The International Journal of Robotics Research

From caging to grasping

Alberto Rodriguez[1], Matthew T Mason[1] and Steve Ferry[2]

Abstract

This paper digs into the relationship between cages and grasps of a rigid body. In particular, it considers the use of cages as waypoints to grasp an object. We introduce the concept of pregrasping cages, caging configurations from which an object can be grasped without first breaking the cage. For two-fingered manipulators, all cages are pregrasping cages and, consequently, useful waypoints to grasp an object. A contribution of this paper is to show that the same does not hold for more than two fingers. A second contribution is to show how to overcome that limitation. We explore the natural generalization of the well-understood squeezing/stretching characterization of two-finger cages to arbitrary workspace dimension, arbitrary object shapes without holes, and arbitrary number of point fingers, and exploit it to give sufficient conditions for a cage to be a pregrasping cage. As a product of that generalization, we introduce grasping functions: scalar functions defined on the finger formation that control the process of going from a cage to a grasp. We finish the paper by establishing an analogy between the role of grasping functions in grasping and that of Lyapunov functions in stability theory.

1. Introduction

To cage an object is to bound its mobility. From the object's point of view, the manipulator constitutes a set of constraints in the object's configuration space that prevents it from escaping arbitrarily far.

In contrast to the prehensile approach to manipulation, caging provides a way to control an object without immobilizing it. If the manipulator moves as a rigid body, a cage on the object is enough to guarantee that the object will 'follow'. Non-prehensile manipulation is the most widely referenced application of caging (Sudsang and Ponce 2000; Blind et al. 2001; Pereira et al. 2004).

Caging is also interesting for a second reason: when an immobilizing grasp is required, or preferred, a cage might constitute a useful waypoint to a grasp. In some cases, the grasping problem becomes trivial once the caging problem is solved. It is on this less referenced application of caging, the relationship between grasping and caging configurations, that we set the focus of this paper.

The most studied case is that of two-fingered manipulators. Vahedi and van Der Stappen (2008) introduced the squeezing and stretching conditions for two-finger cages of polygons in the plane. An object is squeezing caged if it

remains caged when the fingers are given the freedom of moving closer together, see Figure 1(a). Conversely, it is stretching caged if it remains caged when the fingers are given the freedom of moving apart, see Figure 1(b).

The squeezing/stretching conditions, first introduced as algorithmic constructions for the problem of finding cages, have enabled a deep understanding of the configuration space of two-fingered manipulators: all two-finger cages are either squeezing caging, stretching caging, or both.

This characterization gives rise to a fundamental relationship between cages and grasps. Given a cage of an object, there is always an infallible blind strategy to grasp it: either close or open the fingers, depending on whether the cage is a squeezing or a stretching cage, see Figure 2. If we plan to grasp the object beginning from a cage, there is no need for accurate finger positioning or feedback for error control. The squeezing/stretching conditions guarantee that the

[1]Robotics Institute at Carnegie Mellon University, Pittsburgh, PA, USA
[2]Department of Mathematics at Rutgers University, Piscataway, NJ, USA

Corresponding author:
Alberto Rodriguez, Robotics Institute, Carnegie Mellon University, 5000 Forbes Avenue, Pittsburgh, PA 15213, USA.
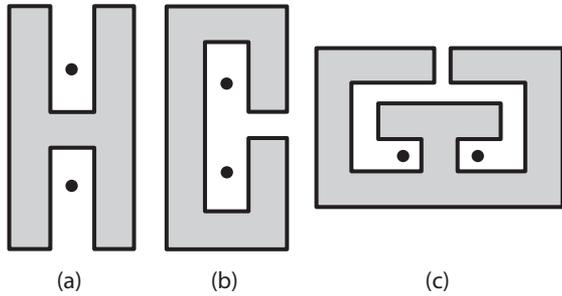Email: albertor@cmu.edu

Fig. 1. Examples of (a) squeezing cage, (b) stretching cage, (c) both stretching and squeezing.
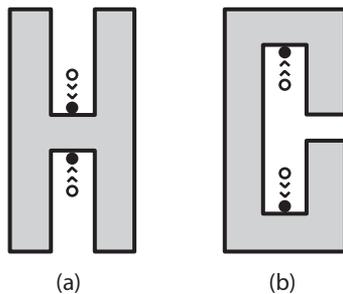

Fig. 2. Pregrasping cages: (a) infallible grasp from a squeezing cage; (b) infallible grasp from a stretching cage.

object will not escape while the manipulator closes/opens the fingers. In this paper we refer to any cage with such a grasping strategy as a *pregrasping cage*.

Does that characterization generalize to cases other than two-finger cages of polygonal objects in the plane? Can we always infallibly grasp an object beginning from a cage? The answer to the first question is yes. In this paper, we generalize the squeezing/stretching characterization to arbitrary object shapes without holes, arbitrary workspace dimension, and arbitrary number of fingers. Unfortunately, the answer to the second question is no. We show in this paper that not every cage is an adequate waypoint to grasp an object, that is, not all cages are pregrasping cages. However, the construction of the proposed generalization builds enough machinery to overcome that limitation and gives sufficient conditions for a cage to be a pregrasping cage.

Section 2 describes previous work on caging and its relation to grasping. In Section 3 we give precise definitions for all concepts related to configuration space, caging, and grasping used in the paper. We discuss in Section 4 the relationship between caging and grasping and generalize the concepts of squeezing and stretching caging to arbitrary number of fingers. For that generalization we introduce *grasping functions*, scalar functions F that control the finger formation, resulting in the concepts of F-cage, F-squeezing cage, and F-stretching cage. A grasping function plays the open/close role that the finger-to-finger distance plays in the two-finger case.

We use grasping functions in Section 5 to prove the generalization of the squeezing/stretching characterization and

explore its implications for the existence of pregrasping cages. We further show that the general formulation, when the chosen caging function is the distance between two fingers, is equivalent to the prior formulation of the two-finger case in the literature. The proposed generalization with grasping functions yields two main consequences:

i. If the manipulator forms a pregrasping cage on the object with a grasping function F, the object remains caged for as long as the manipulator maintains the value of F.
ii. There is always a path from the pregrasping cage to a grasp of the object that never breaks the cage. When the function F is convex (or concave), the process to go from the cage to a grasp is controlled by the value of the function F. We refer to the type of grasp where the grasping process ends as a *grasping cage*, described in detail in Section 3.

Section 6 shows three examples of such grasping functions with associated pregrasping cages. We finish in Section 7 with a review of the implications of the presented results and an analogy between the role played by grasping functions in grasping and that of Lyapunov functions in stability analysis.

2. Related work

Interest in caging an object appears in the literature as early as Besicovitch (1957, 1961) with the analysis of trapping a sphere with a net or a cage. However, it is not until Kuperberg (1990) that we find a formal definition for the caging problem:

> 'Let P be a polygon in the plane, and let C be a set of n points in the complement of the interior of P. The points capture P if P cannot be moved arbitrarily far from its original position without at least one point of C penetrating the interior of P. Design an algorithm for finding a set of capturing points for P.'

Since then, the caging condition has been studied from different perspectives and applied to a number of manipulation problems such as grasping (Davidson and Blake 1998b; Sudsang et al. 1998, 1999; Gopalakrishnan and Goldberg 2002), part feeding (Blind et al. 2001), and non-prehensile manipulation (Pereira et al. 2004; Diankov et al. 2008; Fink et al. 2008).

Relevant to our work are the concepts of squeezing and stretching caging proposed in recent algorithmic approaches to the two-finger caging problem (Pipattanasomporn and Sudsang 2006; Vahedi and van Der Stappen 2008; Pipattanasomporn and Sudsang 2011). Squeezing and stretching caging are stricter conditions than caging that expect the object's mobility to be bounded even when the shape of the manipulator is allowed some partial freedom.

Rimon and Blake (1999) were the first to implicitly study squeezing cages. They were interested in determining the caging set of a grasp: the maximal connected set of caging configurations containing a given immobilization of the object. They showed for hands with one-dimensional configuration spaces that the aperture of the hand at which the cage breaks always corresponds to a frictionless equilibrium grasp, and used that characterization to develop caging algorithms for two-fingered gripping systems. Although they only explicitly considered external or squeezing grasps, they already looked into the connection between immobilizations of the object and caging configurations. Davidson and Blake (1998a) later extended the technique to the three-fingered case.

Pipattanasomporn and Sudsang's works on caging non-convex polygons (Pipattanasomporn and Sudsang 2006) and polytopes (Pipattanasomporn and Sudsang 2011) with two fingers has a similar goal to Rimon and Blake's but with an algorithmic rather than topological approach. They studied the closely related problem of finding maximal two-finger cages on concavities of planar polygons. They introduced the concepts of equivalent squeezed and stretched finger trajectories that allowed them to describe equivalences between paths in the free space of the manipulator and transform the caging problem into a graph search problem in a finite discrete space.

The concepts of squeezing and stretching caging were formally developed by Vahedi and van Der Stappen (2008) as an algorithmic technique to find all two-finger caging placements of planar polygons. They showed that, in that case, squeezing and stretching cages account for all possible cages, and used that characterization to develop the first complete algorithm to report them. Rodriguez and Mason (2008) extended the characterization of two-finger squeezing and stretching cages to all compact, connected and contractible object shapes in arbitrary workspace dimension.

Following Vahedi and van Der Stappen (2008), in this paper we extend the concepts of squeezing and stretching caging to manipulators with more than two fingers, for what we define the concept of grasping functions. Closely related is the work of Pipattanasomporn et al. (2008) on caging polytopes via finger dispersion control. They proposed to cage polytopes by maintaining a scalar function defined on the finger formation over a critical value. Our work with grasping functions can be seen as a formalization and extension of the same idea to a more general set of objects. We present it here as the natural generalization of the two-finger squeezing/stretching characterization of cages to the *n*-finger case.

There is considerably less previous work studying caging with an arbitrary number of fingers. Some examples are the works by Pereira et al. (2004) on conservative decentralized control strategies to maintain caging in multi-robot formations for manipulation of planar objects, and that by Fink et al. (2008) who developed decentralized feedback controllers for collaborative transportation of objects in planar environments with obstacles.

The main idea in this paper is that certain caging configurations can be used as waypoints to reliably grasp an object. The closest work in the literature is that of error-tolerant grasping techniques for two-fingered manipulators developed by Davidson and Blake (1998b) and Gopalakrishnan and Goldberg (2002). Both use, in some way or another, the notion of caging to robustify the grasping process.

3. Terminology

In this section, we introduce concepts and definitions related to configuration space, paths, caging, and grasping.

3.1. Configuration space

Let the manipulator be a set of point fingers $p_1 \dots p_n$ in a workspace $\mathcal{W}$ homeomorphic to $\mathbb{R}^d$, and let $\mathcal{M} = \mathcal{W}_{p_1} \times \dots \times \mathcal{W}_{p_n}$ homeomorphic to $\left(\mathbb{R}^d\right)^n$ be the configuration space of the manipulator.

Let $\mathcal{O} \subset \mathcal{W}$ be the object to manipulate, which we assume to have no holes. Formally, we assume there is an homeomorphism $\mathcal{H} : \mathbb{R}^d \to \mathcal{W}$ mapping a d-dimensional Euclidean closed ball in $\mathbb{R}^d$ to the object. We refer to such an object as a standard topological ball. Note that the condition is a bit stronger than being homeomorphic to a Euclidean closed ball, since the homeomorphism is defined in the entire space. In particular, the exterior of the object is also homeomorphic to the exterior of a ball. This will avoid degeneracies when considering paths in the exterior of the object.

We note by $\mathcal{O}^{\mathcal{M}} = \{(p_1 \dots p_n) \in \mathcal{M} \mid \exists \, p_i \in \mathcal{O}\}$ the obstacle induced by $\mathcal{O}$ in the configuration space of the manipulator. Following the convention in Kuperberg (1990), Pereira et al. (2004), Rimon and Burdick (1996), Rodriguez and Mason (2008), and Vahedi and van Der Stappen (2008) the free workspace of each point finger p_i is defined as $\mathcal{W}_{p_i}^{free} = \mathcal{W}_{p_i} \setminus \text{int}[\mathcal{O}]$, the complement of the interior of $\mathcal{O}$, and equivalently the free space of the manipulator as $\mathcal{M}^{free} = \mathcal{M} \setminus \text{int}[\mathcal{O}^{\mathcal{M}}]$.

3.2. Paths in the configuration space

A path of the manipulator in the free space is a parametrized curve $\alpha : [0, 1] \longrightarrow \mathcal{M}^{free}$. We say the path is contractible in $\mathcal{M}^{free}$ if it is closed ($\alpha(0) = \alpha(1)$) and homotopic to a point $c \in \mathcal{M}^{free}$, that is, the path can be retracted to a point without leaving the free space. The contraction can be written as a homotopy of paths:

$$H(t, s) : [0, 1] \times [0, 1] \longrightarrow \mathcal{M}^{free} \qquad (1)$$

with $H(t, 0) = \alpha(t)$ and $H(t, 1) = c$, where t is the path parameter and s is the contraction parameter.

We define the natural projection operator:

$$\begin{array}{cccc} \Pi_i & : & \mathcal{M} & \longrightarrow & \mathcal{W}_{p_i} \\ & & (p_1 \dots p_n) & \longrightarrow & p_i \end{array} \qquad (2)$$

that projects a configuration of the manipulator $c = (p_1 \ldots p_n)$ onto the ith point p_i. The following proposition characterizes the contractibility of paths in $\mathcal{M}^{free}$ in terms of the contractibility of the projected paths described by each finger in the corresponding free workspace.

Proposition 1 (Characterization of contractible paths). *A path α in $\mathcal{M}^{free}$ is contractible if and only if for each i, $\Pi_i(\alpha)$ is contractible in $\mathcal{W}_{p_i}^{free}$.*

The proof is in Appendix A. Proposition 1 provides a constructive method to build contractible curves in $\mathcal{M}^{free}$ by looking at the paths described by each individual point finger.

3.3. Caging

A *cage* is a configuration c of the manipulator that bounds the mobility of the object. In Kuperberg's original formulation of the caging problem (Kuperberg 1990), point fingers $p_1 \ldots p_n$ are interpreted as obstacles in the configuration space of the object, which are meant to bound the set of the object's free paths. In this paper, as it has been the rule in the literature on caging (Rimon and Burdick 1996; Pipattanasomporn et al. 2008; Rodriguez and Mason 2008; Vahedi and van Der Stappen 2008; Rodriguez et al. 2011), we use the equivalent and more convenient definition in terms of the mobility of the manipulator.

Given a configuration of the manipulator $c \in \mathcal{M}$, let $\mathcal{M}_c$ be the set of all configurations obtained by rigid transformations from c, i.e. the set of configurations reachable by the manipulator while moving as a rigid body. We refer to a path in $\mathcal{M}_c$ as a rigid path.

We define a cage, or a caging configuration, as follows.

Definition 1 (Cage). A cage is a configuration c of the manipulator that lies in a compact connected component of $\mathcal{M}^{free} \cap \mathcal{M}_c$.

When the manipulator cages the object, all rigid paths of the manipulator are bounded in $\mathcal{M}^{free}$. When the manipulator does not cage the object there is a rigid escaping path in $\mathcal{M}^{free} \cap \mathcal{M}_c$ that brings the manipulator arbitrarily far from $\mathcal{O}$.

The concepts of squeezing cages and stretching cages for two fingered manipulators are defined in a similar fashion. Let $D : \mathcal{M} \longrightarrow \mathbb{R}$ be the distance between the two fingers. $\mathcal{M}_c$ is the level set induced by the initial distance between the fingers:

$$\mathcal{M}_c = \{q \in \mathcal{M} \mid D(q) = D(c)\}. \tag{3}$$

Let $\mathcal{M}_c^+$ and $\mathcal{M}_c^-$ be the regions in the free space 'above' and 'below' $\mathcal{M}_c$:

$$\mathcal{M}_c^+ = \{q \in \mathcal{M} \mid D(q) \geq D(c)\} \tag{4}$$

$$\mathcal{M}_c^- = \{q \in \mathcal{M} \mid D(q) \leq D(c)\}. \tag{5}$$

Analogous to Definition 1 we define squeezing/stretching caging configurations as follows.

Definition 2 (Squeezing cage). A squeezing cage is a configuration c of the manipulator that lies in a compact connected component of $\mathcal{M}^{free} \cap \mathcal{M}_c^-$.

Definition 3 (Stretching cage). A stretching cage is a configuration c of the manipulator that lies in a compact connected component of $\mathcal{M}^{free} \cap \mathcal{M}_c^+$.

In a squeezing/stretching cage, the manipulator cannot escape arbitrarily far from the object, even when given the freedom to decrease/increase the distance between the fingers.

3.4. Grasping

One of the goals of this paper is to characterize, among the set of all cages, those that are useful waypoints to lead the way into a grasp of an object. Before describing more formally the notions of 'useful waypoint' and 'leading the way into a grasp', we detail first what we mean by grasp.

Let $c \in \mathcal{M}^{free}$ be a configuration of the manipulator in contact with the object $\mathcal{O}$. We think here of the free configuration space of the object as equivalent to the subspace of admissible rigid transformations of the manipulator $\mathcal{M}_c \cap \mathcal{M}^{free}$. Asada and By (1985) and Bicchi (1995) define the free subset of object motions, U_c^{free}, as the subset of velocities of the object in the tangent space $T\mathcal{M}_c|_c$ that do not imply any contact violation. They define a configuration to be accessible and detachable as follows.

Definition 4 (Accessible and detachable). A configuration of the manipulator c is accessible and detachable if $U_c^{free} \neq \emptyset$ and strongly accessible and detachable if $\text{int}[U_c^{free}] \neq \emptyset$.

In the context of accessibility and detachability, complete object immobilization or form closure is defined as follows.

Definition 5 (Form closure). A form closure grasp is a configuration of the manipulator $c \in \mathcal{M}^{free}$ that is not accessible and detachable, i.e. $U_c^{free} = \emptyset$.

When the manipulator holds a form closure grasp, the manipulator lies in an isolated configuration in the free configuration space, $\mathcal{M}_c \cap \mathcal{M}^{free} = \{c\}$, see the example in Figure 3(a). There is no path of the object that does not result in penetration of one of the finger constraints. Form closure is a strong condition, unattainable for certain objects with symmetries, and not always useful from a practical point of view. Similarly, we define the weaker condition weak form closure as:

Definition 6 (Weak form closure). A weak form closure grasp is a configuration of the manipulator c that is not strongly accessible and detachable, i.e. $\text{int}[U_c^{free}] = \emptyset$.

As illustrated in Figure 3(b), a weak form closure grasp does not prevent all escaping paths of the object. However, all of them imply contact with the object locally at c. Rimon and Burdick (1998) refer to those paths as first-order roll–slide motions.
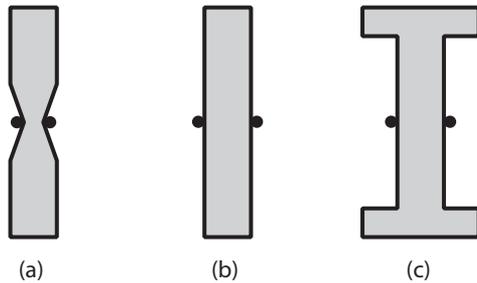
Fig. 3. Different grasp types. (a) Form closure grasp, full immobilization of an object. (b) Weak form closure grasp. The object is accessible and detachable, but not strongly accessible and detachable. The object can escape with a roll–slide motion. (c) Grasping cage. The object has some freedom, but there is no free path that detaches the object from the manipulator.

In the context of this paper, we introduce a stronger type of grasp, a condition that, from a local perspective, is equivalent to weak form closure, but at the same time voids all macroscopic escaping paths that detach the object from the manipulator. We refer to it as *grasping cage*, because it is the type of grasp that we will reach when closing into the object beginning from a cage.

Definition 7 (Grasping cage). A grasping cage is a configuration of the manipulator $c \in \mathcal{M}^{free}$ such that neither c nor any other configuration q reachable from c are not strongly accessible and detachable, i.e. $\text{int}[U_q^{free}] = \emptyset$ for all $q \in \mathcal{M}^{free} \cap \mathcal{M}_c$.

Note that, by definition, a grasping cage is always a cage. Note also that a grasping cage is not necessarily a full immobilization of the object, but is stronger than other grasp conditions in the literature such as equilibrium grasp (Rimon and Burdick 1998) or partial form closure (Bicchi 1995). Figure 3(c) shows an example.

4. From caging to grasping

In this section we put forward the idea that not all cages are equally suited to grasp an object, and introduce the tools to overcome that limitation.

4.1. Pregrasping cages

The first step is to state what it means for a cage to be 'suited' to grasp an object. In that sense, a key concept in this paper is that of a pregrasping cage.

Definition 8 (Pregrasping cage). A pregrasping cage is a cage of the manipulator for which there is a caged path to a grasping cage of the object.

Pregrasping cages are configurations of the manipulator from which the object can be grasped without ever breaking the cage. There is a grasping strategy that yields absolute certainty that the object cannot escape the manipulator.
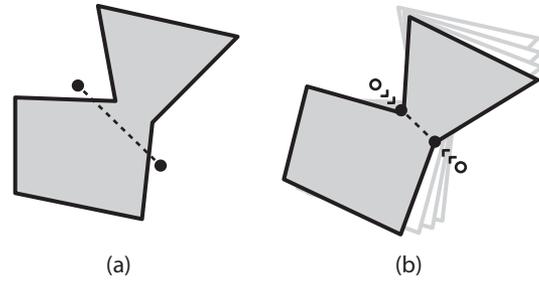

Fig. 4. (a) Pregrasping cage of a planar polygon with two fingers. (b) While moving the fingers closer together, the manipulator never breaks the cage on the object, and the pregrasping cage turns into a grasping cage.
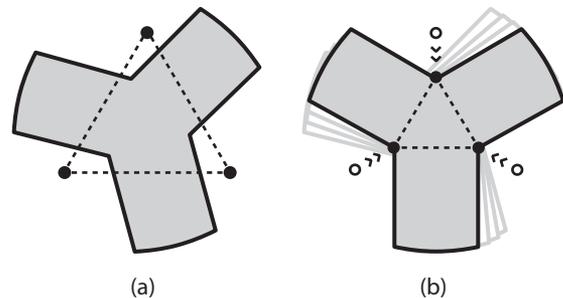

Fig. 5. (a) Multi-finger pregrasping cage. (b) The object remains caged through the process of bringing the fingers together.

Figure 4 shows a simple example of a pregrasping cage with two fingers. Beginning from a caging configuration, the manipulator closes the point fingers to reach a grasping cage without losing the cage on the object.

Thanks to the characterization of two-finger cages by Vahedi and van Der Stappen (2008), we know that all two-finger cages of polygonal objects in the plane are pregrasping cages. Because all two-finger cages of a planar polygon are either squeezing or stretching, it is inevitable that by closing (in the case of squeezing) or opening (in the case of stretching) the fingers, the manipulator will eventually grasp the object. The property, extended by Rodriguez and Mason (2008) to two-finger cages of compact connected and contractible objects in arbitrary dimension, is described in Section 5.3 as a corollary to the more general formulation provided in this paper.

The same idea can be extended to grasping with multi-fingered manipulators. Figure 5 shows an example where a set of three fingers closes to infallibly grasp an object. The manipulator cages the object throughout the grasp process.

Note, however, that there is a key difference between two- and multi-finger cages. Unlike in the case of two fingers, for multi-fingered manipulators not all cages are pregrasping cages. Figure 6 shows a cage that cannot be extended to a grasp without breaking loose of the cage. The triad of points in Figure 6(a) forms an interior cage on the object.
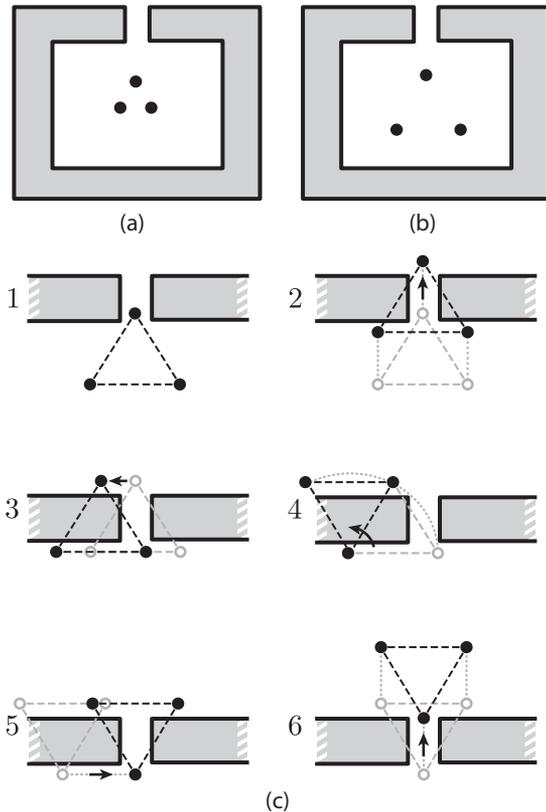
Fig. 6. (a) Example of a cage with three fingers that is not a pregrasping cage. As soon as the fingers stretch to reach a grasp they get to configurations like (b) where the object is no longer trapped. (c) Step-by-step detail of an escaping path showing that the object is not trapped in (b).

However, when the fingers open, they reach configurations as in Figure 6(b) where a slightly larger triad of points no longer forms a cage. In this paper, we develop the necessary machinery to filter out those situations.

4.2. F-caging

We are interested in identifying multi-finger pregrasping cages and characterizing their associated grasping strategies, in the same way that squeezing and stretching motions characterize the relationship between cages and grasps of two-fingered manipulators.

The conditions of cage, squeezing cage and stretching cage for the case of two fingers, as introduced in Section 3.3, are defined by the topology of the spaces $\mathcal{M}_c \cap \mathcal{M}^{free}$, $\mathcal{M}_c^- \cap \mathcal{M}^{free}$, and $\mathcal{M}_c^+ \cap \mathcal{M}^{free}$ respectively. These, in turn, are based on the function D, the distance between the fingers. Here D is a scalar function invariant with respect to rigid transformations. To generalize to the multi-finger case, analogously, we consider functions invariant to rigid transformations of the manipulator, which we refer to as shape functions.

Definition 9 (Shape function). A shape function is a semi-algebraic scalar function $F : \mathcal{M} \longrightarrow \mathbb{R}$ invariant with respect to rigid transformations of the manipulator.

An algebraic function $y(x)$ is a function that satisfies a polynomial equation $f(y) = \sum_{i=0}^{N} c_i(x) \cdot (y)^i$ whose coefficients are themselves polynomials with rational coefficients $c_i(x) = \sum_{j=0}^{N_i} k_j \cdot (x)^j$. A semialgebraic function is a function defined by a finite sequence of algebraic equations and algebraic inequalities. For example, this allow us in Section 6 to use the operators $\max[\cdot]$ and $\min[\cdot]$ as part of the definition of F.

The purpose of the semialgebraicity assumption is to avoid degeneracies in the level sets of F and simplify the derivation of the proofs in the paper, but we do not mean to imply its necessity. From a practical perspective, however, this does not impose any hard constraint, since algebraic functions are dense in the space of continuous functions (Bartle 1964), which means that we can always approximate as close as we want any continuous non-algebraic function by an algebraic one.

With the use of shape functions, and analogous to the definitions of cage, squeezing cage and stretching cage in the case of two fingers, we introduce here the concepts of F-cage, F-squeezing cage, and F-stretching cage. For that, we replace the distance between the fingers D by a generic shape function F. As in the case of two fingers, we consider the level sets of F:

$$\mathcal{M}_k = \{c \in \mathcal{M} | F(c) = k\}. \tag{6}$$

Given a shape function F, we define an F-cage as follows.

Definition 10 (F-cage). An F-cage is a configuration c of the manipulator that lies in a compact connected component of $\mathcal{M}^{free} \cap \mathcal{M}_k$, with $k = F(c)$.

The manipulator is F-caged if it cannot escape when having the freedom to move while F remains constant. As a result, F-cagedness implies the non-existence of escaping paths in $\mathcal{M}_k \cap \mathcal{M}^{free}$.

Note that for the case of two fingers, the definition of an F-cage is equivalent to the definition of a cage when $F = D$, with the implication that all two-finger cages are F-cages for one very specific shape function. On the other hand, when $n \geq 2$, the definition of F-cage is stricter than that of a cage. If $n \geq 2$ $\mathcal{M}_k$ is larger than $\mathcal{M}_c$. Consequently, more free paths need to be bounded for c to be F-caged than to be caged.

We introduce now the concepts of F-squeezing cage and F-stretching cage. Given a configuration of the manipulator c, let $\mathcal{M}_k^+$ and $\mathcal{M}_k^-$ be the sets 'above' and 'below' $\mathcal{M}_k$:

$$\mathcal{M}_k^+ = \{c \in \mathcal{M} | F(c) \geq k\}, \tag{7}$$
$$\mathcal{M}_k^- = \{c \in \mathcal{M} | F(c) \leq k\}. \tag{8}$$

Then we make the following definitions.

Definition 11 (*F*-squeezing cage). An *F*-squeezing cage is a configuration c of the manipulator that lies in a compact connected component of $\mathcal{M}^{free} \cap \mathcal{M}_k^-$, with $k = F(c)$.

Definition 12 (*F*-stretching cage). An *F*-stretching cage is a configuration c of the manipulator that lies in a compact connected component of $\mathcal{M}^{free} \cap \mathcal{M}_k^+$, with $k = F(c)$.

The manipulator is in an *F*-squeezing (*F*-stretching) caging configuration if and only if it cannot escape when given the freedom to move maintaining the value of F below (above) its initial value $F(c)$. Note again that for the case of a two-fingered manipulator, the definitions of *F*-squeezing cage and *F*-stretching cage are equivalent to the definitions of squeezing and stretching cages when $F = D$.

In the same way that squeezing and stretching cages characterize the set of all cages of a two-fingered manipulator, we show in Section 5 that we can characterize the set of all *F*-cages of an object in terms of *F*-squeezing and *F*-stretching cages.

4.3. Grasping functions

Function D has a key property that enables the relationship between two-fingered cages and grasps: for large enough and small enough values of D, the manipulator cannot cage the object. If the distance between the fingers is larger than the diameter of the object, the object can always escape. If the distance becomes zero, both point fingers are coincident, which cannot cage an object without holes.

When beginning from a squeezing or stretching caging configuration, those bounds limit the squeezing and stretching motions, which will inevitably end in a grasp. We mimic the same behavior in multi-fingered manipulators by imposing an equivalent bounding condition to general shape functions F. We name the resulting functions grasping functions.

Definition 13 (Grasping function). A grasping function is a shape function F such that $F^{-1}([-\infty, m])$ and $F^{-1}([M, \infty])$ are non-empty subsets of the non-*F*-caging region of the configuration space for some $m < M$.

If F is a grasping function, the value of F at all *F*-cages is bounded in between m and M. In Section 6 we show three examples of grasping functions with corresponding grasping cages.

We see in Section 5 that, if F is a grasping function, all cages constructed as *F*-cages are pregrasping cages, and that F controls the process to go from the cage to a grasp. Constructing cages as *F*-cages for certain grasping functions effectively filters out the ones that are not good waypoints to grasp an object, such as the counterexample in Figure 6. A consequence of that construction is that there is no grasping function F under which that counterexample is an *F*-cage.

5. The characterization of *F*-squeezing and *F*-stretching cages

We begin in Section 5.1 generalizing the theorem by Vahedi and van Der Stappen (2008) from two-finger cages of polygons in the plane to arbitrary object shapes without holes, arbitrary workspace dimension, and arbitrary number of point fingers. In Section 5.2 we analyze its implications for the existence of pregrasping cages, and finally we give in Section 5.3 a brief description of the specialization of the general theorem to the case of two fingers.

5.1. The *F*-squeezing and *F*-stretching theorem

Given a shape function F, the following theorem characterizes the set of all *F*-cages in terms of *F*-squeezing and *F*-stretching cages.

Theorem 1 (*F*-squeezing/*F*-stretching caging). *Let F be a shape function. Then, given an F-cage of a standard topological ball in $\mathbb{R}^d$ it is either an F-squeezing cage, an F-stretching cage or both.*

The detailed proof is in Appendix B. Although the details are long, the idea, illustrated in Figure 7, is simple. The proof follows the contrapositive. Suppose that a configuration c of the manipulator is neither an *F*-squeezing cage nor an *F*-stretching cage. By definition, there are *F*-squeezing and *F*-stretching escaping paths $\alpha^- \subset \mathcal{M}_k^-$ and $\alpha^+ \subset \mathcal{M}_k^+$, with $k = F(c)$, going from c to non-caging configurations c^+ and c^- arbitrarily far from c, Figure 7(a).

The key idea is to join the ends of those escaping paths in such a way that the resulting closed curve β is contractible in $\mathcal{M}^{free}$ and all of its crossings through $\mathcal{M}_k$, other than c, are non-*F*-caging configurations. We make use of the characterization of contractible paths in Proposition 1 to complete β by specifying the path of each point finger independently.

As illustrated in Figure 7(b), once β is constructed, the intersection of its contraction with $\mathcal{M}_k$ yields a path α_0 inside $\mathcal{M}_k \cap \mathcal{M}^{free}$ connecting c to a non-caging configuration c_0. This establishes the non-cagedness of c which contradicts the hypotheses and proves the theorem. From a topological point of view, we can understand the constructed escape path α_0 as an 'average' of the *F*-squeezing and *F*-stretching escaping paths.

5.2. Implications

Theorem 1 says that for any arbitrary combination of shape function F and sufficiently well-behaved object, we can decompose the set of *F*-cages as the union of all *F*-squeezing and *F*-stretching cages. This result is analogous to the two-finger counterpart described in the introduction, when F is the distance between the fingers.

In the two-finger case, the division of cages as squeezing or stretching cages has an important implication for the relationship between cages and grasps: The freedom of
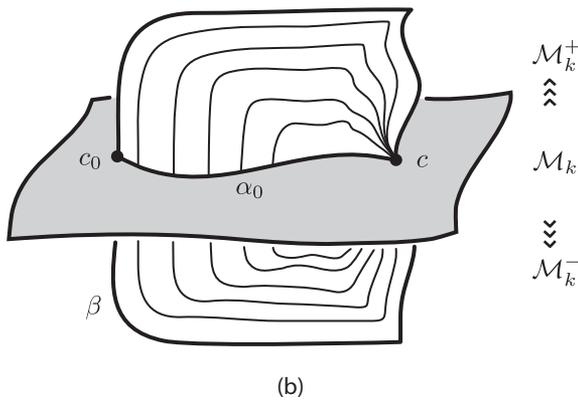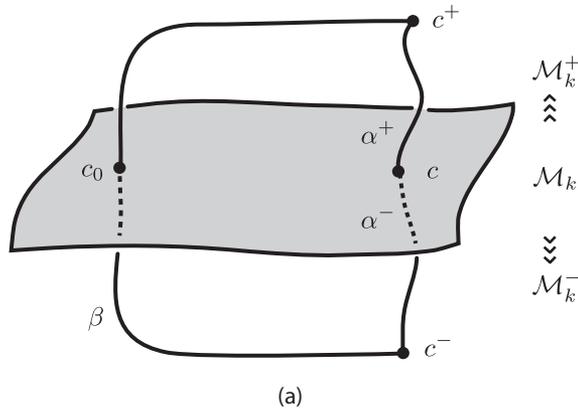
(a)

(b)

Fig. 7. Diagram of the proof of Theorem 1. (a) Escaping paths α^+ and α^- follow from the assumption that c is neither squeezing nor stretching caging. We join their ends c^+ and c^- in such a way that: The concatenation with α^+ and α^- gives a closed path contractible in $\mathcal{M}^{free}$; and c_0, the intersection with $\mathcal{M}_k$, is a non-caging configuration. (b) The intersection of the contraction of the closed path β with $\mathcal{M}_k$ gives the escaping path α_0 that connects c to a non-caging configuration.

an object in a two-finger cage can always be removed by following a squeezing or a stretching path, that is, all two-finger cages are pregrasping cages. Corollary 1 generalizes the same implication to the case of n fingers and arbitrary grasping function.

Before diving into the details of the generalization, we introduce the concept of a limit configuration of a function $F: \mathcal{M} \to \mathbb{R}$, used in the proof of Corollary 1:

Definition 14 (Limit configuration)**.** A limit configuration of F is a configuration of the manipulator c, with $F(c) = k$, for which there is an $\epsilon > 0$ such that the connected component of $F^{-1}([k - \epsilon, k + \epsilon])$ containing c is not empty and is equal to the same connected component of either $F^{-1}([k, k + \epsilon])$ or $F^{-1}([k - \epsilon, k])$.

That is, restricted to the connected component containing c, the preimage of F lies entirely to one side of the level set $F^{-1}(k)$.
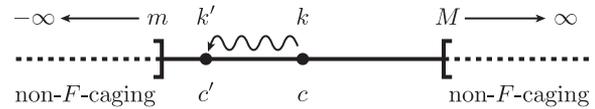

Fig. 8. Representation of the value of F in the real line $\mathbb{R}$. Because F is a grasping function there are M and m upper and lower bounds of F, such that $F^{-1}((\infty, m])$ and $F^{-1}([M, \infty))$ are all non-F-cages. We assume the initial configuration of the manipulator, c, to be an F-squeezing cage with $F(c) = k$. By definition, the manipulator remains F-caged even when the value of F decreases. This implies that all paths that decrease the value of F must be bounded.

Corollary 1 establishes the desired relationship between F-cages and pregrasping cages.

Corollary 1. *Let F be a grasping function. Then, all F-cages of a standard topological ball in $\mathbb{R}^d$ are pregrasping cages.*

Proof. Let c be an F-cage of the object with $k = F(c)$. By hypothesis, as illustrated in Figure 8, there are values m and M, with $m \leq k \leq M$, such that $F^{-1}((-\infty, m])$ and $F^{-1}([M, \infty))$ are non-empty sets containing only non-F-cages.

Since grasping functions are by definition shape functions, it follows from Theorem 1 that c must be either an F-squeezing of F-stretching cage. Without loss of generality, we assume c to be an F-squeezing cage (the derivation for the F-stretching case is analogous). The initial configuration of the manipulator is an F-cage, hence it lies in a compact connected component $\mathcal{C}$ of $\mathcal{M}^{free} \cap \mathcal{M}_k^-$. We note by $F_{\mathcal{C}}$ the restriction of F to $\mathcal{C}$. Note that $\mathcal{C}$ can be decomposed into the union of its level sets:

$$\mathcal{C} = \bigcup_{t=-\infty}^{k} \{F_{\mathcal{C}}^{-1}(t)\} = \bigcup_{t=-\infty}^{k} \mathcal{M}^{free} \cap \mathcal{M}_t \cap \mathcal{C}.$$

We will show that there is always at least one grasping cage inside $\mathcal{C}$.

Consider now the limit configurations of F and $F_{\mathcal{C}}$ as potential candidates to be grasping cages. The limit configurations of F are configurations of the manipulator where F becomes stationary. In the process of constraining F from $\mathcal{M}$ to $\mathcal{C}$ other limit configurations appear, due to the interaction between object and manipulator. We are interested in the limit configurations of $F_{\mathcal{C}}$ that are not due to stationarity of F. Note that there always must be limit configurations of this kind. The value of F at $\mathcal{C}$ would not be bounded by m otherwise, given that by hypothesis $F^{-1}(-\infty, m] \neq \emptyset$.

Consider now $c' \in \mathcal{C}$, one of those limit configurations of $F_{\mathcal{C}}$, and let $k' = F(c')$. By definition, there is $\epsilon > 0$ such that the connected component where c' lies in $F_{\mathcal{C}}^{-1}([k' - \epsilon, k' + \epsilon])$ is equal to that of $F_{\mathcal{C}}^{-1}([k' + \epsilon, t])$. This implies that the level sets $\mathcal{M}^{free} \cap \mathcal{M}_t$ restricted to $\mathcal{C}$, are empty for $k' - \epsilon \leq t < k'$ and not empty for $k' < t \leq k' + \epsilon$. Since F is semialgebraic, it is regular enough that the restriction to

$\mathcal{C}$ of the level set $t = k'$ at the limit configuration $\mathcal{M}^{free} \cap \mathcal{M}_{k'}^{-1}$ must have empty interior when considered as a subset of $\mathcal{M}_{k'}$, so that it can transition to empty when t becomes smaller than k'.

Let now $\mathcal{M}_{c'}$ be the set of configurations reachable from c' through rigid transformations of the manipulator. We shall see that $\mathcal{M}^{free} \cap \mathcal{M}_{c'}$ also has empty interior as a subset of $\mathcal{M}_{c'}$. Note that if that were to be the case, and given that the manipulator has never stopped caging the object, the limit configuration would be a grasping cage. Suppose otherwise that there is a path in $\mathcal{M}^{free} \cap \mathcal{M}_{c'}$ that detaches the manipulator from the object. Because F is a shape function, we know that $\mathcal{M}_{c'} \subset \mathcal{M}_{k'}$. Therefore, the same path that detaches manipulator from object in $M_{c'}$ also detaches the manipulator in $M_{k'}$, which contradicts the fact that the interior of $\mathcal{M}^{free} \cap \mathcal{M}_{k'}$ is empty. $\square$

If c is an F-cage, Corollary 1 shows that there is always a path inside either $\mathcal{M}^{free} \cap \mathcal{M}_k^-$ (F-squeezing cage) or $\mathcal{M}^{free} \cap \mathcal{M}_k^+$ (F-stretching cage) that begins at c and ends in a grasping cage.

In the general case, and without making any extra assumptions on F, the process to go from c to the grasping cage can be arbitrarily complicated. Limit configurations of F difficult the process of 'descending' (F-squeezing) or 'ascending' (F-stretching) into the grasp, since they manifest as local extremum to the grasp process. The next corollary notes that the process becomes trivial when F does not have limit configurations.

Corollary 2. *Let F be a grasping function, and c an F-squeezing (F-stretching) cage of a standard topological ball. Note by $\mathcal{C}$ the compact connected component of $\mathcal{M}^{free} \cap \mathcal{M}_k^-$ ($\mathcal{M}^{free} \cap \mathcal{M}_k^+$) where c lies. If F has no limit configurations in $\mathcal{C}$, all squeezing (stretching) paths of the manipulator end in a grasping cage.*

Proof. The proof follows from the construction in the proof of Corollary 1. Since F has no limit configurations, all limit configurations of $F_{\mathcal{C}}$ must be grasping cages. $\square$

As a consequence of Corollary 1, if F does not have limit configurations, we can look at F as a potential function whose gradient leads the cage to a grasp. If beginning from an F-cage, the process of increasing/decreasing the value of F allows one to blindly grasp the object, assuming we know first whether the cage is either F-squeezing or F-stretching. The following corollary shows that the conditions on Corollary 1 are met, it F is convex or strictly concave.

Corollary 3. *Let F be a convex (concave) grasping function, and c an F-squeezing (F-stretching) cage of a standard topological ball. Then, all squeezing (stretching) paths of the manipulator end in a grasping cage.*

Proof. The proof follows from Corollary 2. If F is convex, it only has one local minimum. By hypothesis, there is m

such that $F^{-1}((-\infty, m])$ is non-empty and non-F-caging. Consequently the unique local minimum of F must be outside $\mathcal{C}$, the compact connected component of $\mathcal{M}^{free} \cap \mathcal{M}_k^-$ where c lies. The derivation for the case of F concave is analogous. $\square$

5.3. *The two-finger case*

We see now that the squeezing/stretching characterization of two-finger cages follows directly from the more general Theorem 1.

Theorem 2 (Squeezing/stretching caging). *Given a two-finger cage of a standard topological ball in $\mathbb{R}^d$, it is either a squeezing cage, a stretching cage or both.*

Proof. Let F be D, the distance between the two fingers. Here D is an algebraic function invariant with respect to rigid transformations, hence a shape function. Consequently, the definitions of F-caging, F-stretching caging and F-squeezing caging are equivalent to that of caging, squeezing caging and stretching caging, and the theorem follows directly. $\square$

From Theorem 2 and Corollary 1 it follows that all two-finger cages are actually pregrasping cages.

Corollary 4. *All two-finger cages are pregrasping cages.*

Proof. We only need to check that D is a grasping function and the corollary will follow directly from Corollary 1.

When $D = 0$ both point fingers coincide and, since the object has no holes, the object cannot be trapped. We choose then $m = 0$. Since the object is a standard topological ball, it must be compact and we can always find a ball that contains it. We define M to be the diameter of such a containing ball. When the distance between the fingers is larger than M, the object will always be able to escape arbitrarily far. $\square$

Provided that D is convex, Corollary 3 implies that squeezing and stretching motions will inevitably lead a cage into a grasp. When the fingers cannot close any more (for the case of a squeezing cage) or open any more (for the case of a stretching cage), both point fingers will be in contact with the object and will form a grasping cage.

Corollary 4 reflects the asymmetry between the two- and multi-finger cases. While all two-finger cages are suitable starting points to grasp an object, not all multi-finger cages are. All two-finger cages are F-cages with respect to the grasping function D. This turns them automatically into pregrasping cages. In the general case, though, the definition of F-cage is stricter than that of a cage.

6. Examples of grasping functions

In this section we show three examples of grasping functions F and associated F-squeezing/F-stretching cages.

For F to be a grasping function, it must be a shape function, i.e. invariant with respect to rigid transformations

of the manipulator. In the following three examples F is not directly a function of the coordinates of the point fingers, but of the distances between them. This makes F automatically a shape function.

$$F_e = \frac{1}{2} \sum_{\substack{i,j\in 1...n \\ i\neq j}} d^2\left(p_i, p_j\right), \tag{9}$$

$$F_{max} = \max_{\substack{i,j\in 1...n \\ i\neq j}} d\left(p_i, p_j\right), \tag{10}$$

$$F_{min} = \min_{\substack{i,j\in 1...n \\ i\neq j}} d\left(p_i, p_j\right). \tag{11}$$

All three F_e, F_{max}, and F_{min} are semialgebraic since they can be expressed in terms of sums, multiplications, and inequalities. We finally verify that they are grasping functions by finding the upper and lower bounds on the value of F for all F-cages:

F_e If $F_e = 0$, all point fingers are coincident, thus, the manipulator neither cages the object nor F_e-cages it, since the object is contractible and does not have holes. If F_e is larger than $\frac{n \cdot (n-1)}{2}$ times the squared diameter of the object, it does not F_e-cage it either, since the manipulator is allowed to move freely while maintaining the value of F_e until the distance between every pair of points is at least the diameter of the object.

F_{max} If $F_{max} = 0$ all point fingers are coincident and the manipulator does not F_{max}-cage the object. If F_{max} is larger than the diameter of the object, all distances between the point fingers can grow freely up to the diameter while F_{max} maintains its value. Hence, the object is not F_{max}-caged either.

F_{min} If $F_{min} = 0$, all point fingers except two that coincide have complete freedom for F_{min} to maintain its value. The object is not F_{min}-caged then. If F_{min} is larger than the diameter of the object, all distances between point fingers are larger than the diameter of the object, thus, it is not F_{min}-caged either.

The set of F-cages induced by a specific grasping function, and therefore the induced set of pregrasping cages, depends on the selection of F. Figure 9 shows examples of F-cages with F_e, F_{max}, and F_{min}.

We call F_e the energy of the finger formation because of the resemblance to the form of the elastic energy that would be stored by springs connecting the point fingers. We maintain an F_e-cage over an object by controlling that energy level. Figure 9(a) shows an example of an object that requires more than two fingers to be caged. If the initial value of F_e is smaller than a critical value ($F_e < 2d_c^2$), the configuration is an F_e-squeezing cage. We descend from the initial cage to a grasp of the object simply by decreasing the value of F_e.

Figure 9(b) shows an example of an F_{max}-squeezing cage of a sphere with four fingers, as long as at the initial configuration F_{max} is smaller than the diameter of the sphere d_c.
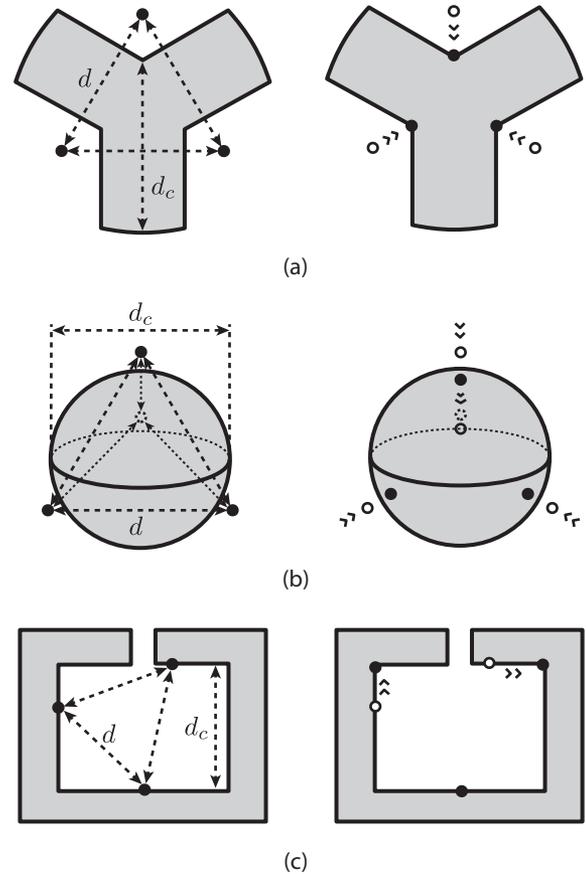

Fig. 9. Three examples of F-cages and associated grasping cages. (a) The manipulator is in an F_e-squeezing cage if $F_e < 2d_c^2$. (b) The manipulator is in an F_{max}-squeezing cage if $F_{max} < d_c$. (c) The manipulator is in an F_{min}-stretching cage if $F_{min} > d_c$. Setting that threshold avoids the counterexample in Section 4.1.

We descend from the cage to a grasping cage by decreasing the value of F_{max}. The minimum value of F_{max} when we reach the grasp, corresponds to the side length of the inscribed regular tetrahedron $\sqrt{\frac{2}{3}}d_c$. Note that F_{max} is not differentiable everywhere, which prevents us from directly following the gradient to reach a grasp. However, we can use instead an appropriate subgradient and still converge to the minimum of F_{max}.

Figure 9(c) illustrates the use of F_{min} in addressing the counterexample in Section 4.1. Beginning from the configuration at the left in Figure 9(c), all configurations with $F_{min} > d_c$ are F_{min}-stretching cages, and consequently pregrasping cages. From one of those configurations we can blindly increase the value of F_{min} to reach a grasp of the object.

Here F_{max} and F_{min} are well suited to be controlled in a decentralized manner. They resemble the conservative decentralized strategies used by Pereira et al. (2004) to cage an object. They derive caging guarantees by keeping the distances between independent mobile robots below a certain

safety threshold. We can think about that metric as a grasping function similar to F_{max}, with the only difference that they only keep track of distances between adjacent robots in a circular formation.

Guaranteeing caging by keeping track of the values of F_e and F_{max} is also very similar to the concepts of dispersion and concentration proposed by Pipattanasomporn et al. (2008) for caging rigid polytopes. They also consider only distances between adjacent robots in a circular formation.

7. Discussion

This paper begins from one key observation: that some cages are especially suited as waypoints to grasp an object. We coin the term *pregrasping cage* to name those cages that yield a caged path to a grasp of the object, and *grasping cage* to the type of grasp we expect to reach at the end of that path. A grasping cage is not necessarily a full immobilization, but more constraining than an equilibrium grasp or a partial form closure grasp.

We generalize the strategy used in two-fingered manipulators to characterize cages as squeezing or stretching cages, to general object shapes, workspace dimension, and number of point fingers. More formally, given a standard topological ball of arbitrary dimension and a *grasping function F*, there is a set of configurations of the manipulator, referred to as F-cages, for which:

i   The object cannot escape the manipulator as long as the value of F is maintained.

ii There is a path to a grasping cage of the object that does not require breaking the cage. If F is not stationary in the relevant F-caging region of the configuration space, e.g. if F is convex or concave, the grasp process is blind and can be controlled by F. Knowledge of the location of the object is not required to guarantee reaching a grasping cage.

This gives rise to the concepts of F-cage, F-squeezing cage, and F-stretching cage, configurations of the manipulator that trap the object even when given the freedom to move when maintaining, decreasing or increasing the value of F correspondingly. These generalize the concepts of cage, squeezing cage, and stretching cage for the case of two fingers. In this paper we show that all F-cages of a standard topological ball are either F-squeezing cages, F-stretching cages, or both. A key consequence is that, if F is a grasping function, all F-cages are pregrasping cages, and therefore are useful waypoints to grasp an object.

The following diagram summarizes the relationship between cages, pregrasping cages and F-cages:

$$F\text{-cages} \subseteq \text{Pregrasping cages} \subset \text{Cages}.$$

The set of cages that ease the grasping problem is a proper subset of the set of all cages. For any given choice of grasping function F, all F-cages are pregrasping cages. Given any configuration c of the manipulator, the existence of a grasping function under which c is an F-cage, directly implies that c is a pregrasping cage. It is important to note, however, that with a specific choice of grasping function F, we only recover a portion of all pregrasping cages. The question of whether the union of all F-cages induced by all possible choices of grasping function covers the entire set of pregrasping cages remains open.

The results in this paper admit two interpretations. From a synthesis point of view, the interest is in finding pregrasping cages of an object. Corollary 1 says that we can choose a particular grasping function F and constrain the search problem to that of finding F-cages. By virtue of Theorem 1, this is equivalent to searching for F-squeezing and F-stretching cages. The same strategy has been shown to be computational advantageous for the two-finger case (Vahedi and van Der Stappen 2008).

From an analysis point of view, the interest is in deciding whether a particular cage is a pregrasping cage or not. Corollary 1 tells us that we can prove a particular cage to be a pregrasping cage by finding a grasping function F under which is also an F-cage.

In that sense, we can understand the role played by grasping functions in grasping as analogous to that played by Lyapunov functions in stability analysis. We prove that a cage can be driven into a grasp the same way one proves that a system is stable: by finding an appropriate scalar function that bounds the freedom of the system. If the manipulator is in a pregrasping cage and we want to grasp the object, we can control the value of that bounding function by leading the manipulator to an extremum of F, which we can call a grasp.

References

Asada H and By AB (1985) Kinematic analysis of workpart fixturing for flexible assembly with automatically reconfigurable fixtures. *IEEE Journal of Robotics and Automation* 1: 86–94.

Bartle RG (1964) Sequences of continuous functions. In *The Elements of Real Analysis*. New York: John Wiley & Sons, p. 183.

Besicovitch A (1957) A net to hold a sphere. *The Mathematical Gazette* 41: 106–107.

Besikovitch A (1961) A cage to hold a unit-sphere. In *7th Symposium in Pure Mathematics*, pp. 19–20.

Bicchi A (1995) On the closure properties of robotic grasping. *The International Journal of Robotics Research* 14: 319–334.

Blind SJ, McCullough CC, Akella S and Ponce J (2001) Manipulating parts with an array of pins: a method and a machine. *The International Journal of Robotics Research* 20: 808–818.

Davidson C and Blake A (1998a) Caging planar objects with a three-finger one-parameter gripper. In *IEEE International Conference on Robotics and Automation (ICRA)*, pp. 2722–2727.

Davidson C and Blake A (1998b) Error-tolerant visual planning of planar grasps. In *International Conference on Computer Vision (ICCV)*, pp. 911–916.

Diankov R, Srinivasa SS, Ferguson D and Kuffner J (2008) Manipulation planning with caging grasps. In *IEEE International Conference on Humanoid Robots*.

Fink J, Hsieh MA and Kumar V (2008) Multi-robot manipulation via caging in environments with obstacles. In *IEEE International Conference on Robotics and Automation (ICRA)*. Piscataway, NJ: IEEE, pp. 1471–1476.

Gopalakrishnan KG and Goldberg K (2002) Gripping parts at concave vertices. In *IEEE International Conference on Robotics and Automation (ICRA)*, pp. 1590–1596.

Kuperberg W (1990) Problems on polytopes and convex sets. In *DIMACS: Workshop on polytopes*, pp. 584–589.

Lee JM (2000) *Introduction to Topological Manifolds*. New York: Springer.

Lee JM (2002) *Introduction to Smooth Manifolds*. New York: Springer.

Milnor JW (1965) *Topology from the Differentiable Viewpoint*. Princeton, NJ: Princeton University Press.

Pereira G, Campos M and Kumar V (2004) Decentralized algorithms for multi-robot manipulation via caging. *The International Journal of Robotics Research* 23: 783–795.

Pipattanasomporn P and Sudsang A (2006) Two-finger caging of concave polygons. In *IEEE International Conference on Robotics and Automation (ICRA)*. Piscataway, NJ: IEEE, pp. 2137–2142.

Pipattanasomporn P and Sudsang A (2011) Two-finger caging of nonconvex polytopes. *IEEE Transactions on Robotics* 27: 324–333.

Pipattanasomporn P, Vongmasa P and Sudsang A (2008) Caging rigid polytopes via finger dispersion control. In *IEEE International Conference on Robotics and Automation (ICRA)*. Piscataway, NJ: IEEE, pp. 1181–1186.

Rimon E and Blake A (1999) Caging planar bodies by one-parameter two-fingered gripping systems. *The International Journal of Robotics Research* 18: 299–318.

Rimon E and Burdick JW (1996) On force and form closure for multiple finger grasps. In *IEEE International Conference on Robotics and Automation (ICRA)*. Piscataway, NJ: IEEE, pp. 1795–1800.

Rimon E and Burdick JW (1998) Mobility of bodies in contact. I. A 2nd-order mobility index for multiple-finger grasps. *IEEE Transactions on Robotics and Automation* 14: 696–708.

Rodriguez A and Mason MT (2008) Two finger caging: squeezing and stretching. In *Algorithmic Foundations of Robotics (WAFR)*. New York: Springer.

Rodriguez A and Mason MT (2011) Path-connectivity of the free space. *IEEE Transactions on Robotics*, conditionally accepted.

Rodriguez A, Mason MT and Ferry S (2011) From caging to grasping. In *Robotics: Science and Systems (RSS)*.

Sudsang A and Ponce J (2000) A new approach to motion planning for disc-shaped robots manipulating a polygonal object in the plane. In *IEEE International Conference on Robotics and Automation (ICRA)*, pp. 1068–1075.

Sudsang A, Ponce J and Srinivasa N (1998) Grasping and in-hand manipulation: experiments with a reconfigurable gripper. *Advanced Robotics* 12: 509–533.

Sudsang A, Ponce J and Srinivasa N (1999) Grasping and in-hand manipulation: geometry and algorithms. *Algorithmica* 26: 466–493.

Vahedi M and van Der Stappen AF (2008) Caging polygons with two and three fingers. *The International Journal of Robotics Research* 27: 1308–1324.

Appendix A: Contractible paths

Proposition (Characterization of contractible paths)**.** *A path α in $\mathcal{M}^{free}$ is contractible if and only if for each i, $\Pi_i(\alpha)$ is contractible in $\mathcal{W}_{p_i}^{free}$.*

Proof. The proposition is a direct consequence of $\mathcal{M}$ being constructed as the Cartesian product of the configuration spaces of each point finger, $\mathcal{M} = \mathcal{W}_{p_1} \times \ldots \times \mathcal{W}_{p_n}$. We prove both directions of the implication.

[$\Rightarrow$] Suppose that α is contractible in $\mathcal{M}^{free}$. Let $H(t, s)$ be the corresponding path homotopy. The natural projections of α are the closed curves $\alpha_i = \Pi_i(\alpha) \subset \mathcal{W}_{p_i}^{free}$. We need to show that α_i is contractible in $\mathcal{W}_{p_i}^{free}$ for all i. Consider then the natural projections of the homotopy $H(s, t)$:

$$
H_i : \quad \begin{array}{ccccc} [0,1]^2 & \overset{H}{\to} & \mathcal{M}^{free} & \overset{\Pi_i}{\to} & \mathcal{W}_{p_i}^{free} \\ (t, s) & \to & H(t, s) & \to & \Pi_i(H(t, s)). \end{array}
$$

Here H_i is the composition of continuous maps, therefore it is continuous; H_i is also a homotopy of paths since:

$$
\begin{aligned}
H_i(t, 0) &= \Pi_i(H(t, 0)) = \Pi_i(\alpha(t)) = \alpha_i(t) \\
H_i(t, 1) &= \Pi_i(H(t, 1)) = \Pi_i(c).
\end{aligned}
$$

We conclude that α_i is contractible to $\Pi_i(c)$ in $\mathcal{W}_{p_i}^{free}$.

[$\Leftarrow$] Suppose that each natural projection $\alpha_i = \Pi_i(\alpha)$ is contractible in its corresponding free workspace $\mathcal{W}_{p_i}^{free}$. Let $H_i(t, s)$, defined inside $\mathcal{W}_{p_i}^{free}$, be the corresponding path homotopy. Consider now the combined path in $\mathcal{M}^{free}$, $\alpha = (\alpha_1 \ldots \alpha_n)$, which maps into $\mathcal{M}^{free}$ by construction. Consider also the combined map $H = (H_1(t, s) \ldots H_n(t, s))$,

which by construction maps into $\mathcal{M}^{free}$. We show now that H is a homotopy of paths for α:

$$H(t,0) = (\alpha_1(t)\ldots\alpha_n(t)) = \alpha(t),$$
$$H(t,1) = (\alpha_1(0)\ldots\alpha_n(0)) = c,$$

which means that α is contractible to c in $\mathcal{M}^{free}$. □

Appendix B: *F*-squeezing and *F*-stretching theorem

Theorem (*F*-squeezing/*F*-stretching caging)**.** *Let F be a shape function. Then, given an F-cage of a standard topological ball in $\mathbb{R}^d$ it is either an F-squeezing cage, an F-stretching cage or both.*

Proof. The proof follows the contrapositive. Suppose that a certain configuration c of the manipulator is neither an F-squeezing cage nor an F-stretching cage. This implies the existence of F-squeezing and F-stretching escaping paths $\alpha^- \subset \mathcal{M}_k^-$ and $\alpha^+ \subset \mathcal{M}_k^+$, with $k = F(c)$, that go from c to non-caging configurations c^+ and c^- arbitrarily far from c. We use the existence of these two escape paths to construct a third escape path, this time in $\mathcal{M}_k$, establishing the non-F-cagedness of c. The proof consists of two steps, both illustrated in Figure 7, that we address in following subsections:

1. Use α^+ and α^- to build a closed contractible curve in $\mathcal{M}^{free}$. The contractible curve will necessarily cross $\mathcal{M}_k$. We want those crossings not to be F-caging configurations.
2. Construct a path inside $\mathcal{M}_k$ from c to one of such crossings by intersecting the contraction of the curve with $\mathcal{M}_k$. By construction, the contraction of the closed curve and, in particular, its intersection with $\mathcal{M}_k$ must live in $\mathcal{M}^{free}$. This establishes the non-F-cagedness of c and concludes the proof. □

B.1. Construction of the contractible curve

The contractible curve must satisfy two requirements:

- All crossings of the curve through $\mathcal{M}_k$, except for c, must be non-F-caging.
- Both the curve and its contraction must live in $\mathcal{M}^{free}$.

Let $B \subset \mathbb{R}^d$ be a ball containing O such that any placement of the point fingers outside B is a non-F-caging configuration of the manipulator. We say that any configuration with the point fingers outside B is at infinity. Note that, since $\mathcal{O}$ is compact, B must exist. If the manipulator has n fingers, we can choose B to be a ball containing $\mathcal{O}$ with diameter $d_B > \frac{n \cdot d_{\mathcal{O}}}{\pi}$, where $d_{\mathcal{O}}$ is the diameter of the object. All configurations of the manipulator with all fingers in the exterior of B are non-cages, and consequently also non-F-cages. We will assume then the escaping paths α^+ and α^- terminate at configurations c^+ and c^- at infinity, respectively.
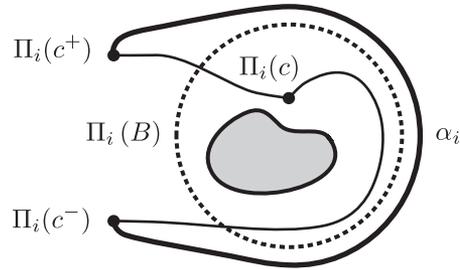

Fig. 10. Completion of the contractible curve in the workspace of point finger p_i. In the case of dimension 2 we choose α_i outside the ball $\Pi_i(B)$ such that the resultant winding number of the closed curve formed by the concatenation of $\Pi_i(\alpha^+)$, α_i, and $\Pi_i(\alpha^-)$ is zero.

We construct the contractible path by closing the concatenation of α^+ and α^- outside B, which will guarantee that all crossings through $\mathcal{M}_k$ will correspond to non-F-caging configurations. Following Proposition 1, if we want to construct a contractible curve in $\mathcal{M}^{free}$, we can do it independently for each point finger. We illustrate in Figure 10 the process of completing the path for an individual finger.

Each point finger $p_i^+ = \Pi_i(c^+)$ must be joined with the corresponding point finger $p_i^- = \Pi_i(c^-)$ outside B and in a contractible way. If the dimension of the workspace is $d = 2$, as illustrated in Figure 10, the completion path goes along the boundary of B enough times to undo the winding number of the concatenation $\Pi_i(\alpha^+ \oplus \alpha^-)$. If $d > 2$, since the object is a standard topological ball, the exterior of the object must be homeomorphic to the exterior of an Euclidean ball, and in particular must be simply connected. Consequently, we can always connect the ends of the escaping paths such that the resultant closed curve is contractible. We will call α to such a completion curve and β to the resultant contractible curve $\alpha^+ \oplus \alpha \oplus \alpha^-$.

B.2. Intersection of the contraction with $\mathcal{M}_k$

The contractibility of β implies the existence of an homotopy $H : [0,1] \times [0,1] \longrightarrow \mathcal{M}^{free}$ that contracts the closed path to c. We want to construct the intersection of $H(S)$ with $\mathcal{M}_k$. We show that H can always be approximated by a well-behaved contraction that yields a non-degenerate intersection.

The boundary of the square $S = [0,1] \times [0,1]$ maps into the curve β, with the inconvenience that three of the sides of the square map to c. To avoid this, and simplify posterior derivations, we consider the quotient map π that identifies them into a single point in the boundary of S. The map π transforms S into a disc D whose boundary has a one to one mapping onto the contractible curve β.

The characteristic properties of the quotient topology (Lee 2000), expressed in Lemma 1 induce a unique continuous map $\tilde{H} : D \longrightarrow \mathcal{M}^{free}$ that commutes the diagram in Figure 11.
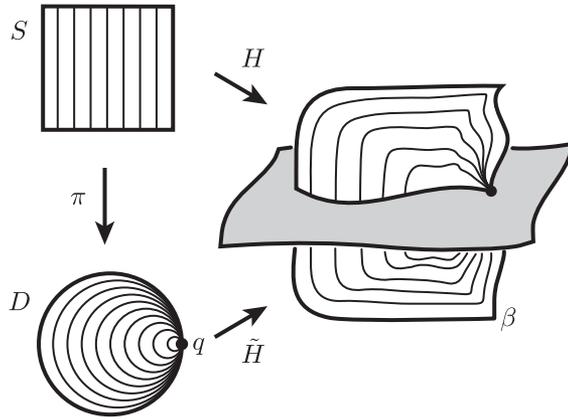
Fig. 11. Construction of the homotopy $\tilde{H}$ induced by the quotient map. Note that $\tilde{H}(q) = c$.

Lemma 1 (Passing to the quotient). *Suppose that $\pi : X \longrightarrow Y$ is a quotient map, B is a topological space, and $f : X \longrightarrow B$ is any continuous map that is constant on the fibers of π (i.e. if $\pi(p) = \pi(q)$, then $f(p) = f(q)$). Then there exists a unique continuous map $\tilde{f} : Y \longrightarrow B$ such that $f = \tilde{f} \circ \pi$:*

$$\begin{array}{ccc} X & & \\ \pi \downarrow & \searrow^{f} & \\ Y & \xrightarrow{\tilde{f}} & B \end{array}$$

From now on when mentioning the contraction we will refer to the quotient version $\tilde{H}$ of the homotopy. We call q the point in the boundary of D that maps into c.

The path that allows c to escape inside $\mathcal{M}_k$ lives in $\tilde{H}(D) \cap \mathcal{M}_k$. The construction of that intersection relies on Lemma 2, borrowed from Milnor (1965).

Lemma 2 (Milnor's lemma). *Let M be an m-dimensional manifold and N an n-dimensional manifold, with $m \geq n$. If $f : M \longrightarrow N$ is smooth, and if $y \in N$ is a regular value, then the set $f^{-1}(y) \subset M$ is a smooth manifold of dimension $m - n$.*

If M is a manifold with boundary ∂M and y is also regular for the restriction $f \mid \partial M$, then $f^{-1}(y)$ is a smooth $(m - n)$ manifold with boundary. Furthermore, the boundary $\partial\left(f^{-1}(y)\right)$ is precisely equal to the intersection of $f^{-1}(y)$ with ∂M.

Let f be the composition $f : D \xrightarrow{\tilde{H}} M^{\text{free}} \xrightarrow{F} \mathbb{R}$. Then $f^{-1}(k)$, as a subset of D, is mapped by $\tilde{H}$ into the mentioned intersection $\tilde{H}(D) \cap \mathcal{M}_k$. Proving the existence of the escaping path in $\mathcal{M}_k$ is equivalent then to showing the existence of a path in $f^{-1}(k)$ from q to another point q_0 in the boundary of D. By construction, $\tilde{H}$ maps that path into an escaping path in $\mathcal{M}_k$ to a non-F-caging configuration.

If f is smooth and k is a regular value, the application of Milnor's Lemma says that $f^{-1}(k)$ is a one dimensional

smooth manifold with $f^{-1}(k) \cap \partial D$ as boundary. As a consequence, q is connected through a continuous and smooth path to a different point in ∂D within the level set $f^{-1}(k)$, which finishes the proof. All that remains to see is what happens when f is not smooth or k is not a regular value of f.

Smoothness of f In order to grant the smoothness of f we ϵ-approximate it. This approximation needs to guarantee that the contraction remains in $\mathcal{M}^{\text{free}}$ and that all crossings of the contractible path β through $\mathcal{M}_k$ still remain non-F-caging. We use Whitney's approximation theorem (Lee 2002) to find smooth approximations both of the curve β, and the contraction that satisfy both constraints.

Lemma 3 (Whitney approximation theorem). *Let M be a smooth manifold and let $F : M \longrightarrow \mathbb{R}^k$ be a continuous function. Given any positive continuous function $\epsilon : M \longrightarrow \mathbb{R}$, there is a smooth function $\hat{F} : M \longrightarrow \mathbb{R}^k$ that is ϵ-close to F ($\|F(x) - \hat{F}(x)\| < \epsilon(x) \; \forall x \in M$). If F is smooth on a closed subset $A \subset M$, then $\hat{F}$ can be chosen to be equal to F in A.*

The challenge is to preserve two key properties of $\tilde{H}$ when we construct the approximation:

- All crossings of the boundary of $\tilde{H}(D)$ through $\mathcal{M}_k$, other than c, must remain non-F-caging.
- The approximation of $\tilde{H}$ must still live in $\mathcal{M}^{\text{free}}$.

The first property was originally satisfied by making the contractible curve go to infinity before crossing $\mathcal{M}_k$. Spurious crossings through $\mathcal{M}_k$ due to the approximation must be prevented. This is especially awkward near the crossing at c. No matter how small an ϵ we choose, the ϵ-approximations of β and $\tilde{H}$ could cross again. Similarly, where $\tilde{H}$ makes contact with the obstacle, the ϵ-approximation might violate the second property by leaving the free space $\mathcal{M}^{\text{free}}$. The following procedure produces a smooth approximation while avoiding the problems:

1. Replace the contractible curve β locally at c by a smooth patch in $\mathcal{M}^{\text{free}}$, as in Figure 12. The free space is defined as $\mathcal{M}^{\text{free}} = \mathcal{M} \setminus \text{int}\left[\mathcal{O}^{\mathcal{M}}\right]$. Rodriguez and Mason (2011) show that this definition correctly regularizes the space, ensuring the free space has no thin bits. Even if c is in contact with $\mathcal{O}^{\mathcal{M}}$, there is still freedom to smoothly escape the contact through half of the directions on the tangent space.
2. Apply a similar patch wherever the contractible curve contacts the object. Thanks to the regularity of the free space, Proposition 3 of Rodriguez and Mason (2011) guarantees that these contact points are isolated configurations.
3. Apply the Whitney approximation theorem to approximate the contractible curve by a smooth curve, equal to the original curve on the patches.
4. Define the contraction $\tilde{H}$ as before, but using the smoothed contractible curve.
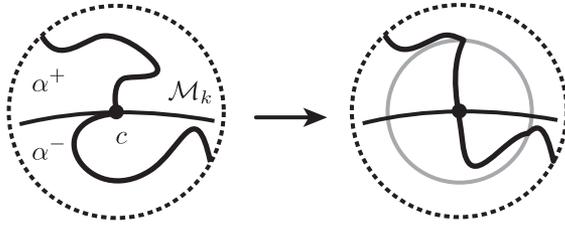
Fig. 12. Smooth patch for replacing the contractible curve in a neighborhood of c and eliminate possible non-smoothness at the crossing with $\mathcal{M}_k$.

5. If the contraction makes contact with the object, the approximation could violate the second key property. Repeat the previous strategy of defining smooth patches.

6. Apply Whitney approximation theorem once more to approximate the contraction by a smooth map $\hat{H}$ that is equal to the original contraction on the closed set $\partial D \subset D$ and on any patches, and otherwise lives in the free space $\mathcal{M}^{free}$. This guarantees that the boundary of the contraction will map to the approximated contractible curve.

Regularity of k If k is not a regular value of f, $f^{-1}(k)$ is not necessarily a manifold and Milnor's lemma cannot be used to build the intersection of the contractible curve and $\mathcal{M}_k$. We see that, even if that is the case, if the level set $f^{-1}(k)$ is well behaved, the escape path connecting q with a boundary point of $f^{-1}(k)$ still exists.

We make use of Sard's theorem (Milnor 1965) which characterizes the critical points of smooth functions.

Lemma 4 (Sard's theorem). *Let $f : U \longrightarrow \mathbb{R}^p$ be a smooth map, with U open in R^n and let C be the set of critical points; that is, the set of all $x \in U$ with $\operatorname{rank} df_x < p$. Then $f(C) \subset \mathbb{R}^p$ has measure zero.*

The Sard's theorem says that the set of regular values is dense on the image of f. Consequently, if k is a critical value, there is a monotonic sequence of regular values $\{k_n\}_n$, converging to k.
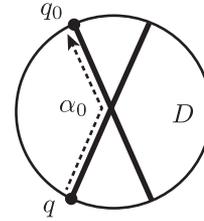
(a)
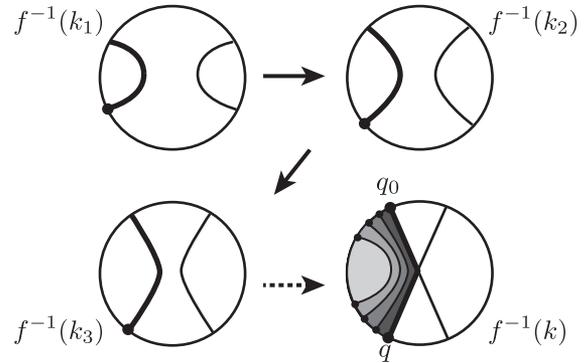
(b)

Fig. 13. (a) When k is a critical value, $f^{-1}(k) \subset D$ might not be a smooth manifold. Still q is path-connected to a point $q_0 \in \partial D$ mapped by $\tilde{H}$ to a non-caging configuration. (b) To prove it we consider the limit of a series of level sets $\left\{f^{-1}(k_n)\right\}_n$ of regular points where the path is guaranteed.

By virtue of Milnor's lemma, $\left\{f^{-1}(k_n)\right\}_n$ is a series of smooth one-dimensional manifolds. This level set induces a sequence of smooth paths that gradually approach the level set $f^{-1}(k)$, as illustrated in Figure 13. In general, the limit of a sequence of continuous paths does not necessarily converge to a continuous path but, if we assume that we constructed the smooth approximation of the contraction as a semialgebraic approximation, the function $f = F \circ \tilde{H}$, level sets $\left\{f^{-1}(k_n)\right\}_n$ and $f^{-1}(k)$ will also be semialgebraic. Consequently, the approximating paths and the limit, that lives in a semialgebraic set, must also be continuous and semialgebraic. Which concludes the proof.   □