About spherical kinematics

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- There is a close connection to planar kinematics. Let the radius of the sphere approach infinity . . .

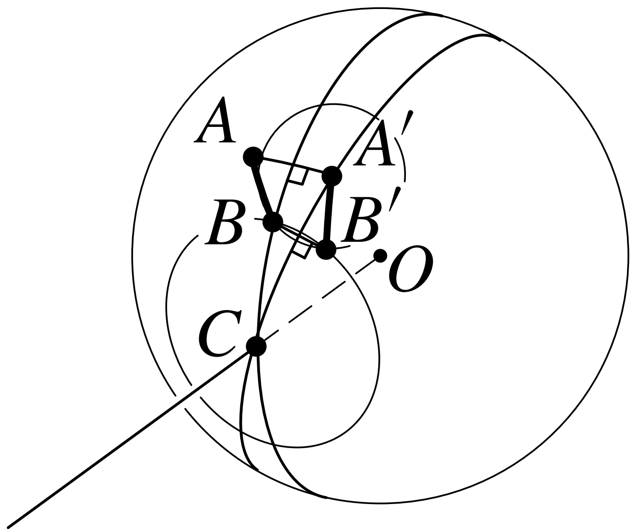
Two not-antipodal points enough

Theorem 2.5: A displacement of the sphere is completely determined by the motion of any two points that are not antipodal.

Proof: Construct a coordinate frame . . .

Euler's theorem

Theorem 2.6: For every spatial rotation, there is a line of fixed points. In other words, every rotation about a point is a rotation about a line, called the *rotation axis*.



Proof:

> Prove that every displacement of the sphere has a fixed point.
>
> Define A, $\perp AA'$, B, B', $\perp BB'$.
>
> Define C to be either intersection of $\perp AA'$ with $\perp BB'$.
>
> Let R be the rotation mapping A to A' and C to itself.
>
> Show R maps B to B', so R is the given displacement.

Chasles's theorem

Theorem 2.7: Every spatial displacement is the composition of a rotation about some axis, and a translation along the same axis.

Proof:

Assume arbitrary displacement D is given.

Use theorem 2.2 to decompose $D = R \circ T$.

Decompose T into components parallel to and perpendicular to axis of R: $D = R \circ T_{\perp} \circ T_{\|}$.

Note that $R \circ T_{\perp}$ is planar! Every plane perpendicular to rotation axis is mapped rigidly to itself.

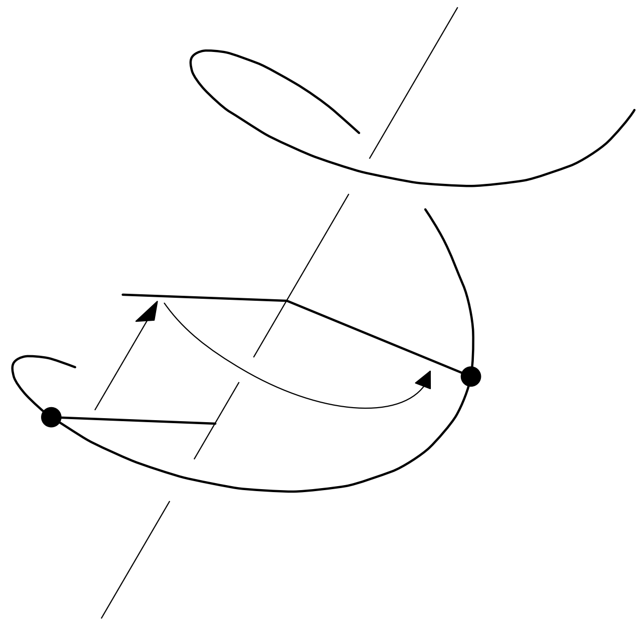
If $R \circ T_{\perp}$ is a translation the theorem follows immediately. Otherwise $R \circ T_{\perp}$ is a rotation about some axis parallel to the rotation axis of R.

So $D = (R \circ T_{\perp}) \circ T_{\|}$ is the desired decomposition.

Screws.

A *screw* is a line in space with an associated pitch, which is a ratio of linear to angular quantities.

A *twist* is a screw plus a scalar magnitude, giving a rotation about the screw axis plus a translation along the screw axis. The rotation angle is the twist magnitude, and the translation distance is the magnitude times the pitch. Thus the pitch is the ratio of translation to rotation.



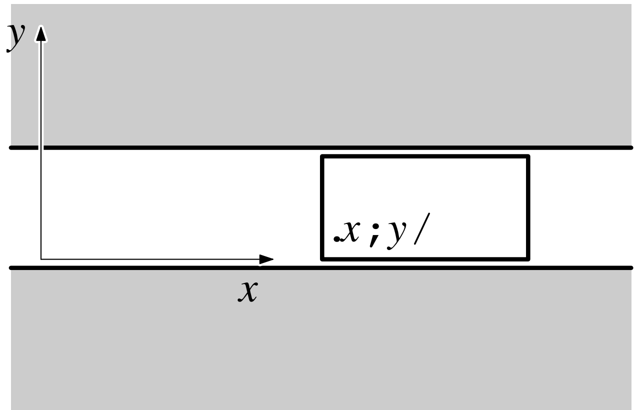
Constraint: taxonomy and examples

bilateral

Expressed as an equation. Two sided.

$$y = 0$$

$$\theta = 0$$



unilateral

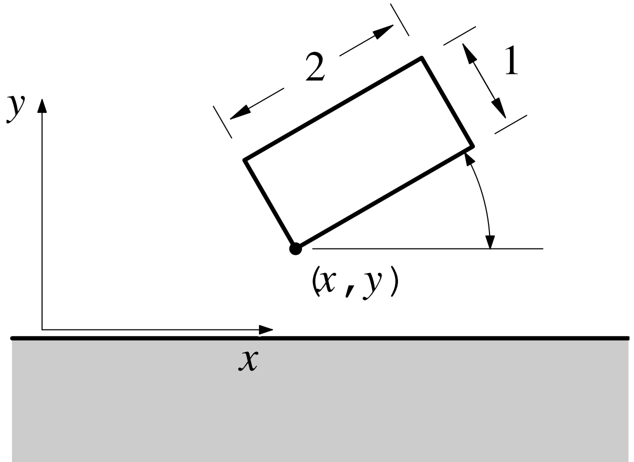
Expressed as an inequation. One sided.

$$y \geq 0$$

$$y + 2\sin\theta \geq 0$$

$$y + 2\sin\theta + \cos\theta \geq 0$$

$$y + \cos\theta \geq 0$$



Constraint: taxonomy and examples

scleronomic

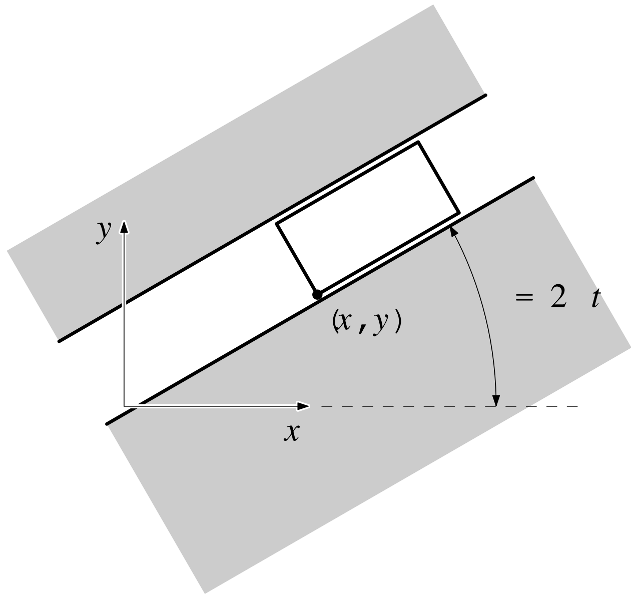
Independent of t. Stationary.

rheonomic

Depends on t.

$$x \sin(2\pi t) - y \cos(2\pi t) = 0$$

$$\theta = 2\pi t$$

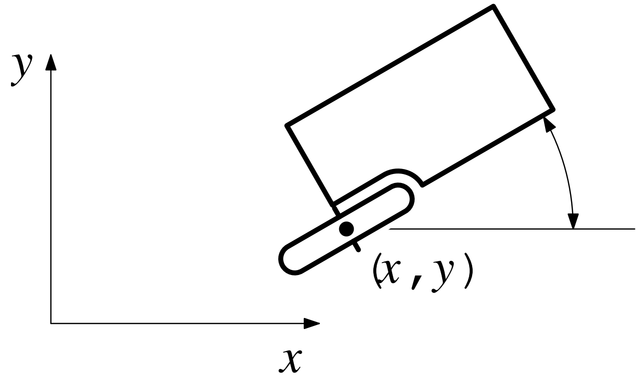


holonomic

Independent of $\dot{q}$ and bilateral.

$$f(q, t) = 0$$

nonholonomic



But watch for false positives