1 Part I

1. I provide you with two proofs. The first one is shorter.

PROOF Top
[A1:] ¬(∀i:0≤i<n:b[i])
[A2:] (∀i:j≤i<n:b[i])
[G:] ¬(∀i:0≤i<j:b[i])

BEGIN
1. { see the subproof below } (∀i:0≤i<j:b[i]) ⇒ false
   PROOF SP1
   [A1:] (∀i:0≤i<j:b[i])
   [G:] false
   BEGIN
   1. { rewriting Top.A1 using Negate-∀ (Theorem A.4.7) }
      (∃i:0≤i<n:¬b[i])
   2. { Applying ∃-elimination on 1 } [SOME i] 0≤i<n ∧ ¬b[i]
   3. { see the subproof below } j≤i ⇒ false
      PROOF SP1.A
      [A1:] j≤i
      [G:] false
      BEGIN
      1. { SP1.2 implies i < n, put this in conjunction to A1 above }
         j≤i<n
      2. { ∀-elimination on Top.A2 using 1 above } b[i]
      3. { contradiction between 2 and the 2nd conjunct of SP1.2 }
         false
      END
   4. { see the subproof below } i<j ⇒ false
      PROOF SP1.B
      [A1:] i<j
      [G:] false
      BEGIN
      1. { SP1.2 implies 0 ≤ i, put this in conjunction to A1 above }
         0≤i<j
      2. { ∀-elimination on SP1.A1 using 1 above } b[i]
      3. { contradiction between 2 and the 2nd conjunct of SP1.2 }
         false
      END
   END

5. { trivial } j≤i ∨ i<j

6. { applying the Case-split Rule (Rule A.1.9) on 3,4,5 } false

END

2. { Contraduction Rule (Rule A.1.3) on 1 } ¬(∀i:0≤i<j:b[i])
END
Below is another way (more lengthy) to prove the same goal. The general idea is that \( A2 \) together with the negation of \( G \) implies \((\forall i : 0 \leq i < n : b[i])\), which is then in contradiction with \( A1 \), and thus closing a proof-by-contradiction. However, the previously mentioned implication actually assumes that \( j \) is in the interval \([0..n]\), which is nowhere assumed in \( A1 \) nor \( A2 \). This creates another proof obligation, namely that we have to prove the goal from the case where \( j \) is actually outside the domain of \([0..n]\).

**PROOF Top**

\[ A1: \neg(\forall i : 0 \leq i < n : b[i]) \]
\[ A2: (\forall i : j \leq i < n : b[i]) \]
\[ G: \neg(\forall i : 0 \leq i < j : b[i]) \]

**BEGIN**

1. \{ see the subproof below \} \((\forall i : 0 \leq j < n : b[i]) \Rightarrow \text{false}\)

**PROOF SP1**

\[ A1: (\forall i : 0 \leq j < n : b[i]) \]
\[ G: \text{false} \]

**BEGIN**

1. \{ see the subproof below \} \( 0 \leq j \leq n \Rightarrow \text{false} \)

**PROOF SP1.A**

\[ A1: 0 \leq j \leq n \]
\[ G: \text{false} \]

**BEGIN**

1. \{ conjunction of Top.A2 and SP1.A1 \} \((\forall i : 0 \leq i < j : b[i]) \land (\forall i : j \leq i < n : b[i])\)

2. \{ rewriting 1 with Domain Split (Theorem A.4.12) \} \((\forall i : 0 \leq i < j \lor j \leq i < n : b[i])\)

3. \{ Domain Merging (Theorem A.4.16) says that \( 0 \leq i < j \lor j \leq i < n \) is equivalent to \( 0 \leq i < n \), provided \( j \) is between 0 and \( n \), but this is justified by \( A1 \) above \} \((\forall i : 0 \leq i < n : b[i])\)

4. \{ contradiction between 4 and top.A1 \} \text{false}

**END**

2. \{ see the subproof below \} \( \neg(\exists j : j \leq n) \Rightarrow \text{false} \)

**PROOF SP1.B**

\[ A1: \neg(0 \leq j \leq n) \]
\[ G: \text{false} \]

**BEGIN**

1. \{ applying de Morgan on \( A1 \) and simplifying the result \} \( j < 0 \lor j > n \)

2. \{ HINTrewriting Top.A1 using Negate-\( \forall \) (Theorem A.4.7) \( (\exists i : 0 \leq i < n : \neg b[i]) \)
3. \{ Applying \exists\text{-elimination on 1 } \} \{\text{SOME } i\} 0 \leq i < n \land \neg b[i] \\

4. \{ see the proof below \} j < 0 \Rightarrow false

PROOF SP1.B.1
[A1:] j < 0
[G:] false
BEGIN
1. \{ SP1.B.3’s left conjunct together with A1 imply: \} j \leq i < n
2. \{ \forall\text{-elimination on Top.A2 using 1 above } \} b[i]
3. \{ contradiction between 2 and the 2nd conjunct of SP1.B.3 \} false
END

5. \{ prove this yourself \} j > n \Rightarrow false

6. \{ applying the Case-split Rule (Rule A.1.9) on 1,4,5 \} false

END

3. \{ applying the Case-split Rule (Rule A.1.9) on 1,2,3 \} false

END

2. \{ Contradiction Rule (Rule A.1.3) on 1 \} \neg(\forall i : 0 \leq i < j : b[i])

END