Kinematic foundations.

We will focus on rigid motions in

- the Euclidean plane ($\mathbb{E}^2$)
- Euclidean three space ($\mathbb{E}^3$)
- the sphere ($\mathbb{S}^2$)

Why the sphere? Rigid motions of the sphere correspond to rotations about a given point in $\mathbb{E}^3$.

Kinematics foundations: some definitions

First, some general definitions. Let $\mathbb{X}$ be the *ambient space*, either $\mathbb{E}^2$, $\mathbb{E}^3$, or $\mathbb{S}^2$.

- A *system* is a set of points in the space $\mathbb{X}$.
- A *configuration* of a system is the location of every point in the system.
- *Configuration space* is a metric space comprising all configurations of a given system.
  (What kind of space is configuration space? Devise a metric.)
  (Note: Every metric for cspace is sort of defective.)
- The *degrees of freedom* of a system is the dimension of the configuration space. (A less precise but roughly equivalent definition: the minimum number of real numbers required to specify a configuration.)

Kinematics foundations: systems, DOFs

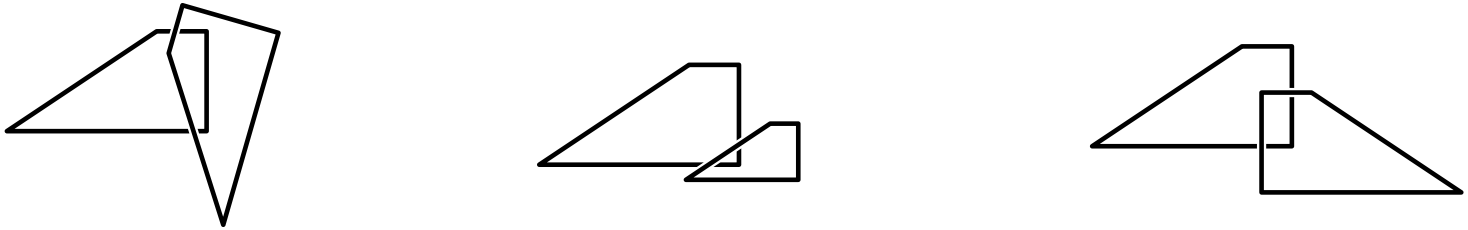
System	**Configuration**	**DOFs**
point in plane	x, y	2
point in space	x, y, z	3
rigid body in plane	x, y, θ	3
rigid body in space	x, y, z, ϕ, θ, ψ	6

Kinematics foundations: rigid bodies, displaceme

Definitions:

A *displacement* is a change of configuration that does not change the distance between any pair of points, nor does it change the handedness of the system.

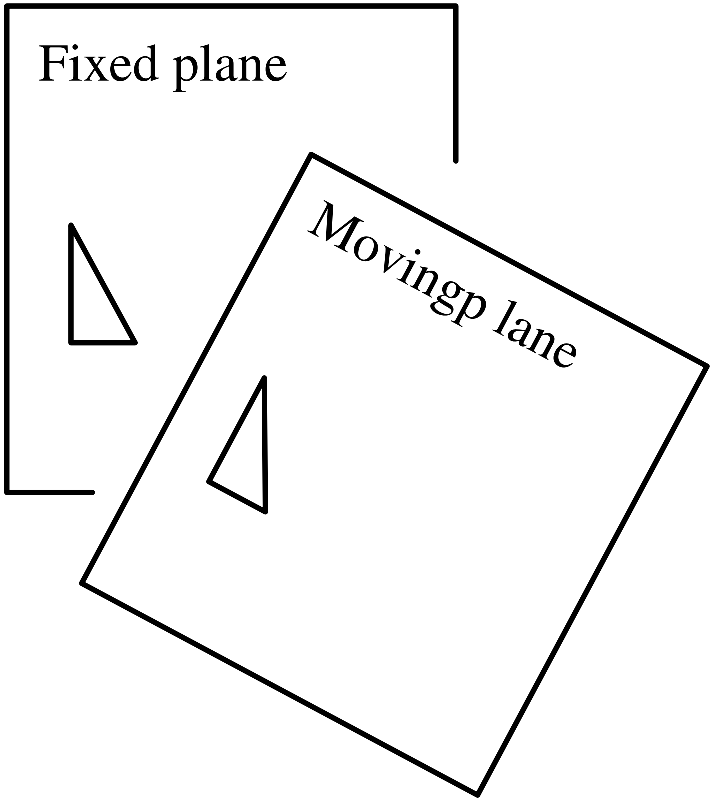
A *rigid body* is a system that is capable of displacements only.



Transformations, rigid and otherwise.

Kinematics foundations: moving and fixed planes

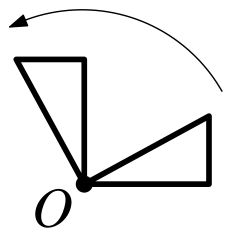
We will consider displacements to apply to *every* point in the ambient space. E.g., displacements are described as motion of *moving* plane relative to *fixed* plane.



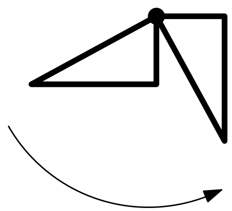
Moving and fixed planes.

Kinematics foundations: rotations and translatio

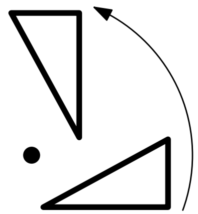
A *rotation* is a displacement that leaves at least one point fixed.
A *translation* is a displacement for which all points move equal distances along parallel lines.



Rotation about O



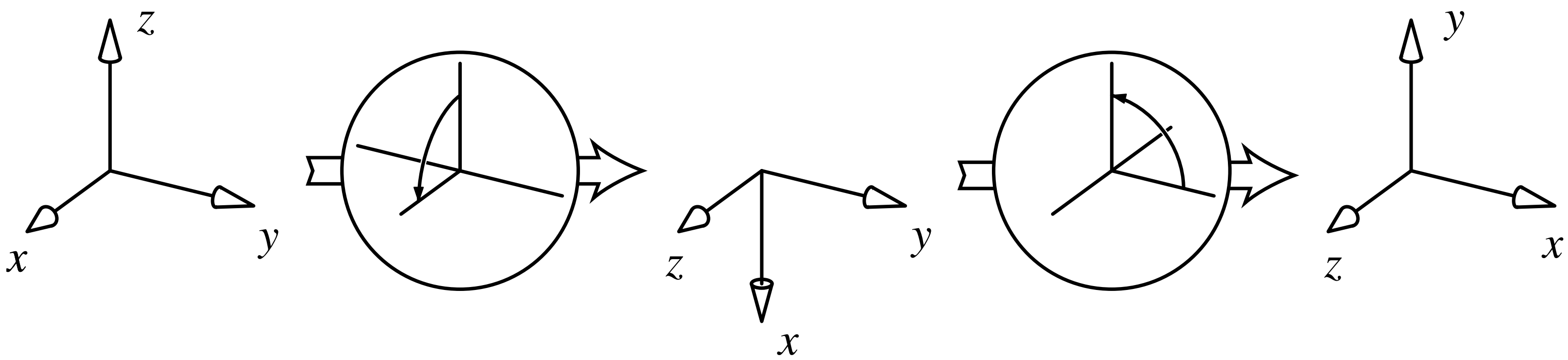
Rotation about a point on the body

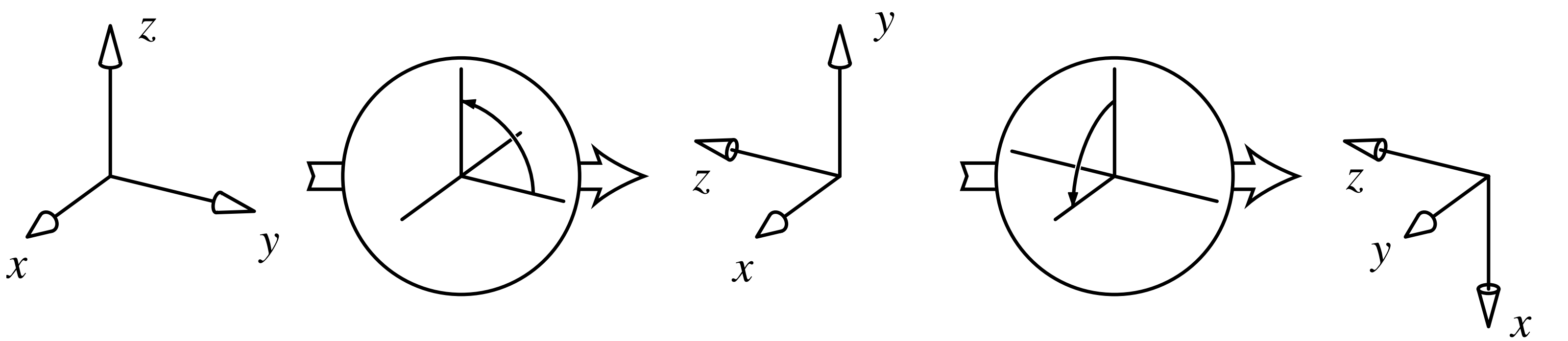


Rotation about a point not on the body

Kinematics foundations: do displacements comm

Does $\mathbf{SO}(3)$ commute? **NO!** No, no, no. (If you have found a commutative way of representing spatial rotations, you are confused.)





The projective plane.

The basic idea:

Start with the Euclidean plane $\mathbb{E}^2$.

Add some points, the **ideal points** or the **points at infinity**.

Call the new structure the **projective plane**—$\mathbb{P}^2$.

You can do it formally by defining an ideal point for each set of parallel lines, but we will employ a more concrete method using **homogeneous coordinates**.

Homogeneous coordinates.

Let the Cartesian coordinates of some point in $\mathbb{E}^2$ be

$$(\eta, \nu)$$

Then we will say that

$$(x, y, w) \triangleq (w\eta, w\nu, w)$$

are the **homogeneous coordinates** of the point, **provided**

$$w \neq 0$$

To go from homogeneous to Cartesian:

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} \mapsto \begin{pmatrix} x/w \\ y/w \end{pmatrix}, w \neq 0 \tag{1}$$

Point in $\mathbb{E}^2$ versus line through origin of $\mathbb{E}^3$

Scaling the homogeneous coordinates does **not** change the point!

$$\begin{pmatrix} ax \\ ay \\ aw \end{pmatrix} \mapsto \begin{pmatrix} ax/aw \\ ay/aw \end{pmatrix} = \begin{pmatrix} x/w \\ y/w \end{pmatrix}, a, w \neq 0 \qquad (2)$$

So, homogeneous coordinates represent a point in $\mathbb{E}^2$ by a line through the origin of $\mathbb{E}^3$.

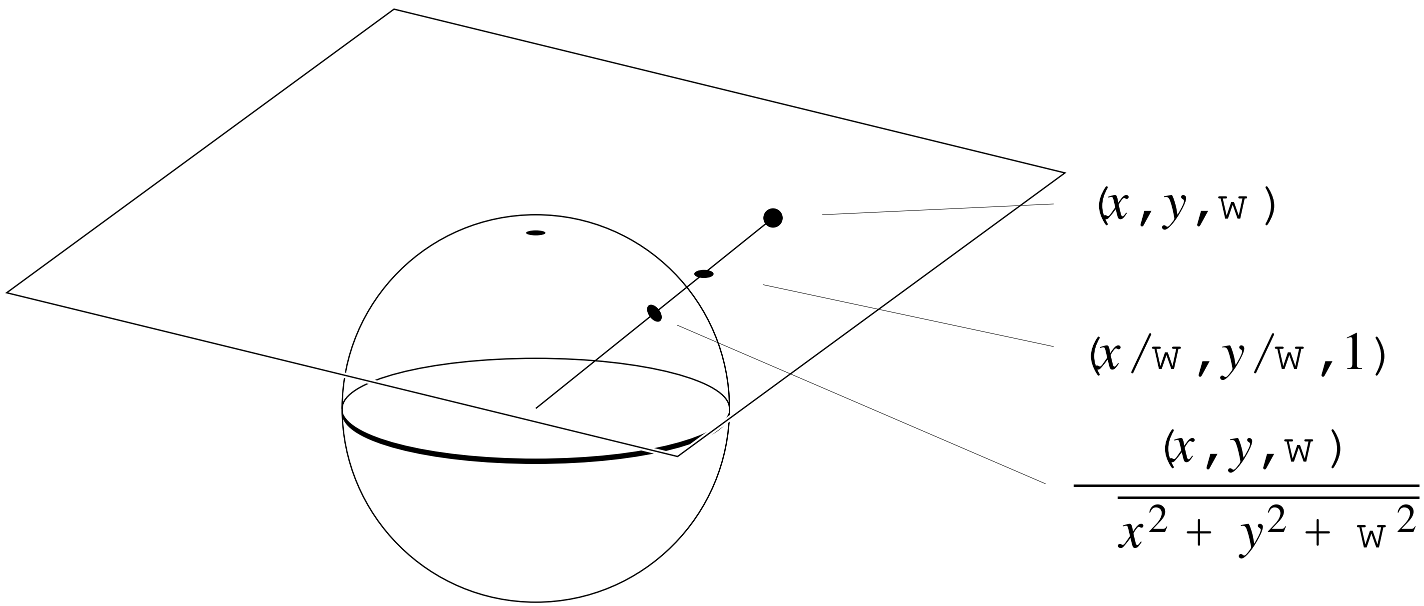
$$\begin{pmatrix} x \\ y \end{pmatrix} \leftrightarrow \left\{ \begin{pmatrix} wx \\ wy \\ w \end{pmatrix} \middle| w \neq 0 \right\}$$

Central projection

The Euclidean plane can be embedded as the $w = 1$ plane.
We can also embed a sphere of points satisfying $x^2 + y^2 + w^2 = 1$.
A line through the origin of $\mathbb{E}^3$

> intersects the sphere in **antipodal points**
>
> intersects the $w = 1$ plane at the appropriate point $(x/w, y/w)$.

These constructions are **central projection**, either to the sphere or to the plane.



Ideal points

The original idea: extend $\mathbb{E}^2$ by adding some ideal points.

Euclidean point: line through origin of $\mathbb{E}^3$ intersecting $w = 1$ plane.

Ideal point: line through origin of $\mathbb{E}^3$ parallel to $w = 1$ plane.

With Cartesian coords, no place to put ideal points. With homogeneous coordinates, there's a big gaping hole!