1 Part I

1. I provide you with two proofs. The first one is shorter.

PROOF Top
[A1:] \neg(\forall i : 0 \leq i < n : b[i])
[A2:] (\forall i : j \leq i < n : b[i])
[G:] \neg(\forall i : 0 \leq i < j : b[i])

BEGIN
1. { see the subproof below } (\forall i : 0 \leq i < j : b[i]) \Rightarrow false

PROOF SP1
[A1:] (\forall i : 0 \leq i < j : b[i])
[G:] false
BEGIN
1. { rewriting Top.A1 using Negate-\forall (Theorem A.4.7) } 
   (\exists i : 0 \leq i < n : \neg b[i])
2. { Applying \exists-elimination on 1 } [SOME i] 0 \leq i < n \land \neg b[i]
3. { see the subproof below } j \leq i \Rightarrow false

PROOF SP1.A
[A1:] j \leq i 
[G:] false
BEGIN
1. { SP1.2 implies i < n, put this in conjunction to A1 above } 
   j \leq i < n
2. { \forall-elimination on Top.A2 using 1 above } b[i]
3. { contradiction between 2 and the 2nd conjunct of SP1.2 } 
   false
END

4. { see the subproof below } i < j \Rightarrow false

PROOF SP1.B
[A1:] i < j 
[G:] false
BEGIN
1. { SP1.2 implies 0 \leq i, put this in conjunction to A1 above } 
   0 \leq i < j
2. { \forall-elimination on SP1.A1 using 1 above } b[i]
3. { contradiction between 2 and the 2nd conjunct of SP1.2 } 
   false
END

5. { trivial } j \leq i \lor i < j

6. { applying the Case-split Rule (Rule A.1.9) on 3,4,5 } false

END

2. { Contraduction Rule (Rule A.1.3) on 1 } \neg(\forall i : 0 \leq i < j : b[i])
END
Below is another way (more lengthy) to prove the same goal. The general idea is that A2 together with the negation of G implies \((\forall i: 0 \leq i < n : b[i])\), which is then in contradiction with A1, and thus closing a proof-by-contradiction. However, the previously mentioned implication actually assumes that \(j\) is in the interval \([0..n]\), which is nowhere assumed in A1 nor A2. This creates another proof obligation, namely that we have to prove the goal from the case where \(j\) is actually outside the domain of \([0..n]\).

**PROOF Top**

[A1:] \(\neg (\forall i: 0 \leq i < n : b[i])\)

[A2:] \((\forall i: j \leq i < n : b[i])\)

[G:] \(\neg (\forall i: 0 \leq i < j : b[i])\)

**BEGIN**

1. \{ see the subproof below \} \((\forall i: 0 \leq i < j : b[i]) \Rightarrow \text{false}\)

**PROOF SP1**

[A1:] \((\forall i: 0 \leq i < j : b[i])\)

[G:] false

**BEGIN**

1. \{ see the subproof below \} \(0 \leq j \leq n \Rightarrow \text{false}\)

**PROOF SP1.A**

[A1:] \(0 \leq j \leq n\)

[G:] false

**BEGIN**

1. \{ conjunction of Top.A2 and SP1.A1 \}

\((\forall i: 0 \leq i < j : b[i]) \land (\forall i: j \leq i < n : b[i])\)

2. \{ rewriting 1 with Domain Split (Theorem A.4.12) \}

\((\forall i: 0 \leq i < j \lor j \leq i < n : b[i])\)

3. \{ Domain Merging (Theorem A.4.16) says that \(0 \leq i < j \lor j \leq i < n\) is equivalent to \(0 \leq i < n\), provided \(j\) is between 0 and \(n\), but this is justified by A1 above \}

\((\forall i: 0 \leq i < n : b[i])\)

4. \{ contradiction between 4 and top.A1 \} false

**END**

2. \{ see the subproof below \} \(\neg (0 \leq j \leq n) \Rightarrow \text{false}\)

**PROOF SP1.B**

[A1:] neg(0 \leq j \leq n)

[G:] false

**BEGIN**

1. \{ applying de Morgan on A1 and simplifying the result \}

\(j < 0 \lor j > n\)

2. HINTrewriting Top.A1 using Negate-\(\forall\) (Theorem A.4.7)

\((\exists i: 0 \leq i < n : \neg b[i])\)
3. { Applying \exists\text{-elimination on 1 } } [\text{SOME } i] \ 0 \leq i < n \land \neg b[i]

4. { see the proof below } \ j < 0 \Rightarrow \text{false}

\text{PROOF SP1.B.1}
[\text{A1:}] \ j < 0
[\text{G:}] \text{false}
BEGIN
1. { SP1.B.3's left conjunct together with A1 imply: } \ j \leq i < n
2. { \forall\text{-elimination on Top.A2 using 1 above } } b[i]
3. { contradiction between 2 and the 2nd conjunct of SP1.B.3 } \text{false}
END

5. { prove this yourself } \ j > n \Rightarrow \text{false}

6. { applying the Case-split Rule (Rule A.1.9) on 1,4,5 } \text{false}
END

3. { applying the Case-split Rule (Rule A.1.9) on 1,2,3 } \text{false}
END

2. { Contradiction Rule (Rule A.1.3) on 1 } \neg(\forall i : \ 0 \leq i < j : b[i])
END