# Outline.

- One more general theorem: Displacement $=$ translation $\circ$ rotation.
- Some fundamental theorems on planar motion.
- Main result: every planar motion has a rotation center in projective plane.
- Centrodes.

# Decomposition of displacements

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So then $T \circ R = T \circ T^{-1} \circ D = D$ is the desired decomposition. QED

# Decomposition of displacements

Note:

- Instead of $R = T^{-1} \circ D$, $D = T \circ R$,
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- The decomposition is not unique: it depends on the choice of $O$.
- The proof extends to arbitrary $\mathbb{E}^n$.
- Note how simple it is to prove using group theory!

# Planar kinematics

That is all we will do on “general” kinematics. On to planar kinematics.
What can we say about rigid motions of $\mathbb{E}^2$?
Theorem 2.3: A planar displacement is completely determined by the motion of any two points.

Proof: Construct a coordinate frame . . .

# Planar kinematics: every $D$ is an $R$ or a $T$

Now for the big one:

Theorem 2.4: Every planar displacement is either a translation or a rotation.

Not a proof:

Pick two points $A$ and $B$.

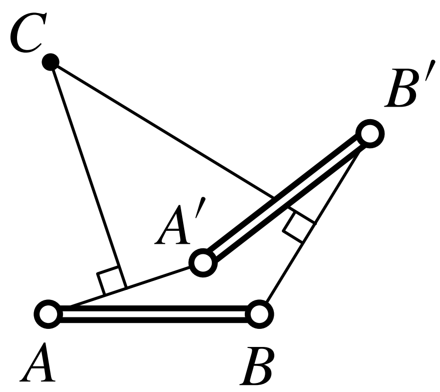
Let $A'$ and $B'$ be the images.

Construct perpendicular bisectors.

Intersection gives fixed point.

Why? Preserves distance from $A$ and from $B$, so ...

Okay, not a proof, but a useful construction.



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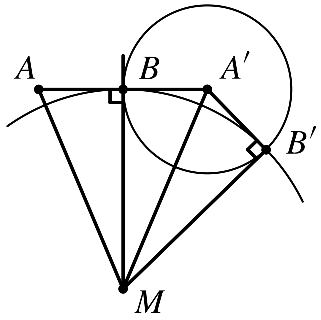



Pick any point $A$. We can assume $A \neq A'$.

Pick $B$ the midpoint of the line segment $\overline{AA'}$. We can assume $B'$ is not on $\overline{AA'}$.

Construct perp to $AB$ at $B$, and perp to $A'B'$ at $B'$. They are not parallel. Let $M$ be their intersection.

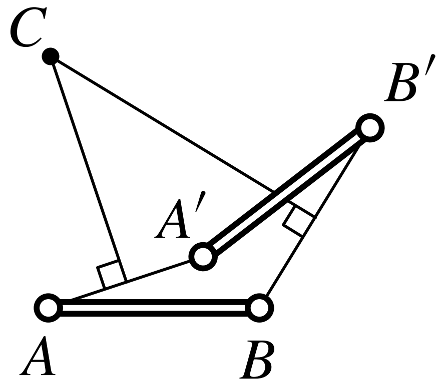
Consider the rotation that maps $A$ to $A'$ and $M$ to itself. Where does it map $B$? Preservation of distance gives two candidates, and we can exclude one.

So our rotation maps $B$ to $B'$. It is the given displacement. QED.

# Planar kinematics. Rotation centers

So, every planar displacement is a rotation or a translation.

Consider again construction of rotation centers from the motion of two points. How does it fail when $\overline{AA'}$ is parallel to $\overline{BB'}$?



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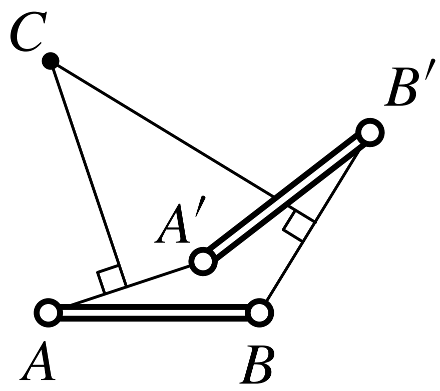

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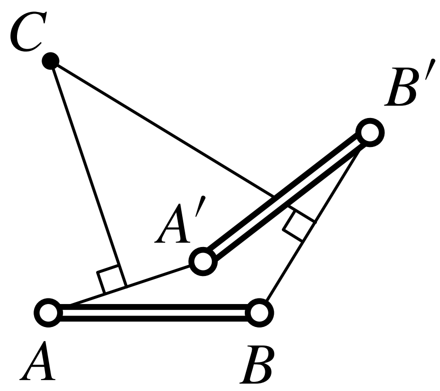

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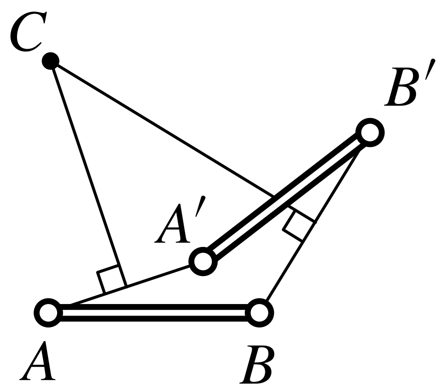


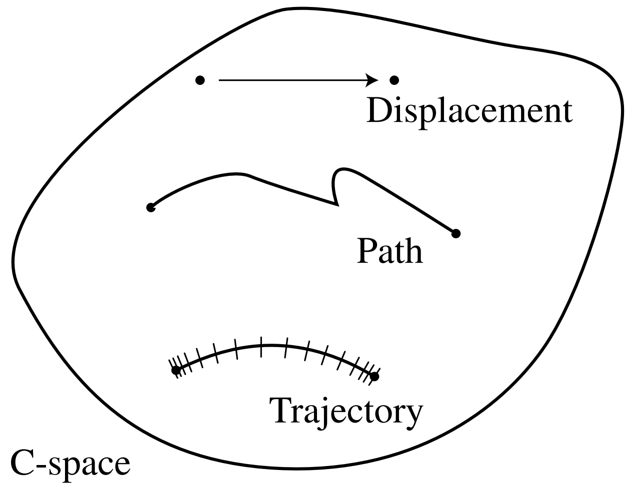

*Every planar displacement is a rotation about a point in the projective plane.*

# Displacements, paths, trajectories.

**Displacement** Discontinuous change of configuration.

**Trajectory** Configuration a continuous function of time: a continuous curve $q(t)$ in configuration space.

**Path** A curve $q(s)$ in configuration space parameterized perhaps by arc length.




For differentiable trajectory $q(t)$ or a path $q(s)$ we have *velocity* $dq/dt$ or *differential change in configuration* $dq$.
To construct rot'n center for diff'l displacement: