge (aL \vee bL)$$

- The standard way to encode CNF expressions for MiniSat is to write each disjunction on one line.
- Also, MiniSat does not support variable names, just numbers. So we enumerate our variables:

$$1\ aH, 2\ aL, 3\ bH, 4\ bL$$

- The input file must also specify in the first line the number of variables and number of clauses.

Example: MiniSat (cont'd)

- The input file will now look like this:

```
p cnf 4 4
-1 -3 0
1 3 0
-2 -4 0
2 4 0
```

- Minus before a variable number means that the variable is negated.
- Zero at the end is just a terminating zero, meaning the end of the clause.
- The task of MiniSat is now to find a set of inputs that can satisfy all lines in the input file.

Example: MiniSat (cont'd)

- Here is how to run MiniSat (input file named `adder.cnf`):

```
% minisat -verb =0 adder.cnf results.txt
SATISFIABLE
```

Example: MiniSat (cont'd)

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```
% minisat -verb =0 adder.cnf results.txt
SATISFIABLE
```

- Results are in `results.txt`:

```
SAT
-1 -2 3 4 0
```

- So, if the first two variables (aH and aL) will be false, and the last two variables (bH and bL) will be set to true, the whole CNF expression is satisfiable.

Example: MiniSat (cont'd)

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- So, if the first two variables (aH and aL) will be false, and the last two variables (bH and bL) will be set to true, the whole CNF expression is satisfiable.
- What about other possible results?

Example: MiniSat (cont'd)

- SAT-solvers (like SMT solvers) produce only one solution (or instance).
- Standard way to work around this: negate the solution clause and add it to the input expression.
- For our exampple, we have -1 -2 3 4, so if we negate all values in it we get 1 2 -3 -4).
- Add this to the end of the input file and run the solver again. Result:

```
SAT
1 2 -3 -4 0
```

Example: MiniSat (cont'd)

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- Add this to the end of the input file and run the solver again. Result:

```
SAT
1 2 -3 -4 0
```

- Do the same with this clause. Result:

```
SAT
-1 2 3 -4 0
```

Example: MiniSat (cont'd)

- And again with this clause:

  ```
  SAT
  1 -2 -3 4 0
  ```

Example: MiniSat (cont'd)

- And again with this clause:

  ```
  SAT
  1 -2 -3 4 0
  ```

- And another time:

  ```
  UNSATISFIABLE
  ```

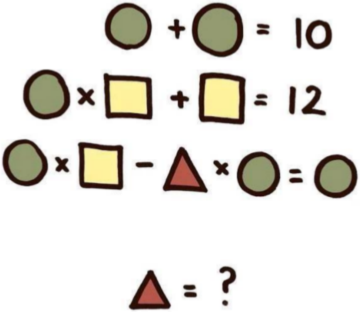
- Obviously, we have found all possible solutions.

List of SAT Solvers

- CryptoMiniSat, `https://github.com/msoos/cryptominisat/`
- The Glucose SAT Solver, based on Minisat, `http://www.labri.fr/perso/lsimon/glucose/`
- gophersat, a SAT solver in Go, `https://github.com/crillab/gophersat`
- microsat, `https://github.com/marijnheule/microsat/`
- MiniSat, `http://minisat.se/`
- PicoSat, PrecoSat, Lingeling, CaDiCaL, `https://github.com/arminbiere/cadical`
- Open-WBO, `http://sat.inesc-id.pt/open-wbo/`

Example: Z3

- Z3 is an SMT solver developed by Microsoft Resarch.
- The tutorial [2] gives you a detailed introduction.
- Here is an example with Z3 from [1] - solving the puzzle below:



Example: Z3 (cont'd)

- If we use Z3's Python interface, all we have to write is:

```
#!/ usr/bin/ python
from z3 import *
circle, square, triangle
    = Ints(' circle square triangle ')
s = Solver()
s.add(circle + circle == 10)
s.add(circle * square + square == 12)
s.add(circle * square - triangle * circle == circle)
print s.check()
print s.model()
```

Example: Z3 (cont'd)

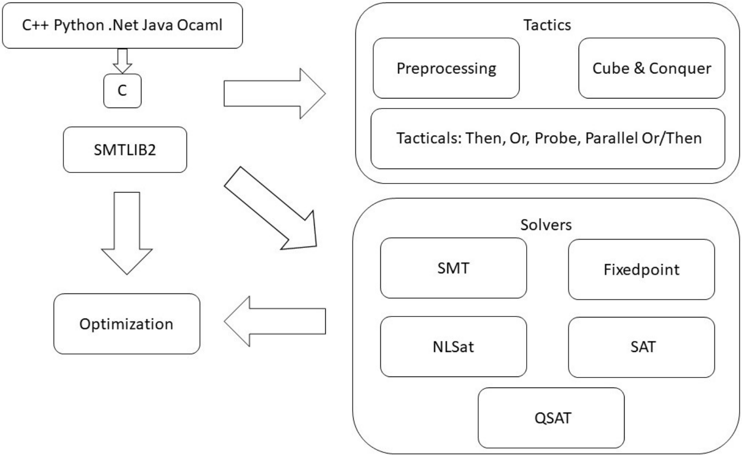
- Output:

```
sat
[triangle = 1, square = 2, circle = 5]
```

Satisfiability Modulo Theories

- SMT solvers take into account background theories, such as the theory of real numbers, the theory of integers, and the theories of various data structures such as lists, arrays, bit vectors, etc.
- Early approaches ("eager approaches") translated SMT instances into Boolean SAT instances to solve them.
- More recent approaches ("lazy approaches") integrate the Boolean reasoning of a DPLL-style search with theory-specific solvers (T-solvers) that handle conjunctions of predicates from a given theory.

Z3 Architecture



Another Example with Z3

- Here is another Z3 example from [1].
- Suppose we have an equation system:

$$\begin{aligned} 3x + 2y - z &= 1 \\ 2x - 2y + 4z &= -2 \\ -x + \tfrac{1}{2}y - z &= 0 \end{aligned}$$

- You can probably imagine how this would look in the Python interface.
- Python (and other high-level programming languages like C#) interfaces are highly popular, because they are practical.
- But actually there is a standard, LISP-like language for SMT-solvers called SMT-LIB.

Another Example with Z3

- The equation system in SMT-LIB:

```
(declare-const x Real)
(declare-const y Real)
(declare-const z Real)
(assert (=( -(+(* 3 x) (* 2 y)) z) 1))
(assert (=(+( -(* 2 x) (* 2 y)) (* 4 z)) -2))
(assert (=( -(+ (- 0 x) (* 0.5 y)) z) 0))
(check-sat)
(get-model)
```

Another Example with Z3

- Z3 will return:

```
% z3 -smt2 example .smt
sat
(model
  (define-fun z () Real
    (- 2.0))
  (define-fun y () Real
    (- 2.0))
  (define-fun x () Real
    1.0)
)
```

A Third Example with Z3

[1] also shows how to solve the following problem from `http://www.cs.utsa.edu/~wagner/knuth/fasc0b.pdf` with Z3:

▸ **56.** [*20*] The satisfiability problem for a Boolean function $f(x_1, x_2, \ldots, x_n)$ can be stated formally as the question of whether or not the quantified formula

$$\exists x_1 \, \exists x_2 \ldots \exists x_n \, f(x_1, x_2, \ldots, x_n)$$

is true; here '$\exists x_j \, \alpha$' means, "there exists a Boolean value x_j such that α holds."

A much more general evaluation problem arises when we replace one or more of the existential quantifiers $\exists x_j$ by the universal quantifier $\forall x_j$, where '$\forall x_j \, \alpha$' means, "for all Boolean values x_j, α holds."

Which of the eight quantified formulas $\exists x \, \exists y \, \exists z \, f(x,y,z)$, $\exists x \, \exists y \, \forall z \, f(x,y,z)$, $\ldots$, $\forall x \, \forall y \, \forall z \, f(x,y,z)$ are true when $f(x,y,z) = (x \vee y) \wedge (\bar{x} \vee z) \wedge (y \vee \bar{z})$?

A Third Example with Z3 (cont'd)

Encoding in Z3:

```
(assert
  (exists ((x Bool)) (forall ((y Bool)) (forall ((z Bool))
    (and
      (or x y)
      (or (not x) z)
      (or y (not z))
    )))
  )
)
(check-sat)
```

A Third Example with Z3 (cont'd)

```
-z3 -smt2 KnuthEAA .smt
```

Output:

```
unsat
```

Analogously for all other combinations of ∀ and ∃ (some result in sat and some in unsat).

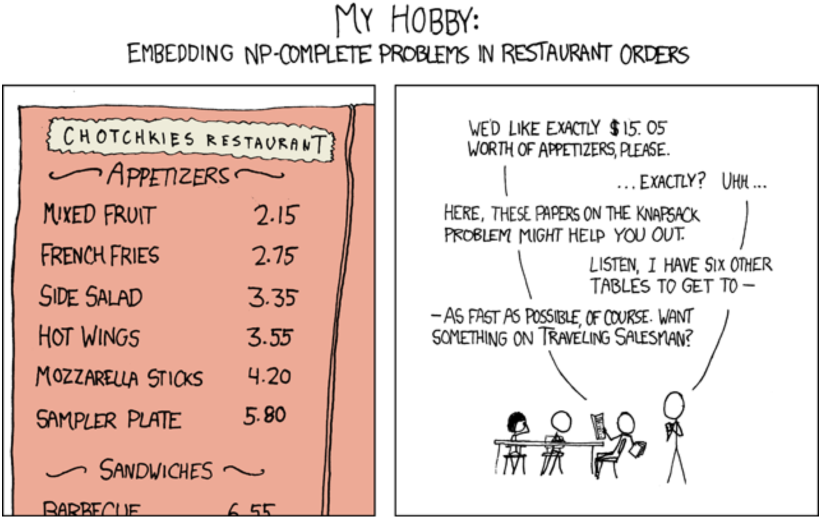
List of SMT Solvers

- Alt-Ergo, `https://alt-ergo.ocamlpro.com/`
- Boolector, `http://fmv.jku.at/boolector/`
- CVC3/CVC4, `http://cvc4.stanford.edu/`
- dReal, `http://dreal.cs.cmu.edu`, `https://github.com/dreal`
- MathSAT, `http://mathsat.fbk.eu/`
- MK85, `https://github.com/DennisYurichev/MK85`
- STP, `https://github.com/stp/stp`
- toysolver, `https://github.com/msakai/toysolver`
- veriT, `http://www.verit-solver.org/`
- Yices, `http://yices.csl.sri.com/`
- Z3, `https://github.com/Z3Prover/z3`

Example: Wolfram Mathematica

- Not in the list on the previous slide, but also Wolfram Mathematica (`https://www.wolfram.com/mathematica/`) can be used as SMT solver.
- Here is an example from [1], showing how to solve XKCD No. 287 (`https://www.xkcd.com/287/`):

Example: Wolfram Mathematica (cont'd)



Example: Wolfram Mathematica (cont'd)

- Input:

```
In[]:= FindInstance [2.15 a + 2.75 b + 3.35 c
+ 3.55 d + 4.20 e + 5.80 f == 15.05 && a >= 0
&& b >= 0 && c >= 0 && d >= 0 && e >= 0 &&
f >= 0, {a, b, c, d, e, f}, Integers , 1000]
```

- 1000 means "find at most 1000 solutions", but only 2 are found:

```
Out[]= {{a -> 1, b -> 0, c -> 0,
d -> 2, e -> 0, f -> 1},
{a -> 7, b -> 0, c -> 0,
d -> 0, e -> 0, f -> 0}}
```

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